Gravitational Positivity Bounds on Dark Sector

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Based on arXiv:2305.10058
in collaboration with
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Junsei Tokuda and Masahito Yamazaki

Contents

Model for New Physics

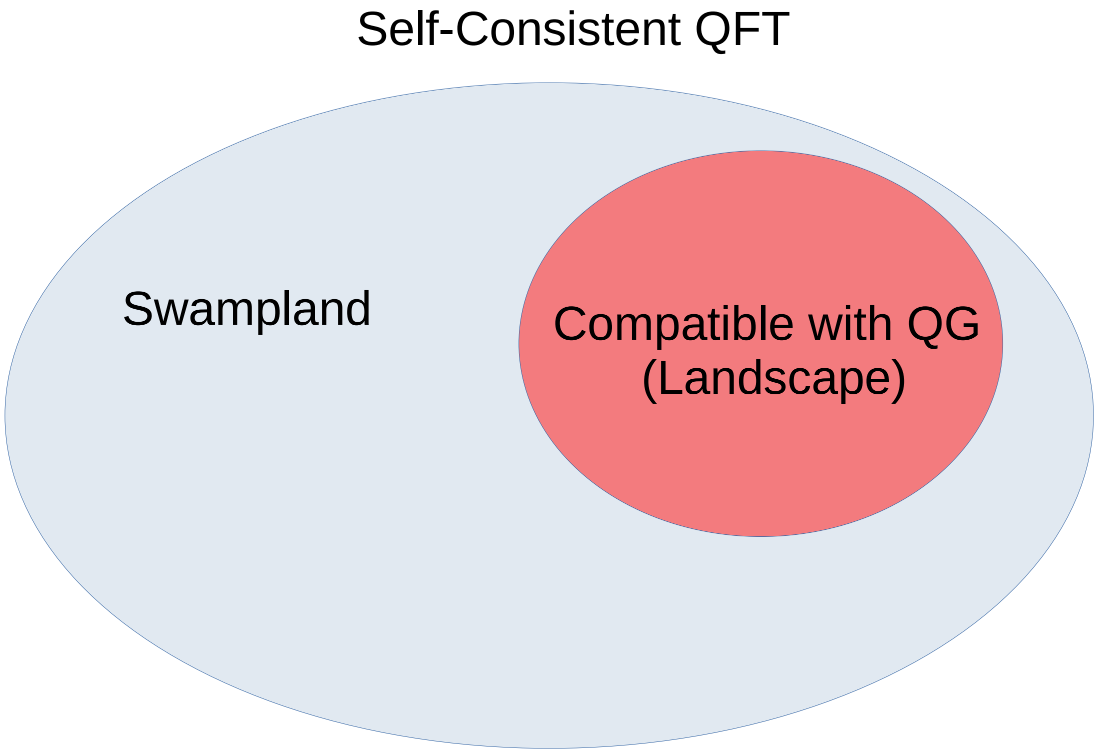
New Physics Model

Inflation
Axion
SUSY
WIMP
PBH
…

Quantum Field Theory (QFT)

Quantum Gravity (QG)

QFT and Quantum Gravity



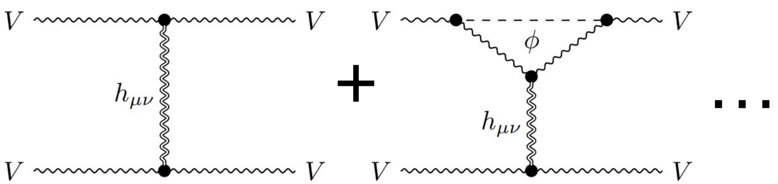
Examples of Swampland

- No global symmetry Banks & Dixon 88
- Weak gravity conjecture Arkani-Hamed et.al. 2006
  - Gravity should be weakest force.
- Distance conjecture Ooguri & Vafa 2006
  - Derivation of filed VEV is less than M_{pl}.
- de Sitter conjecture Obied et.al, 2018
  - Forbidding flat potential with V>0.

Gravitational Positivity

Swampland from comparison of gravitational and non-gravitational interaction.

Gravitational Process



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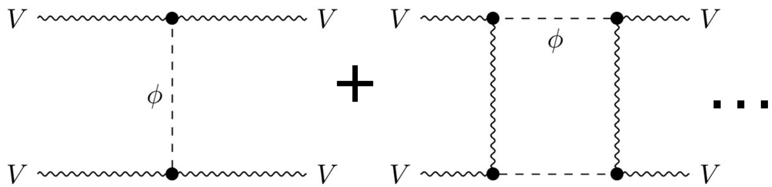
Non-Gravitational Process

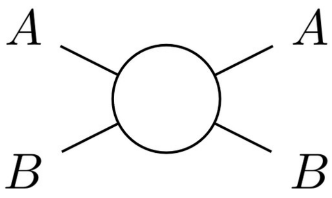


The gravitational process is weaker than non-gravitational process.

