Modular flavor symmetry to the flavor structure of SM

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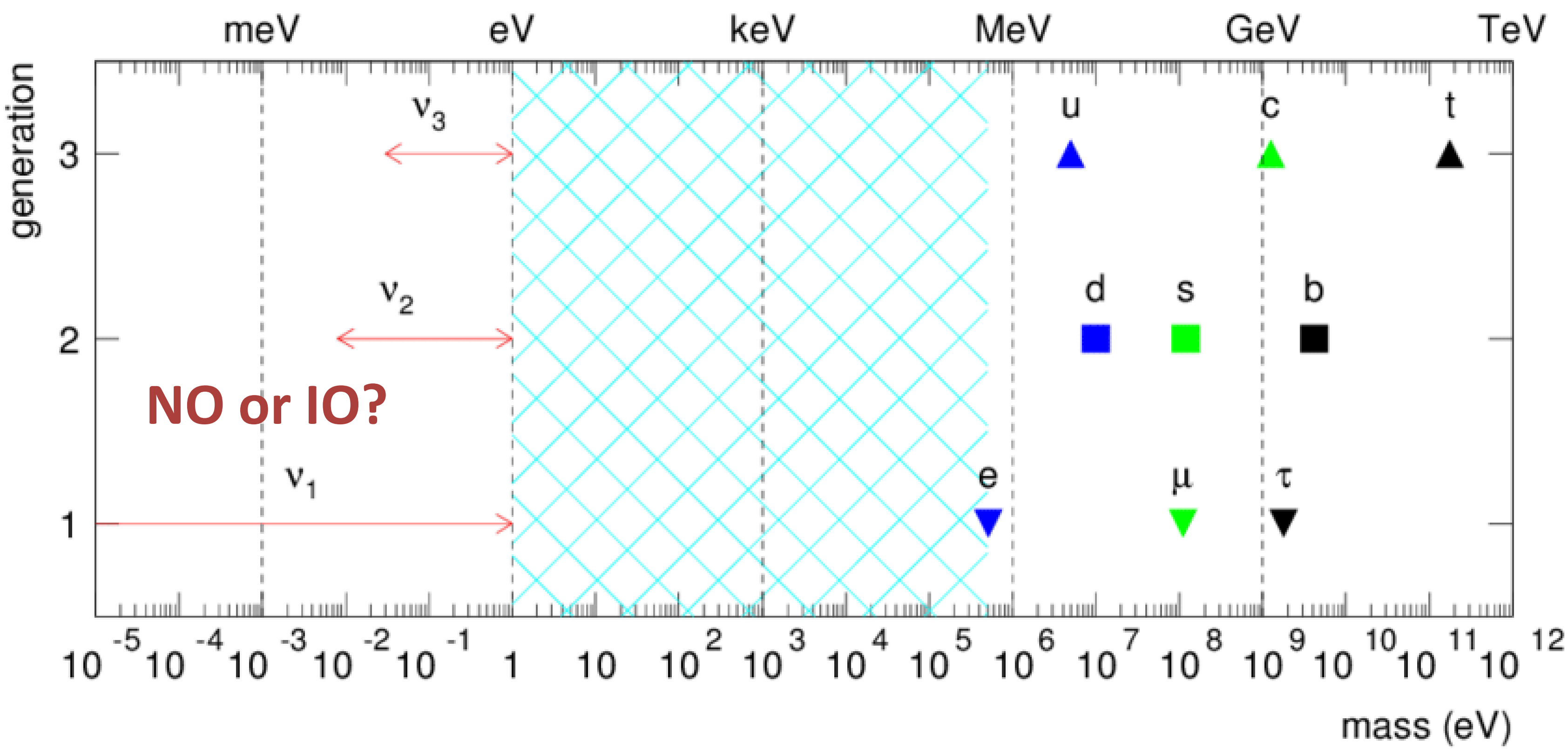
Flavor puzzle 1: why mass hierarchies?



Quarks: from MeV to 100 GeV

Charged leptons: from MeV to GeV

Neutrinos: $\sum_{i=1}^{3} m_i \le 0.12\,\mathrm{eV}$ **from cosmology**

Flavor puzzle 2: why different quark and lepton mixing?

- Quark mixings are small

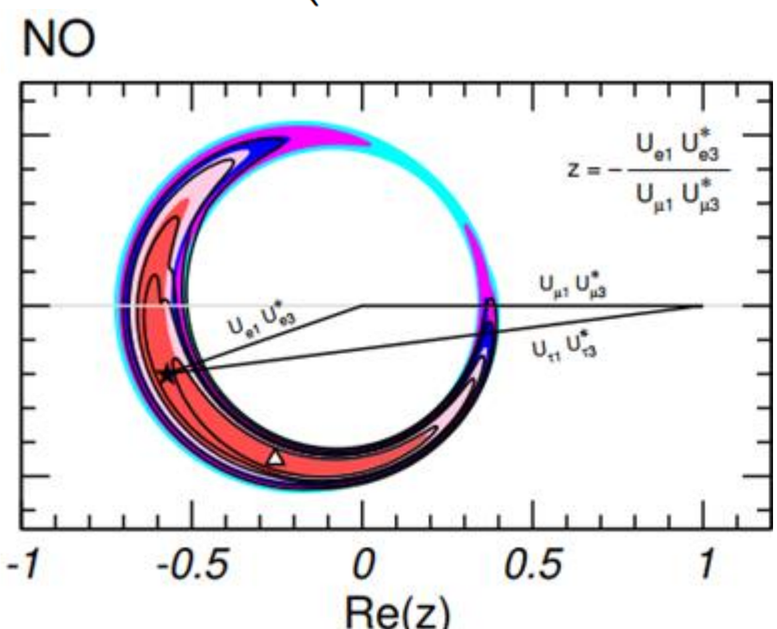
[Particle Data Group 2022]

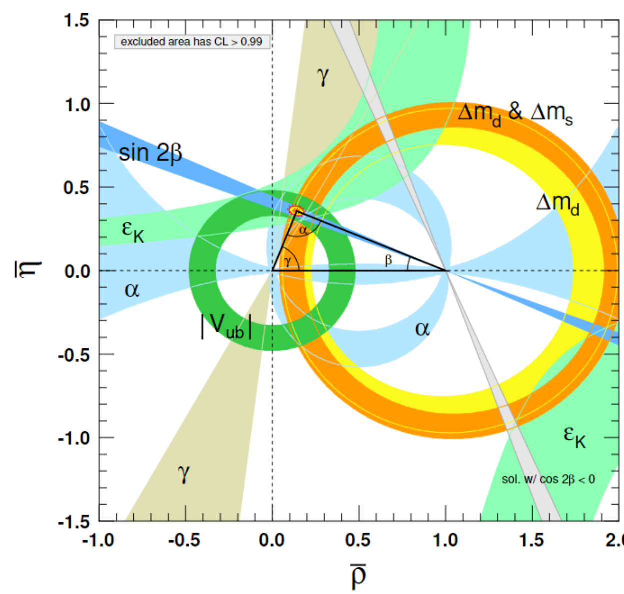
$$V_{\rm CKM} = \begin{pmatrix} 0.97401 \pm 0.00011 & 0.22650 \pm 0.00048 & 0.00361^{+0.00011}_{-0.00009} \\ 0.22636 \pm 0.00048 & 0.97320 \pm 0.00011 & 0.04053^{+0.00083}_{-0.00061} \\ 0.00854^{+0.00023}_{-0.00016} & 0.03978^{+0.00082}_{-0.00060} & 0.999172^{+0.000024}_{-0.000035} \end{pmatrix}$$