Positivity w/o Gravity

Amplitude of forward scattering with s-u symmetry in IR theory :

$$i\mathcal{M}(s, t \to 0) =$$



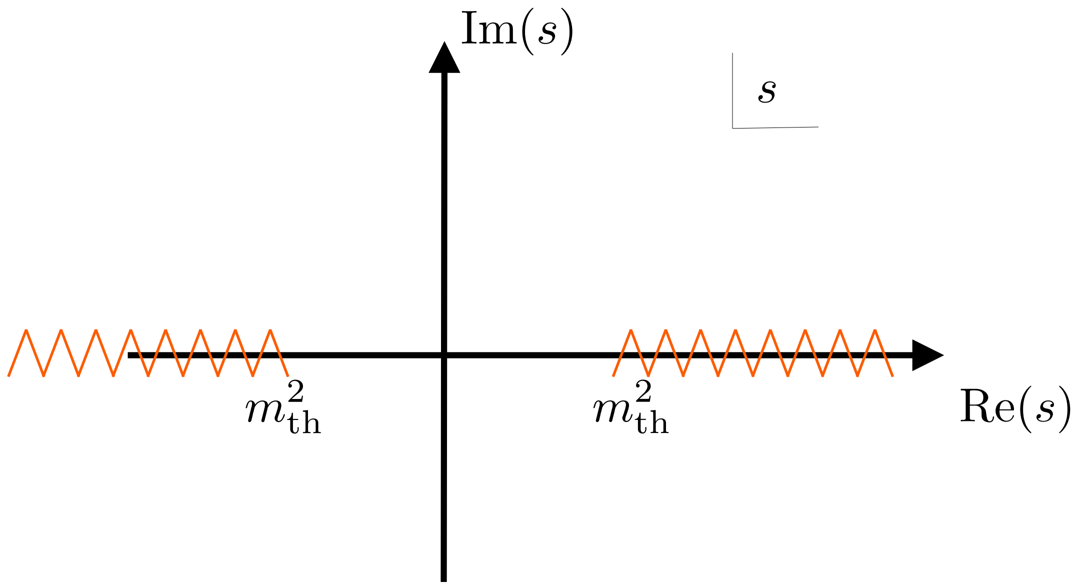
The IR behavior is given by

$$\mathcal{M}(s, t=0) = \sum_{n=1}^{\infty} \frac{a_{2n}}{(2n)!} s^{2n}$$

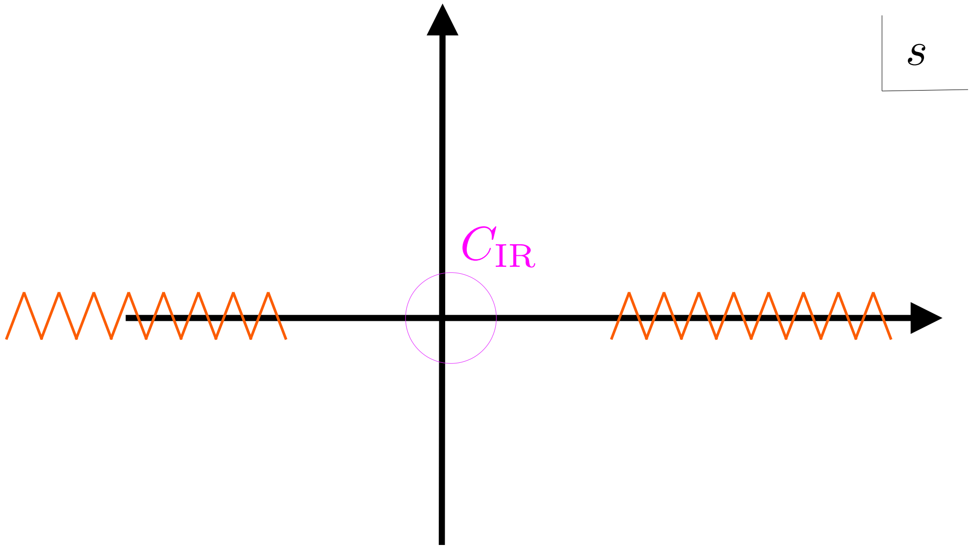
Positivity condition $\boxed{a_{2n} > 0}$

Assumptions

- Analyticity of the amplitude on complex s - plane except for cut and pole
- Unitarity: $\mathrm{Im}\mathcal{M}(s,t=0)>0$ $\sigma_{\mathrm{tot}}(s)\propto \mathrm{Im}\mathcal{M}(s,t=0)$
- High-energy behavior $\lim_{|s|\to\infty}\frac{\mathcal{M}(s,0)}{s^2}\to 0$, c.f., Froissart bound.