- Lepton mixings are large

[NuFIT 5.2 (2022)]

$$|U|^{\rm w/o\ SK\text{-}atm}_{3\sigma} = \begin{pmatrix} 0.801 \to 0.845 & 0.513 \to 0.579 & 0.143 \to 0.156 \\ 0.232 \to 0.507 & 0.459 \to 0.694 & 0.629 \to 0.779 \\ 0.260 \to 0.526 & 0.470 \to 0.702 & 0.609 \to 0.763 \end{pmatrix}$$





“Who orderd that ?”

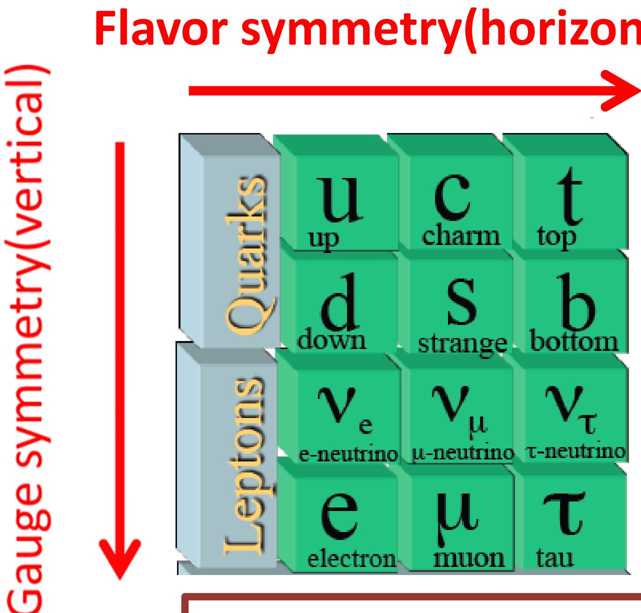
Where do fermion mass hierarchy, flavor mixing, and CP violation come from?

Is there a simple organization principle?

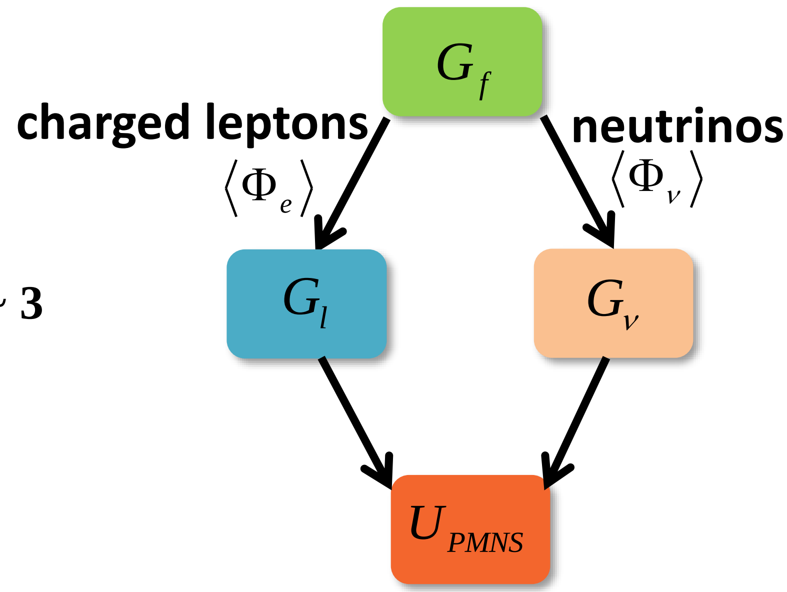
Isidor Issac Rabi

Symmetry as a guiding principle to flavor puzzle

Distinguish three generations by a flavor symmetry G_f, lepton mixing arises from mismatch of the different residual subgroups G_l and G_ν



$$\begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \\ \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L \\ \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \end{pmatrix} \sim \mathbf{3}$$



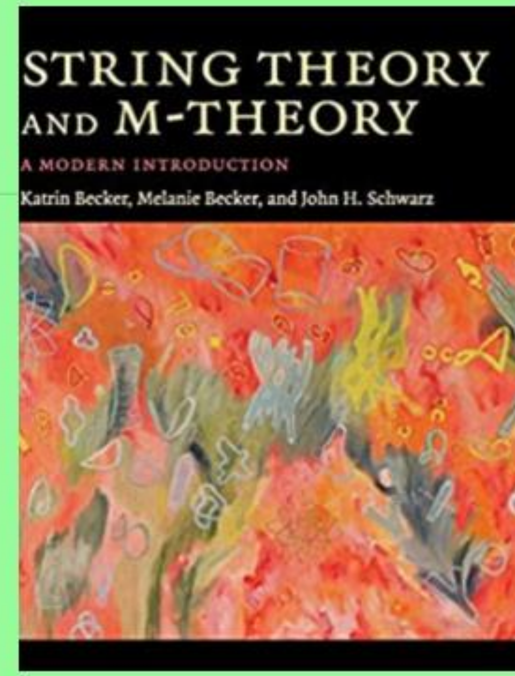
$$\boxed{\mathcal{L}_m = -Y^e_{ij}(\langle\Phi_e\rangle)\overline{L_i}He_{Rj} - \frac{1}{2}Y^\nu_{ij}(\langle\Phi_\nu\rangle)\overline{L^c_i}HH^TL_j + \ldots}$$

G_f	Continuous	Discrete
Abelian	U(1)	Z_n
Non-Abelian	U(2), SU(3),SO(3) …	A_4, S_4, A_5, $\Delta(6n^2)$, …

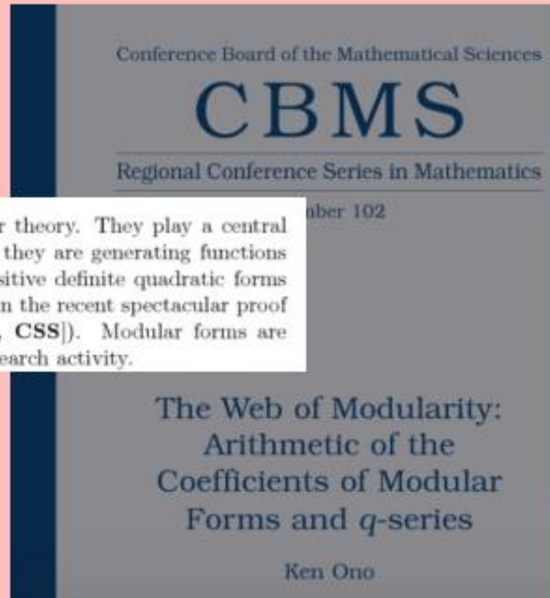
[Altarelli, Feruglio,1002.0211;Tanimoto et al., 1003.3552; King and Luhn, 1301.1340; Xing,1909.09610; Feruglio,Romanino, arXiv:1912.06028]

Modular Symmetries

String Theory



Number Theory



Modular forms appear in many ways in number theory. They play a central role in the theory of quadratic forms; in particular, they are generating functions for the number of representations of integers by positive definite quadratic forms (for example, see [**Gro**]). They are also key players in the recent spectacular proof of Fermat's Last Theorem (see for example, [**Bos**, **CSS**]). Modular forms are presently at the center of an immense amount of research activity.

Credit: Mu-Chun Chen, Bonn2022

Condensed Matter Physics

ELSEVIER

Nuclear Physics B
Volume 474, Issue 3, 2 September 1996, Pages 543-574

Modular invariance, self-duality and the phase transition between quantum Hall plateaus

Eduardo Fradkin [a], Steven Kivelson [b]

Neutrino Physics

arXiv.org > hep-ph > arXiv:1706.08749

High Energy Physics – Phenomenology

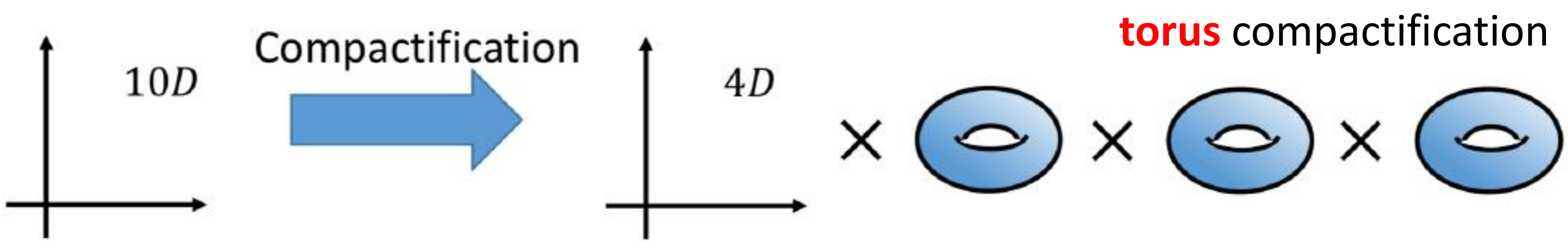
[Submitted on 27 Jun 2017 (v1), last revised 29 Sep 2017 (this version, v2)]

Are neutrino masses modular forms?

Ferruccio Feruglio

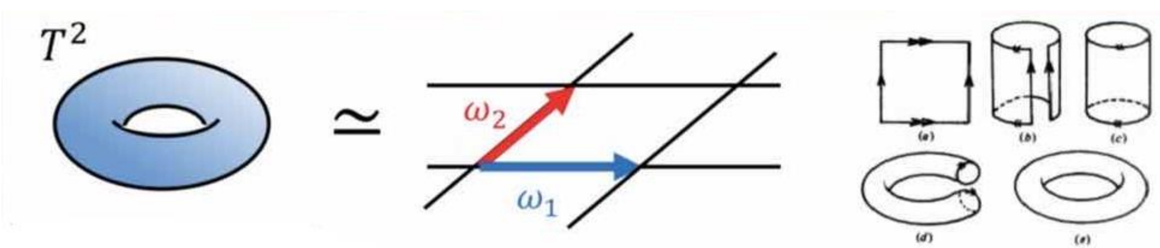
Modular invariance as flavor symmetry

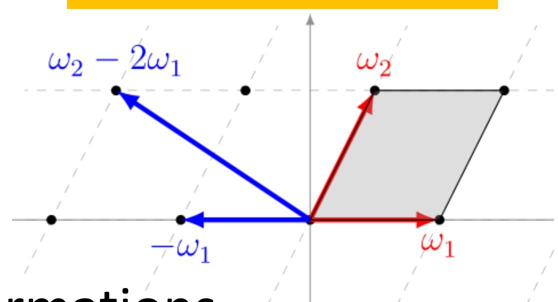
Modular invariance is motivated by more fundamental theory such as string theory at high energy scale [Ferrara et al, 1989; Feruglio, 1706.08749]



4D effective Lagrangian : $S = \int d^4x d^6y \, \mathcal{L}_{10D} \Rightarrow \int d^4x \, \mathcal{L}_{\rm eff}(\varphi, \tau_i)$

The shape of a torus T^2 is characterized by a modulus $\tau = \omega_2 / \omega_1, \, {\rm Im}\, \tau > 0$





The torus (lattice) is left **invariant** by modular transformations

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \Longrightarrow \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

$SL(2,Z):\; ad - bc = 1, \;\; a, b, c, d$ integers

- Finite modular groups as G_f : the quotient over the principal congruence subgroups $\Gamma(N)$

$\mathcal{L}_{\rm eff}(\tau, \Phi) \Longrightarrow \mathcal{L}_{\rm eff}$ **modular invariant**

$\Gamma_N \equiv SL(2,Z) / \pm\Gamma(N), \quad \Gamma_N' \equiv SL(2,Z) / \Gamma(N)$

Modular invariant theory

For N=1 global SUSY, the modular invariant action [Ferrara et al, 1989; Feruglio, 1706.08749]

$$S = \int d^4x d^2\theta d^2\bar{\theta}\; K(\Phi_I, \bar{\Phi}_I, \tau, \bar{\tau}) + \int d^4x d^2\theta\, W(\Phi_I, \tau) + \text{h.c.}$$

- Minimal Kahler potential **(less constrained)**

$$K = -h\Lambda^2 \ln(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\Phi_I|^2$$

[Chen, Sanchez, Ratz 1909.06910]

- Modular invariant superpotential

$$W = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau)\; \Phi_{I_1} \Phi_{I_2} \ldots \Phi_{I_n}$$

Modular transformation: weight k, reps $\rho(\gamma)$ of Γ_N / Γ'_N

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \Phi_I \to (c\tau + d)^{-k_I} \rho_I(\gamma) \Phi_I,$$

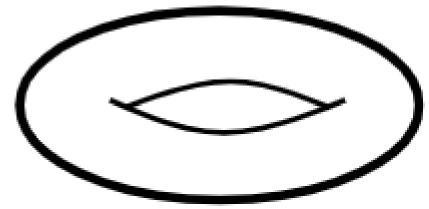
$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

Modular invariance requires

modular weights balance: $k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n}$

containts singlet: $\rho_Y \otimes \rho_{I_1} \otimes \ldots\ldots \otimes \rho_{I_n} \supset \mathbf{1}$