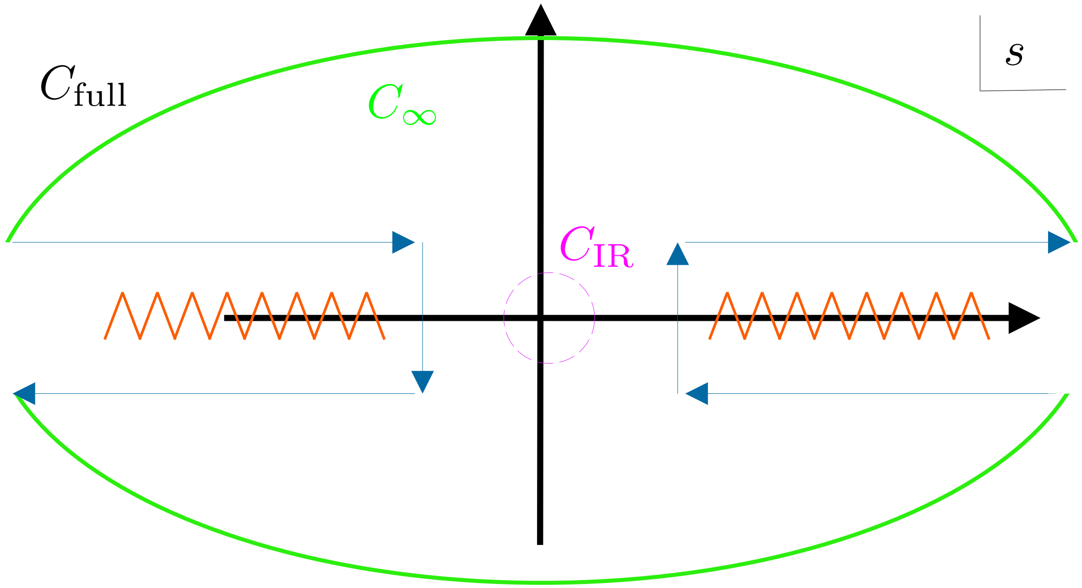
Positivity w/o Gravity



$$\mathcal{M} = \frac{1}{2} a_2 s^2 + \cdots$$

$$a_2 = \frac{1}{i\pi} \oint_{C_{\mathrm{IR}}} \frac{\mathcal{M}(s, t=0)}{s^3}$$

Positivity w/o Gravity



$$\mathcal{M} = \frac{1}{2} a_2 s^2 + \cdots$$

$$a_2 = \frac{1}{i\pi} \oint_{C_{\rm IR}} \frac{\mathcal{M}(s, t=0)}{s^3} = \frac{1}{i\pi} \oint_{C_{\rm full}} \frac{\mathcal{M}(s, t=0)}{s^3}$$

$$a_2 = \frac{4}{\pi} \int_{m_{\rm th}^2}^{\infty} \frac{{\rm Im}\mathcal{M}(s,0)}{s^3} + \frac{1}{i\pi} \oint_{C_\infty} \frac{\mathcal{M}(s,0)}{s^3}$$

Positivity w/o Gravity

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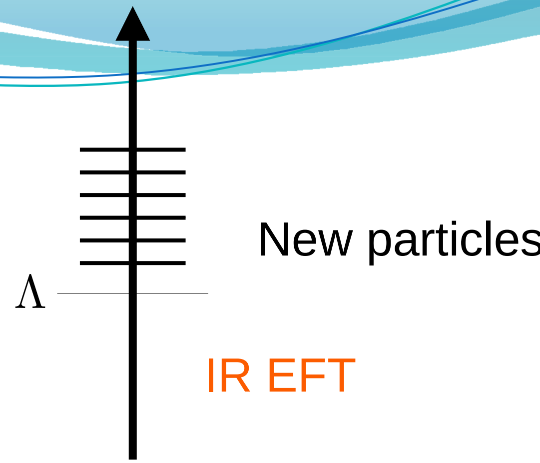
$$>0 \qquad\qquad \to 0$$