Yukawa couplings are modular forms $Y_{I_1 I_2 \ldots I_n}(\tau)$



$SL(2,Z)$ on torus T^2

$T^N \in \Gamma(N)$

$\Gamma(N)$
congruence subgroups

$\Gamma'_N \equiv SL(2,Z)/\Gamma(N)$
or $\Gamma_N \equiv SL(2,Z)/\pm\Gamma(N)$

finite modular groups

general finite modular groups: SL(2,Z)/normal subgroups

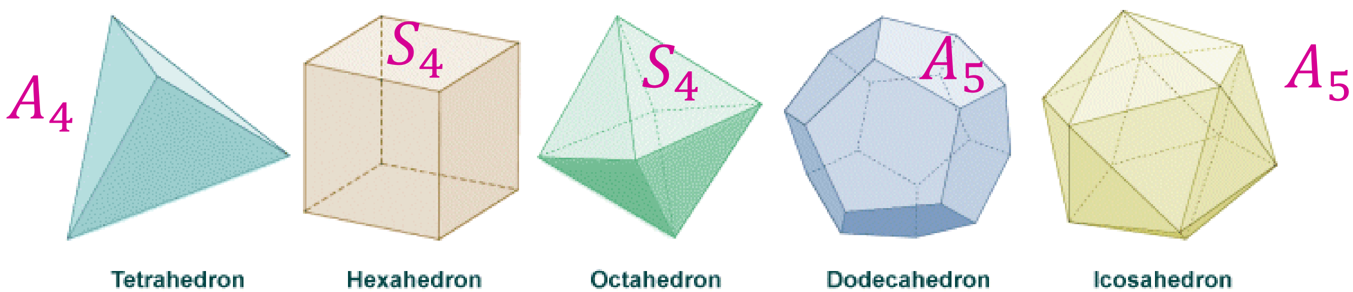
Normal subgroups $\ker(\rho)$			Finite modular groups $\Gamma/\ker(\rho) \cong \mathrm{Im}(\rho)$	
Index	Label	Additional relators	Group structure	GAP Id
6	$\Gamma(2)$	T^2	S_3	$[6,1]$
12	–	S^2T^2	$Z_3 \rtimes Z_4 \cong 2D_3$	$[12,1]$
	$\pm\Gamma(3)$	S^2, T^3	A_4	$[12,3]$
18	–	$ST^{-2}ST^2$	$S_3 \times Z_3$	$[18,3]$
24	$\Gamma(3)$	T^3	T'	$[24,3]$
	–	S^2T^3		
	$\pm\Gamma(4)$	S^2, T^4	S_4	$[24,12]$
	–	$S^2, (ST^{-1}ST)^2$	$A_4 \times Z_2$	$[24,13]$
36	–	$S^3T^{-2}ST^2$	$(Z_3 \rtimes Z_4) \times Z_3$	$[36,6]$
42	–	$T^6, (ST^{-1}S)^2TST^{-1}ST^2$	$Z_7 \rtimes Z_6$	$[42,1]$
	–	$T^6, ST^{-1}ST(ST^{-1}S)^2T^2$		
48	–	S^2T^4	$2O$	$[48,28]$
	–	T^8, ST^4ST^{-4}	$GL(2,3)$	$[48,29]$
	$\Gamma(4)$	T^4	$A_4 \rtimes Z_4 \cong S_4'$	$[48,30]$
	–	$(ST^{-1}ST)^2$	$A_4 \times Z_4$	$[48,31]$
	–	$S^2(ST^{-1}ST)^2$	$T' \times Z_2$	$[48,32]$
	–	T^{12}, ST^3ST^{-3}	$((Z_4 \times Z_2) \rtimes Z_2) \rtimes Z_3$	$[48,33]$
54	–	$T^6, (ST^{-1}ST)^3$	$(Z_3 \times Z_3) \rtimes Z_6$	$[54,5]$
60	$\pm\Gamma(5)$	S^2, T^5	A_5	$[60,5]$
72	–	T^{12}, ST^4ST^{-4}	$S_4 \times Z_3$	$[72,42]$
	$\pm\Gamma(6)$	$S^2, T^6, (ST^{-1}STST^{-1}S)^2T^2$	$A_4 \times S_3$	$[72,44]$

[Liu,Ding, 2112.14761]

modular invariant flavor models

- Finite modular groups: known flavor symmetry S_3, A_4, S_4, A_5

N	2	3	4	5	6	7
Γ_N	S_3	A_4	S_4	A_5	$\Gamma_6 \cong S_3 \times A_4$	$\Gamma_7 \cong \Sigma(168)$
Γ'_N	S_3	$A'_4 = T'$	$S'_4 \cong SL(2, \mathbb{Z}_4)$	$A'_5 \cong SL(2, \mathbb{Z}_5)$	$\Gamma'_6 \cong S_3 \times T'$	$\Gamma'_7 \cong SL(2, \mathbb{Z}_7)$



- Bottom-up models for lepton and quark **[see talks by Ivo Varzielas, Levy Miguel, Xin Wang]**

	Γ_N/Γ'_N	leptons alone	leptons & quarks	$SU(5)$	$SO(10)$
$N=2$	S_3	Kobayashi et al, 1803.10391...	—	Kobayashi et al, 1906.10341...	—
$N=3$	A_4	Feruglio, 1706.08749,1807.01125...	Okada,Tanimoto,1905.13421; King, King, 2002.00969; Yao, Lu, Ding, 2012.13390...	Anda, King,Perdomo, 1812.05620; Chen, Ding, King,2101.12724...	Ding, King,Lu,2108.09655
	T'	Liu, Ding,1907.01488...	Lu, Liu, Ding,1912.07573...	—	—
$N=4$	S_4	Penedo,Petcov,1806.11040 ; Novichkov, Penedo et al,1811.04933...	Qu, Liu et al,2106.11659	Zhao, Zhang,2101.02266 ; Ding, King, Yao,2103.16311...	—
	S'_4	Novichkov,Penedo,Petcov,2006.03058...	Liu, Yao, Ding, 2006.10722...	—	—
$N=5$	A_5	Novichkov, Penedo et al,1812.02158; Ding, King, Liu, 1903.12588...	—	—	—
	A'_5	Wang, Yu, Zhou, 2010.10159 ...	Yao, Liu, Ding,2011.03501	—	—
$N=6$	Γ_6	—	—	Abe,Higaki et al, 2307.01419	—
	Γ'_6	Li,Liu,Ding,2108.02181	—	—	—
$N=7$	Γ_7	Ding, King et al, 2004.12662	—	—	—
	Γ'_7	—	—	—	—

Modular flavor symmetry: significant reduction of the number of parameters

minimal modular lepton model

Modular symmetry allows to construct quite predictive lepton models. The modular flavor symmetry S'_4 which is the double cover of S_4

	L	$E^c_D = (e^c, \mu^c)$	τ^c	N^c	$H_{u,d}$
S'_4	$\mathbf{3}$	$\hat{\mathbf{2}}$	$\hat{\mathbf{1}}'$	$\mathbf{3}$	$\mathbf{1}$
k_I	$-1/2$	$9/2$	$9/2$	$3/2$	0

CP symmetry constrains all coupling constants to be real

[Ding, Liu, Yao, 2211.04546]