$$\Rightarrow \quad a_2>0$$

Improving Bounds



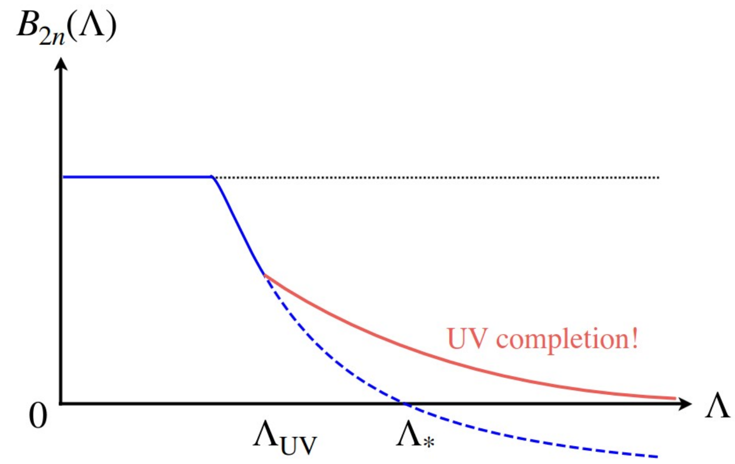
IR behavior of amplitude

$$\mathcal{M} = a_2 s^2 + \cdots \qquad a_2 > 0$$

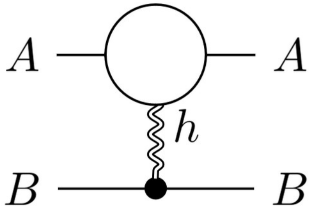
Introduce the cutoff scale of IR EFT Λ

$$B(\Lambda) = a_2 - \frac{4}{\pi}\int_{m_{\rm th}^2}^{\Lambda^2} \frac{{\rm Im}\mathcal{M}(s,0)}{s^3} = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} \frac{{\rm Im}\mathcal{M}(s,0)}{s^3} > 0$$

$B(\Lambda) \sim 0$ indicates the valid cutoff scale of EFT.



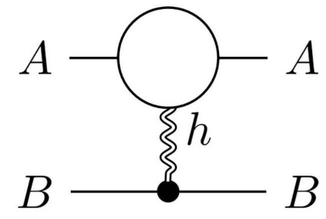
Including Gravity



$$\mathcal{M}(s, t \to 0) = -\frac{s^2}{M_{\mathrm{Pl}} t} + \sum_{n=1}^{\infty} a_{2n} s^{2n}$$

Graviton pole

Including Gravity



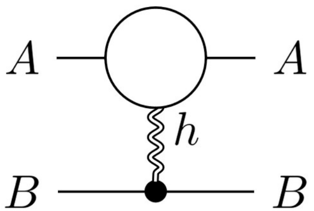
$$\mathcal{M}(s, t \to 0) = -\frac{s^2}{M_{\rm Pl} t} + \sum_{n=1}^{\infty} a_{2n} s^{2n}$$

Graviton pole

$$\Rightarrow \quad B(\Lambda) = \lim_{t \to -0} \left[\frac{2}{M_{\rm Pl}^2 t} + \frac{4}{\pi} \int_{\Lambda^2}^{\infty} ds \, \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{\left(s + \frac{t}{2}\right)^3} \right]$$

Apparently singular, undetermined sign?

Including Gravity



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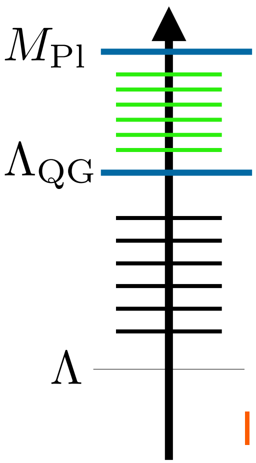
Apparently singular, undetermined sign?

Split into two parts

$$B(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\Lambda_{\rm QG}^2} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3} + \lim_{t\to -0}\left[\frac{2}{M_{\rm Pl}^2 t} + \frac{4}{\pi}\int_{\Lambda_{\rm QG}^2}^{\infty} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{\left(s+\frac{t}{2}\right)^3}\right]$$

Including Gravity



"Stringy" State, Regge Tower

New particles

IR EFT

$$B(\Lambda) = \underbrace{\frac{4}{\pi}\int_{\Lambda^2}^{\Lambda_{\rm QG}^2} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3}}_{\text{IR part}} + \overbrace{\lim_{t\to -0}\left[\frac{2}{M_{\rm Pl}^2 t} + \frac{4}{\pi}\int_{\Lambda_{\rm QG}^2}^{\infty} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{\left(s+\frac{t}{2}\right)^3}\right]}^{\text{Quantum Gravity}}$$

Assumption:
Regge behavior resolves the singularity.

$$=: \frac{\sigma}{M_{\rm Pl}^2 M_{\rm QG}^2} \quad (\sigma = \pm 1)$$

Including Gravity

$$B(\Lambda) = \underbrace{\frac{4}{\pi}\int_{\Lambda^2}^{\Lambda_{\rm QG}^2} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,0)}{s^3}}_{\substack{\text{IR part} \\ O(M_{\rm Pl}^0) + O(M_{\rm Pl}^{-2}) + \cdots}} + \underbrace{\lim_{t\to -0}\left[\frac{2}{M_{\rm Pl}^2 t} + \frac{4}{\pi}\int_{\Lambda_{\rm QG}^2}^{\infty} ds\, \frac{\mathrm{Im}\,\mathcal{M}(s,t)}{\left(s+\frac{t}{2}\right)^3}\right]}_{=: \frac{\sigma}{M_{\rm Pl}^2 M_{\rm QG}^2} \quad (\sigma = \pm 1)}$$

$$\Rightarrow \qquad B_{\rm IR}(\Lambda) \geq \frac{\sigma}{M_{\rm Pl}^2 M_{\rm QG}^2}$$

We do know the exact size of M_{QG}, but it will be around M_{pl}.

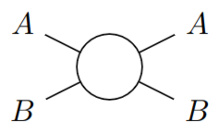
In this case, B is dominated by IR part.

How to: Gravitational Positivity

0. Write down the (renormalizable) Lagrangian of your model, couple the model to gravity, and consider a $2 \to 2$ scattering process $AB \to AB$.

1. Compute the amplitude $\mathcal{M}_{\text{non-grav}}$ of diagrams without gravity:

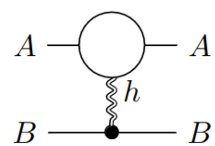
$$i\mathcal{M}_{\text{non-grav}}(s,t) =$$



$$+ \cdots ,$$

   and $\mathcal{M}_{\text{grav},t\text{-channel}}$ of graviton t-channel diagrams:

$$i\mathcal{M}_{\text{grav},t\text{-channel}}(s,t) =$$



$$+ \cdots .$$

2. Compute $B_{\text{non-grav}}$:

$$B_{\text{non-grav}}(\Lambda) = \frac{4}{\pi}\int_{\Lambda^2}^{\infty} ds \, \frac{\mathrm{Im}\,\mathcal{M}_{\text{non-grav}}(s,t=0)}{(s-m_A^2-m_B^2)^3} .$$

3. Compute series expansion of $\mathcal{M}_{\text{grav},t\text{-channel}}$ around $t=0$ and $s = m_A^2 + m_B^2$, and extract B_{grav} from the coefficient of $t^0(s-m_A^2-m_B^2)^2$.

4. Compute $B(\Lambda) = B_{\text{non-grav}}(\Lambda) + B_{\text{grav}}(\Lambda)$ and check its positivity.

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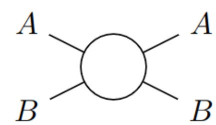
$$i\mathcal{M}_{\text{non-grav}}(s,t) =$$ 

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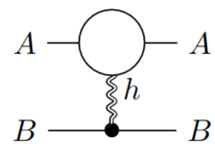
$$i\mathcal{M}_{\text{grav},t\text{-channel}}(s,t) =$$ 