➢ Charged leptons

$$\mathcal{W}_e = \alpha \left(E^c_D L Y^{(3)}_{\hat{\mathbf{3}}'}\right)_{\mathbf{1}} H_d + \beta \left(E^c_D L Y^{(3)}_{\hat{\mathbf{3}}}\right)_{\mathbf{1}} H_d + \gamma \left(E^c_3 L Y^{(3)}_{\hat{\mathbf{3}}}\right)_{\mathbf{1}} H_d$$

$$M_e = \begin{pmatrix} 2\alpha Y^{(3)}_{\hat{\mathbf{3}}',1} & -\alpha Y^{(3)}_{\hat{\mathbf{3}}',3} + \sqrt{3}\beta Y^{(3)}_{\hat{\mathbf{3}},2} & -\alpha Y^{(3)}_{\hat{\mathbf{3}}',2} + \sqrt{3}\beta Y^{(3)}_{\hat{\mathbf{3}},3} \\ -2\beta Y^{(3)}_{\hat{\mathbf{3}},1} & \sqrt{3}\alpha Y^{(3)}_{\hat{\mathbf{3}}',2} + \beta Y^{(3)}_{\hat{\mathbf{3}},3} & \sqrt{3}\alpha Y^{(3)}_{\hat{\mathbf{3}}',3} + \beta Y^{(3)}_{\hat{\mathbf{3}},2} \\ \gamma Y^{(3)}_{\hat{\mathbf{3}},1} & \gamma Y^{(3)}_{\hat{\mathbf{3}},3} & \gamma Y^{(3)}_{\hat{\mathbf{3}},2} \end{pmatrix} v_d$$

➢ Neutrino mass : seesaw mechanism

$$\mathcal{W}_\nu = g_1 \left(N^c L\right)_{\mathbf{1}} H_u + \Lambda \left(\left(N^c N^c\right)_{\mathbf{2},s} Y^{(2)}_{\mathbf{2}}\right)_{\mathbf{1}}$$

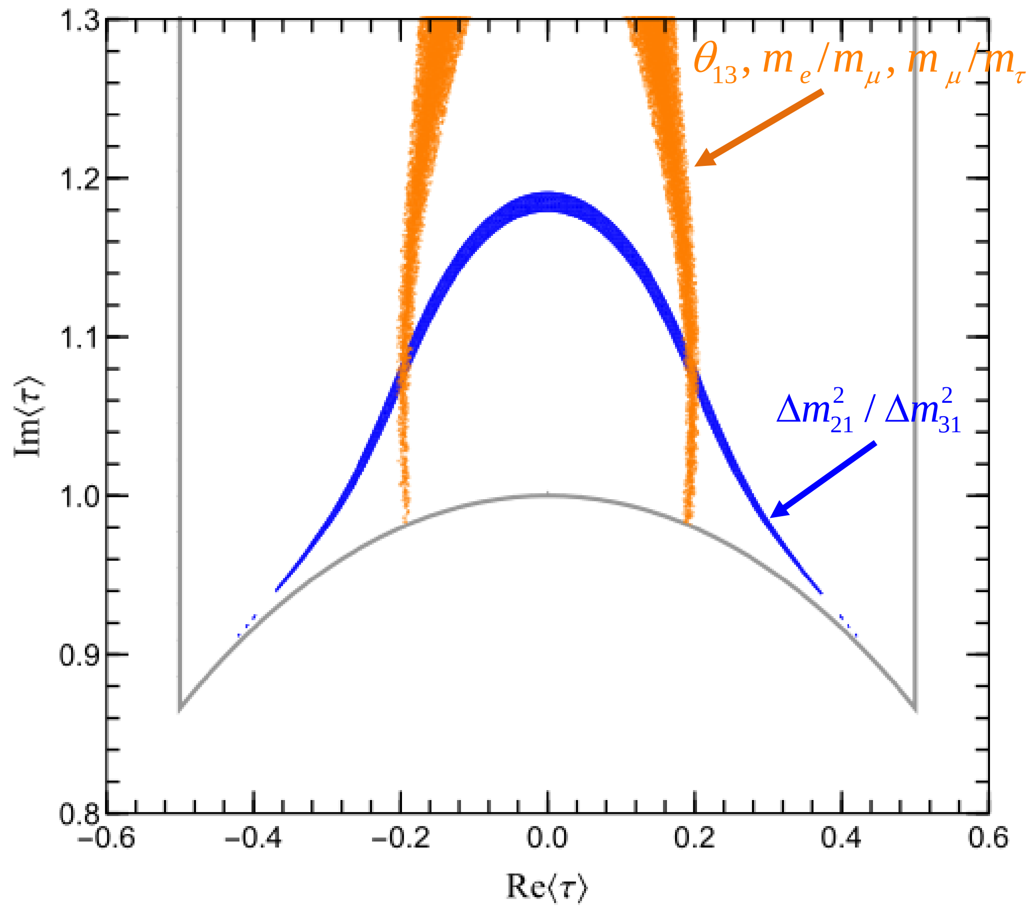
Minimal #p: $\alpha, \beta, \gamma, g^2/\Lambda$

$$M_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u\,, \quad M_N = \Lambda \begin{pmatrix} 2Y^{(2)}_{\mathbf{2},1} & 0 & 0 \\ 0 & \sqrt{3}Y^{(2)}_{\mathbf{2},2} & -Y^{(2)}_{\mathbf{2},1} \\ 0 & -Y^{(2)}_{\mathbf{2},1} & \sqrt{3}Y^{(2)}_{\mathbf{2},2} \end{pmatrix}$$

Light neutrino mass

$$m_1 = \frac{1}{|2Y^{(2)}_{\mathbf{2},1}|}\frac{g^2 v_u^2}{\Lambda}\,, \quad m_2 = \frac{1}{|Y^{(2)}_{\mathbf{2},1} - \sqrt{3}Y^{(2)}_{\mathbf{2},2}|}\frac{g^2 v_u^2}{\Lambda}\,, \quad m_3 = \frac{1}{|Y^{(2)}_{\mathbf{2},1} + \sqrt{3}Y^{(2)}_{\mathbf{2},2}|}\frac{g^2 v_u^2}{\Lambda}$$

only depends on modulus τ up to overall scale



Neutrino mass spectrum is normal ordering

Minimal: only 4 real couplings plus modulus τ can explain 12 observables

$$\langle\tau\rangle = -0.1938 + 1.0832i, \quad \beta/\alpha = 1.7305, \quad \gamma/\alpha = 0.2703,$$

$$\alpha v_d = 244.621\,\text{MeV}, \quad g^2 v_u^2/\Lambda = 29.0744\,\text{meV}$$

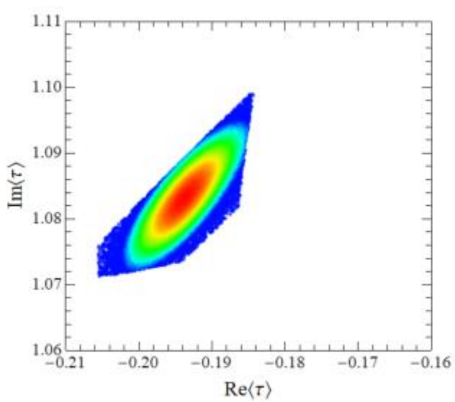
τ is the unique source breaking both modular and CP symmetries. All observables are within the 3σ regions

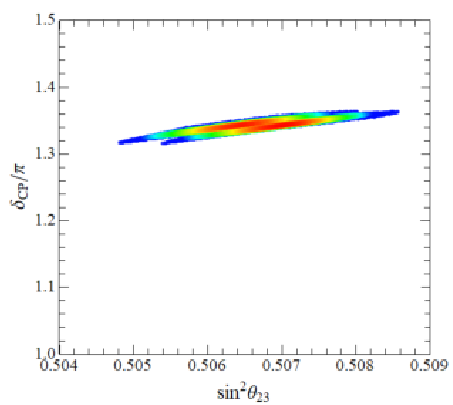
$$\sin^2\theta_{12} = 0.3289, \ \sin^2\theta_{13} = 0.02185, \ \sin^2\theta_{23} = 0.5070, \ \delta_{CP} = 1.3426\pi,$$
$$\alpha_{21} = 1.3287\pi, \ \alpha_{31} = 0.5444\pi, \ m_e/m_\mu = 0.00473, \ m_\mu/m_\tau = 0.0588,$$
$$m_1 = 14.4007\,\text{meV}, \quad m_2 = 16.7803\,\text{meV}, \quad m_3 = 51.7755\,\text{meV}$$

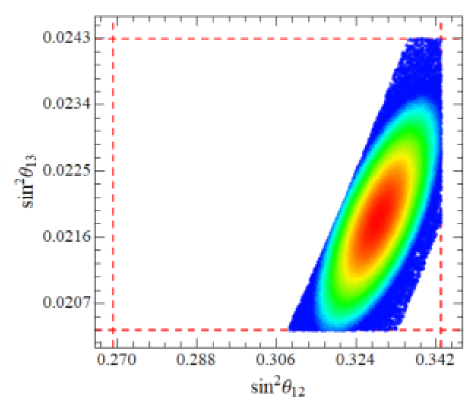
The effective neutrino masses:

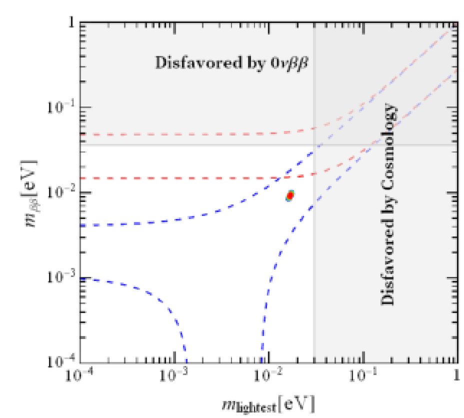
$$m_\beta = 16.891\,\text{meV}, \quad m_{\beta\beta} = 9.253\,\text{meV}$$