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   >0 <0

Abelian Higgs Model

Simple massive gauge boson model

$$\mathcal{L} = |D_\mu \Phi|^2 - \frac{\lambda}{4}(|\Phi|^2 - v^2)^2$$

Massive gauge boson V

$$m_V = \sqrt{2} g_\Phi v$$

Higgs boson ϕ

$$m_\phi = \sqrt{\lambda} v$$

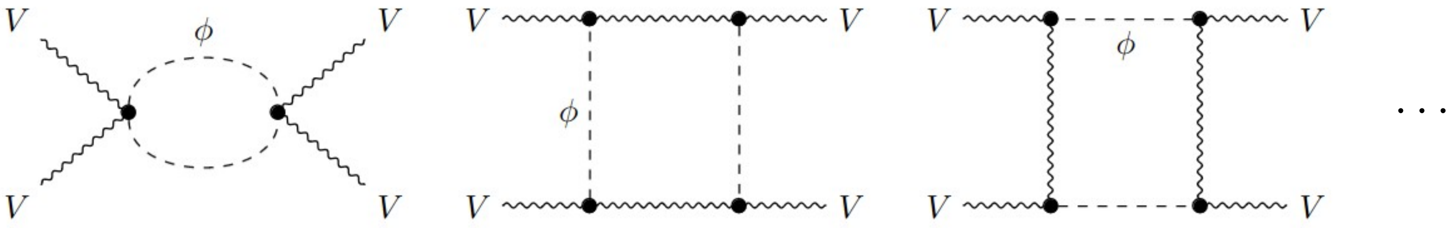
Let’s consider 2-to-2 scattering and get the positivity bound.

$$V_T V_T \to V_T V_T$$

Non-Gravitational Part

Tree-level does not contribute to B.

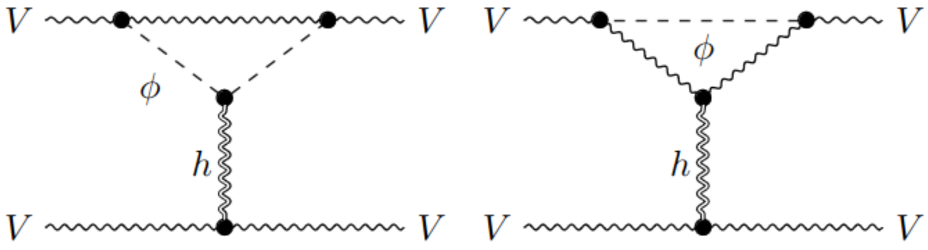
Let's compute one-loop diagrams by usual Feynman diagrams.



$$B^{TT}_{\text{non-grav}} = \frac{g_\Phi^4}{4\pi^2\Lambda^4} + O(\Lambda^{-6})$$

Gravitational Part

Obtain graviton contribution with subtraction of t-pole.



$$B_{\rm grav}^{TT} = -\frac{g_\Phi^2}{72\pi^2 m_\phi^2 M_{\rm Pl}^2} \quad (m_V/m_\phi \to 0)$$

Gravitational Positivity

$$B^{TT}_{\text{non-grav}} = \frac{g_\Phi^4}{4\pi^2\Lambda^4} + O(\Lambda^{-6}) \qquad B^{TT}_{\text{grav}} = -\frac{g_\Phi^2}{72\pi^2 m_\phi^2 M_{\text{Pl}}^2} \quad (m_V/m_\phi \to 0)$$

Gravitational Positivity implies:

$$B^{TT}_{\text{non-grav}} + B^{TT}_{\text{grav}} > 0 \quad \longrightarrow \quad m_\phi > \frac{\Lambda^2}{3\sqrt{2} g_\Phi M_{\text{Pl}}}$$