below the sensitivity of future experiments





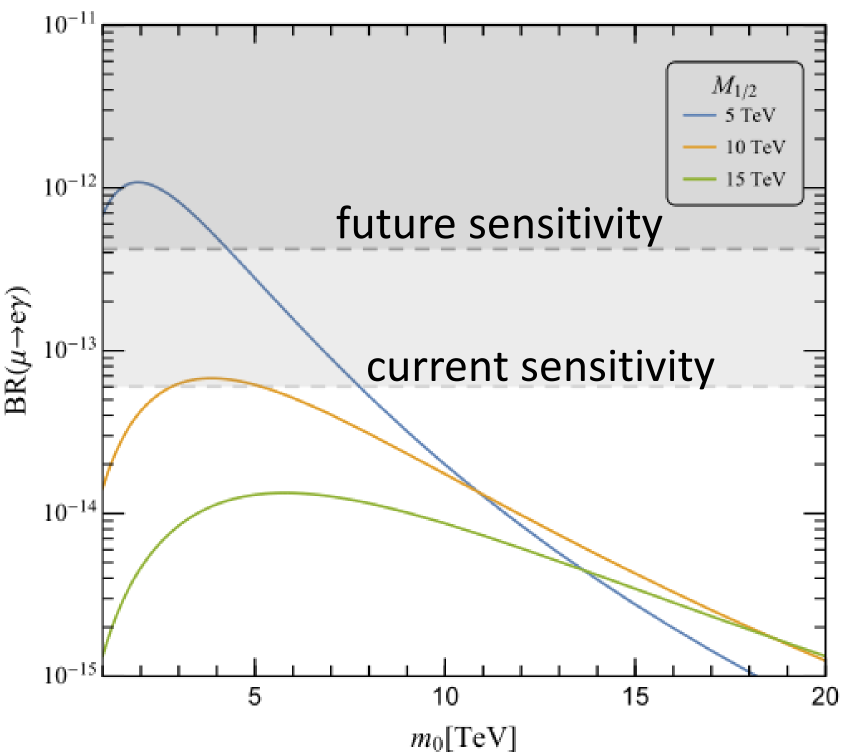


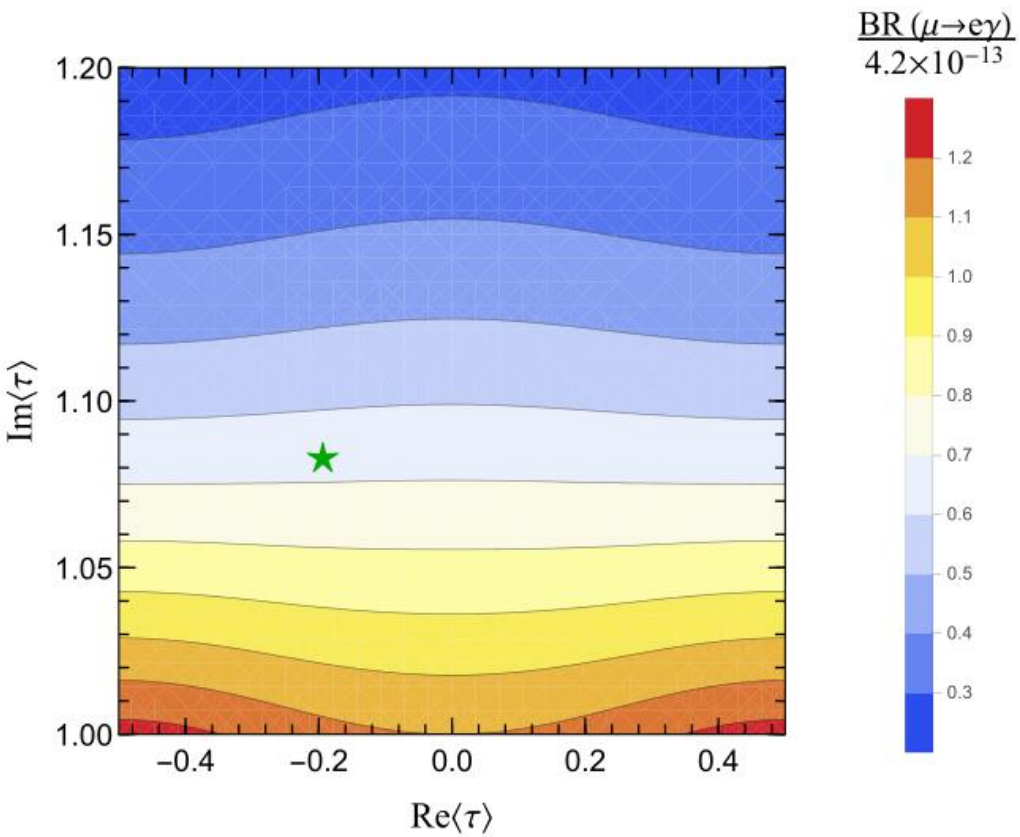


➢ lepton flavor violation $\mu \to e\gamma$

$$\mathcal{M}(\ell_i \to \ell_j \gamma) = m_{\ell_i} \epsilon^{\lambda} \bar{u}_j(p-q)\left[iq^{\nu}\sigma_{\lambda\nu}\left(A_L P_L + A_R P_R\right)\right] u_i(p)$$

$$\frac{\mathrm{BR}\left(\ell_i \to \ell_j \gamma\right)}{\mathrm{BR}\left(\ell_i \to \ell_j \nu_i \overline{\nu_j}\right)} = \frac{48\pi^3 \alpha_e}{G_F^2}\left(\left|A_L^{ij}\right|^2 + \left|A_R^{ij}\right|^2\right)$$





[Ding, Liu, Yao, 2211.04546]

Texture-zero from modular symmetry

- **Texture-zero**: some matrix elements are assumed to be vanishing toreduce the number of free parameters **[S. Weinberg; H. Fritzsch; F. Wilczek & A. Zee, 1977]**

$$M_u = \begin{pmatrix} 0 & A_U & 0 \\ A_U^* & 0 & B_U \\ 0 & B_U^* & C_U \end{pmatrix}, M_d = \begin{pmatrix} 0 & A_D & 0 \\ A_D^* & 0 & B_D \\ 0 & B_D^* & C_D \end{pmatrix} \Rightarrow \tan\theta_C \simeq \sqrt{\frac{m_d}{m_s}}$$

Abelian flavor symmetry is usually used to enforce texture-zero, and non-zero entries are uncorrelated.

- **Modular symmetry origin of texture-zero**

✓ At lower weight, modular forms in certain representations are absent→ texture-zero pattern

$$\sum_{\mathbf{r}} \left(\psi^c \psi Y_{\mathbf{r}}^{(k)} H_{u,d}\right)_{\mathbf{1}} \xrightarrow{\text{missed } \mathbf{r}} m_\psi = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \ldots$$

✓ Odd weight modular forms are in the **``spinor'' irreps $\rho_r(S^2)$=-1**, even weight modular forms are in the **``vector'' irreps $\rho_r(S^2)$=+1**

Note: texture-zero can also arise from fixed points $\tau_f = i\infty, i, \omega$ **[Kikuchi et al,2207.04609]**

Taking level N=3 as an example

Assignment under $\Gamma_3' \cong T'$:

$$Q_D \equiv \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \sim 2\ (\text{or } 2', 2''), \qquad Q_3 \sim 1\ (\text{or } 1', 1'')$$

Left-handed doublets

$$q_D^c \equiv \begin{pmatrix} q_1^c \\ q_2^c \end{pmatrix} \sim 2\ (\text{or } 2', 2''), \qquad q_3^c \sim 1\ (\text{or } 1', 1'')$$

Right-handed singlets

Mass matrix with texture zero

vanishing for $k_{q_D^c} + k_{Q_D}$ odd

vanishing for $k_{q_D^c} + k_{Q_3}$ even

$$M_q = \begin{pmatrix} \text{x} & \text{x} & \text{x} \\ \text{x} & \text{x} & \text{x} \\ \text{x} & \text{x} & \text{x} \end{pmatrix}$$

[Lu, Liu, Ding, 1912.07573]

vanishing for $k_{q_3^c} + k_{Q_D}$ even

vanishing for $k_{q_3^c} + k_{Q_3}$ odd

- **Five** texture zeros of quark mass matrices up to row and column permutations can be achieved from the $\Gamma'_3 \cong T'$ modular symmetry

$$\text{Case } \mathcal{A}: \; M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \text{Case } \mathcal{B}: \; M_q = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}$$

$$\text{Case } \mathcal{C}: \; M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ \times & \times & \times \end{pmatrix}, \quad \text{Case } \mathcal{D}: \; M_q = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & 0 \end{pmatrix}$$

$$\text{Case } \mathcal{E}: \; M_q = \begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & 0 & \times \end{pmatrix}.$$

[Lu, Liu, Ding, 1912.07573]

- Texture-zero patterns of lepton mass matrices for **#p≤9**

Neutrino nature	gCP	Number of textures	Viable
Dirac	no	136	23 (NO)
			27 (IO)
	yes	174	97 (NO)
			57 (IO)
Majorana (Weinberg operator)	no	29	5 (NO)
			7 (IO)
	yes	31	11 (NO)
			10 (IO)
Majorana (Seesaw mechanism)	no	35	6 (NO)
			10 (IO)
	yes	36	13 (NO)
			14 (IO)

[Ding, Joaquim, Lu, 2211.0813]

- ✓ Texture-zeros is enforced by the structure of modular forms
- ✓ Non-zero entries are correlated by modular symmetry, the predictive power of texture-zero is improved greatly.

Mass hierarchies in modular symmetry

- **Weighton mechanism**: modular weight plays the role of Froggatt-Nielsen charge, the couplings are suppressed by power of "weighton" ϕ which is a scalar invariant under modular symmetry

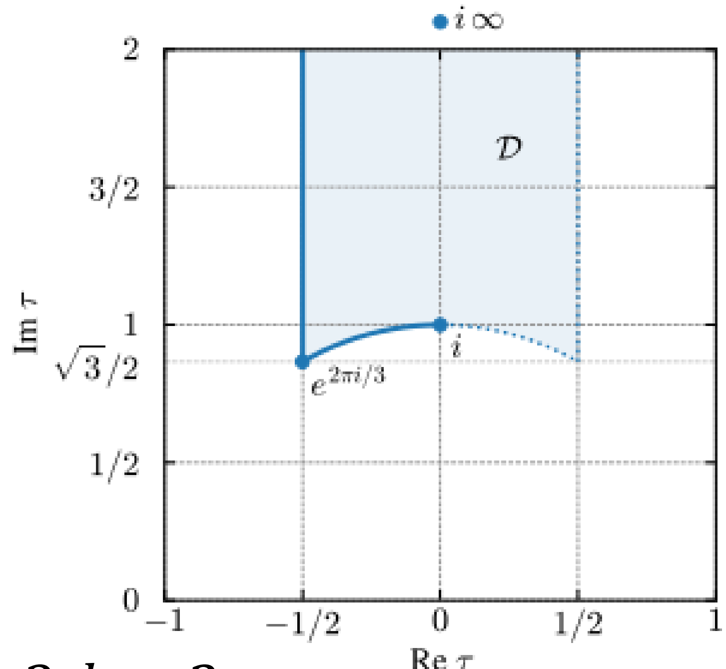
$$\mathcal{W}_e = \alpha e^c \tilde{\phi}^4 (LY_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta \mu^c \tilde{\phi}^2 (LY_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma \tau^c \tilde{\phi} (LY_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d, \quad \tilde{\phi} = \frac{\phi}{\Lambda}$$