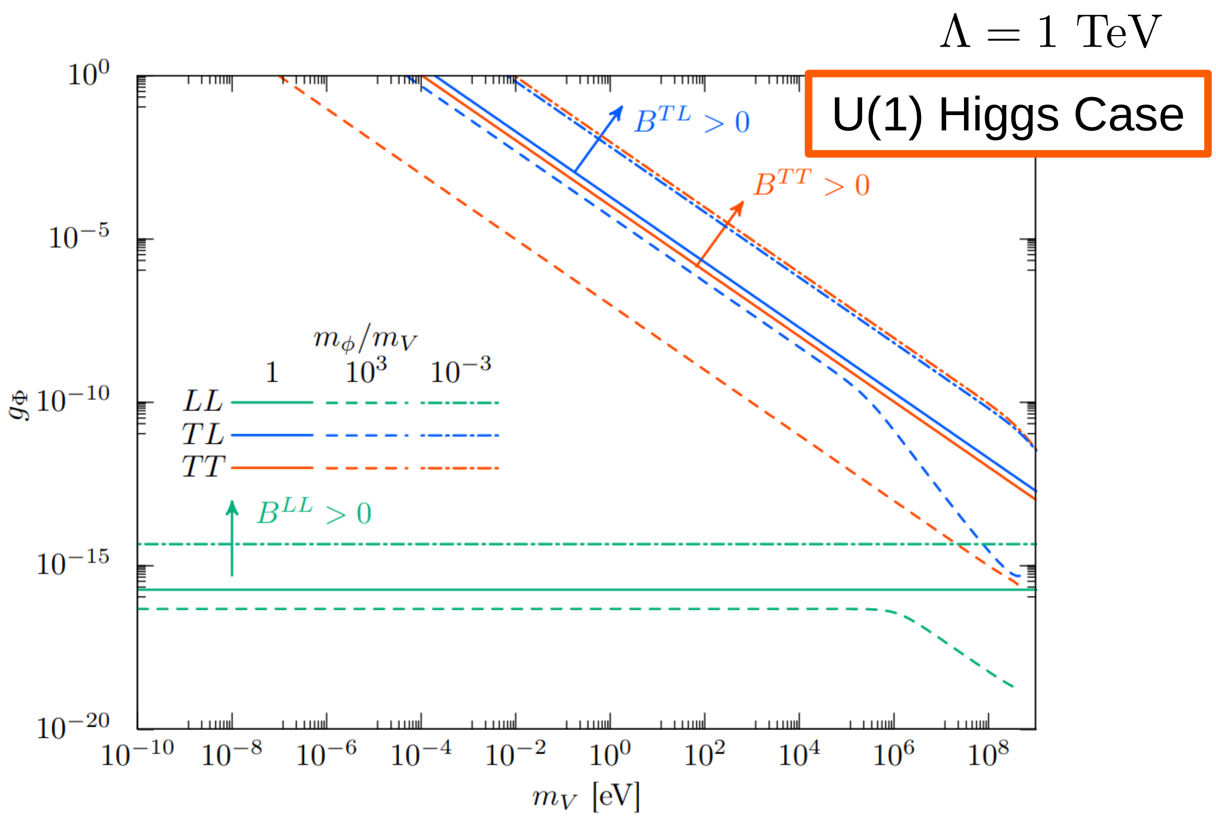
Other Helicity Cases

$$B^{TT}_{\text{non-grav}} + B^{TT}_{\text{grav}} > 0 \quad \longrightarrow \quad m_\phi > \frac{\Lambda^2}{3\sqrt{2} g_\Phi M_{\text{Pl}}}$$

$$B^{TL}_{\text{non-grav}} + B^{TL}_{\text{grav}} > 0 \quad \longrightarrow \quad m_V > \frac{\Lambda^2}{6\sqrt{2} g_\Phi M_{\text{Pl}}}$$

$$B^{LL}_{\text{non-grav}} + B^{LL}_{\text{grav}} > 0 \quad \longrightarrow \quad g_\Phi > \frac{\Lambda}{6\sqrt{2} M_{\text{Pl}}}$$

Gravitational Positivity

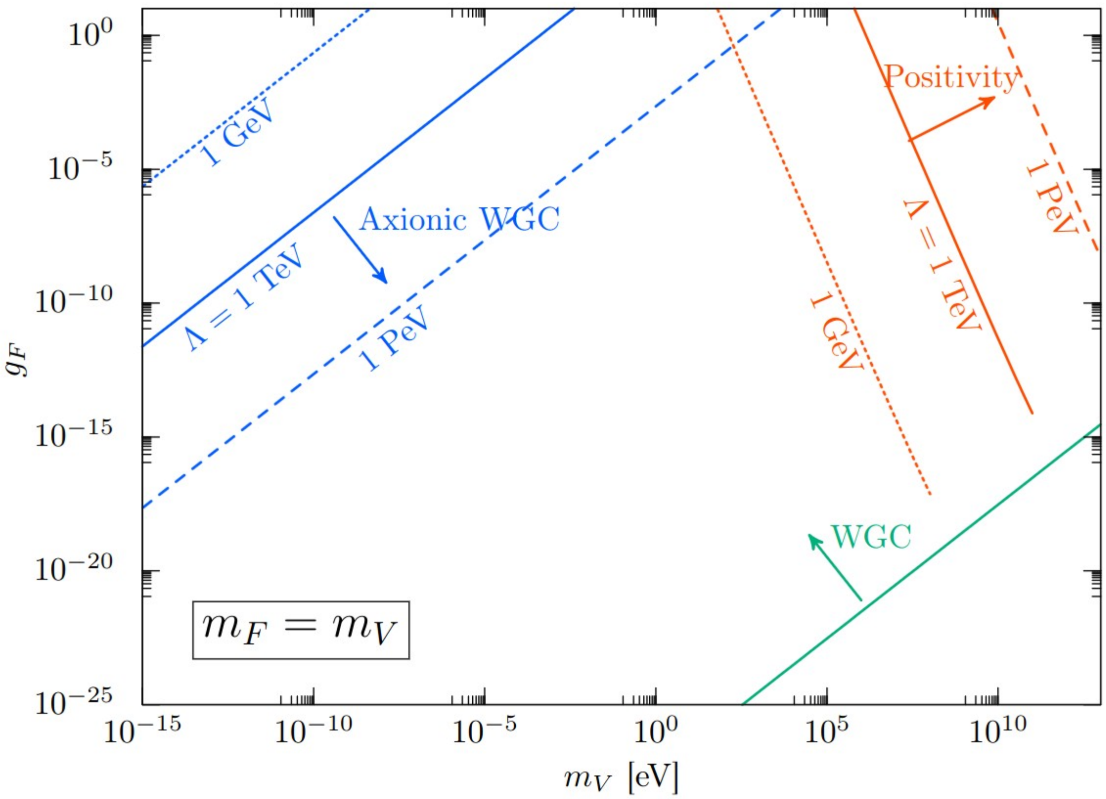


Larger mass and larger gauge couplings are preferred.

Stuckelberg Case

Stuckelberg + fermion

Axionic WGC:
arXiv:1808.09966
by M. Reece



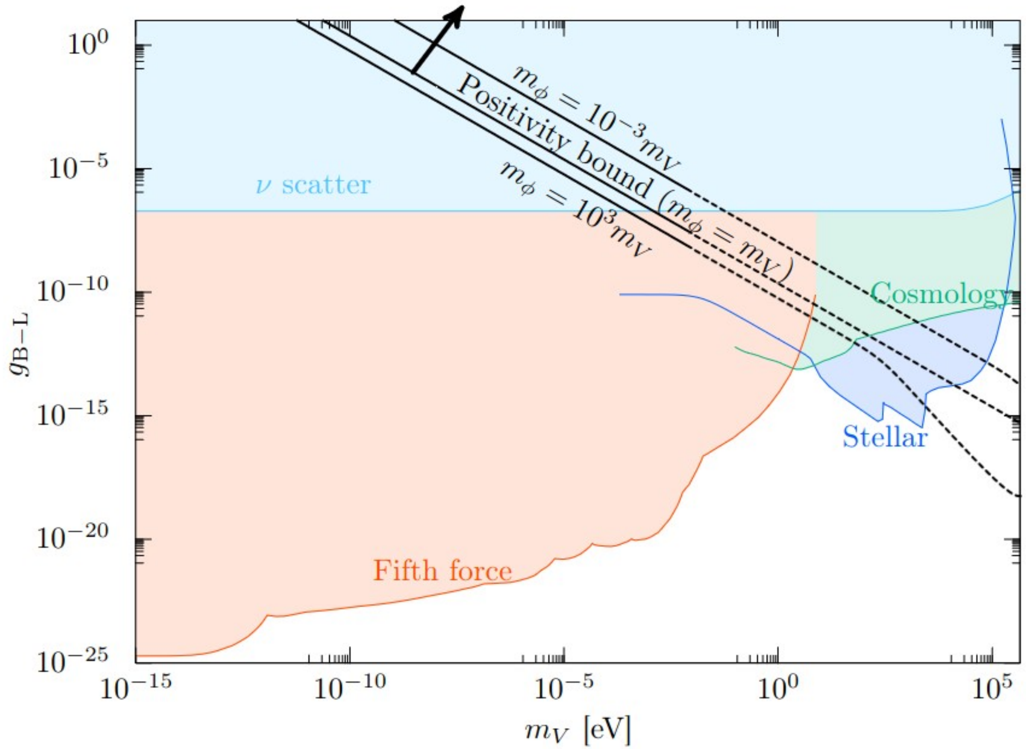
WGC: Weak gravity conjecture

Subtleties for Practical Application

- The Regge scale is still uncertain. $$B(\Lambda) \geq \frac{\sigma}{M_{\mathrm{Pl}}^2 M_{\mathrm{QG}}^2}$$
  - If this scale is much lower than the Planck scale, EFT constraint is weak.
  - Constraint of EFT or Quantum gravity itself?
- S-matrix needs stable asymptotic states
  - In the toy model case, the gauge boson can be stable.
  - Massive gauge boson coupling to SM sector decays into fermions/photons.
  - If the decay rate is super tiny, compared to e.g., age of Universe, can we use the positivity bound?

B-L gauge boson

$\Lambda = 1\ \mathrm{GeV}$



Summary and Discussion

- Gravitational positivity can provide strong constraint on EFT.
  - However, it depends how QG resolve the t-singularity.
- Issues of unstable particles.
  - No asymptotic states in scattering.
  - 3 $\rightarrow$ 3 scattering?
  - Other kinematical choices?
- Various developments in this area of research are ongoing.