[Criado, Feruglio, King, 1908.11867; King, King, 2002.00969]

- Fermion mass hierarchies arise from proximity of τ to the fixed points τ_f

[Okada and Tanimoto, 2009.14242;Feruglio, Gherardi, Romanino,Titov, 2101.08718; Novichkov,Penedo, Petcov,2102.07488]

τ_f	Inv. under	Residual sym.
$i\infty$	$\tau \to \tau + 1$	$\mathbb{Z}_N$
i	$\tau \to -\frac{1}{\tau}$	$\mathbb{Z}_2$
$\omega = e^{2\pi i/3}$	$\tau \to -\frac{1}{\tau+1}$	$\mathbb{Z}_3$



Some modular forms vanish at fixed point $\tau = \tau_f$

e.g. $(Y_1(\tau), Y_2(\tau), Y_3(\tau)) \xrightarrow{\tau = i\infty} (1, 0, 0)$ at $N = 3, k = 2$

- ✓ Some entries of fermion mass matrix are vanishing at fixed points, and they are power of deviation O (ϵ^p) in the vicinity of fixed points

$\tau = \tau_f$

$\epsilon = |\tau - \tau_f| \ll 1$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow M \sim \begin{pmatrix} 1 & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \end{pmatrix}$$

hierarchical fermion masses in powers of ϵ

- ✓ Promising hierarchical patterns for leptons

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

[Novichkov, Penedo, Petcov, 2102.07488]

Kähler problem in modular symmetry

- Kähler potential potential not fixed by modular flavor symmetry

$$\mathcal{K} = \boxed{(-i\tau + i\bar{\tau})^{-k_\psi} \left(\psi^\dagger \psi\right)_{\mathbf{1}}} + \boxed{\sum_{n, \boldsymbol{r_1}, \boldsymbol{r_2}} c^{(n, \boldsymbol{r_1}, \boldsymbol{r_2})} (-i\tau + i\bar{\tau})^{-k_\psi + n} \left(\psi^\dagger Y^{(n)\dagger}_{\boldsymbol{r_1}} Y^{(n)}_{\boldsymbol{r_2}} \psi\right)_{\mathbf{1}}}$$

minimal Kähler potential

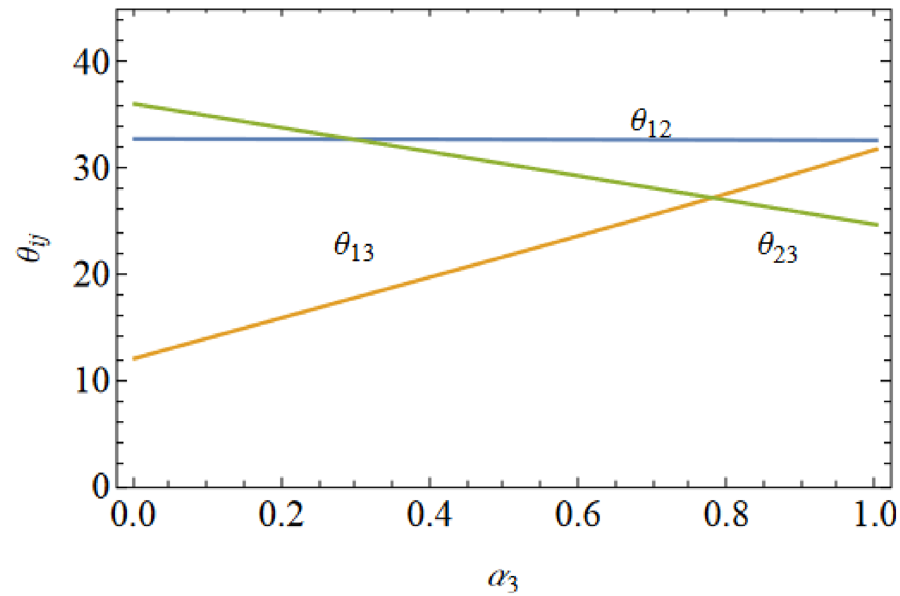
non-canonical terms

Many non-canonical terms on the same footing as the minimal Kähler potential are allowed by modular symmetry

- Modifying Kähler metric and kinetic terms

[Chen, Ramon-Sanchez, Ratz 1909.06910]

$$\mathcal{K}_\psi^{ij} = \frac{\partial^2 \mathcal{K}}{\partial \psi_i^\dagger \partial \psi_j} = \langle -i\tau + i\bar{\tau} \rangle^{-k_\psi} \delta^{ij} + \Delta \mathcal{K}_\psi^{ij}$$



- Back to canonical Basis ➜ **sizable corrections to mixing parameters**

Solution to Kähler problem: eclectic flavor groups

- Modular flavor symmetries from top-down approach (orbifold string compactification) gives [Nilles et al, 2001.01736; 2004.05200]
  - Normal symmetries of extra dimensions ➜traditional flavor symmetries
  - String duality transformations ➜modular flavor symmetries
  - the multiplicative closure of these groups is defined as the eclectic flavor group

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

[see talk by Hans Nilles]

- Traditional flavor symmetry vs. modular symmetry transformations

modular: $\tau \xrightarrow{\gamma} \gamma\tau \equiv \frac{a\tau+b}{c\tau+d}, \quad \psi \xrightarrow{\gamma} (c\tau+d)^{-k_\psi}\rho(\gamma)\psi,$

flavor: $\tau \xrightarrow{g} \tau, \quad \psi \xrightarrow{g} \rho(g)\psi, \quad g \in G_f$

flavor symmetry transformations leave τ invariant

- The interplay of traditional flavor symmetry an modular symmetry can restrict the allowed Kähler potential

➢ Consistency condition [Nilles et al, 2001.01736]

$$
\begin{tikzcd}
 & (c\tau+d)^{-k}\rho(\gamma)\psi \arrow[dr, bend left, "g\in G_f"] & \\
\psi \arrow[ur, bend left, "\gamma\in SL(2,\mathbb{Z})"] \arrow[dr, bend right, "g'=u_\gamma(g)"'] & & (c\tau+d)^{-k}\rho(\gamma)\rho(g)\psi \arrow[dl, bend left, "\gamma^{-1}"] \\
 & \rho(u_\gamma(g))\psi = \rho(\gamma)\rho(g)\rho^{-1}(\gamma)\psi &
\end{tikzcd}
$$

$$\rho(\gamma)\rho(g)\rho^{-1}(\gamma)=\rho(u_\gamma(g))$$

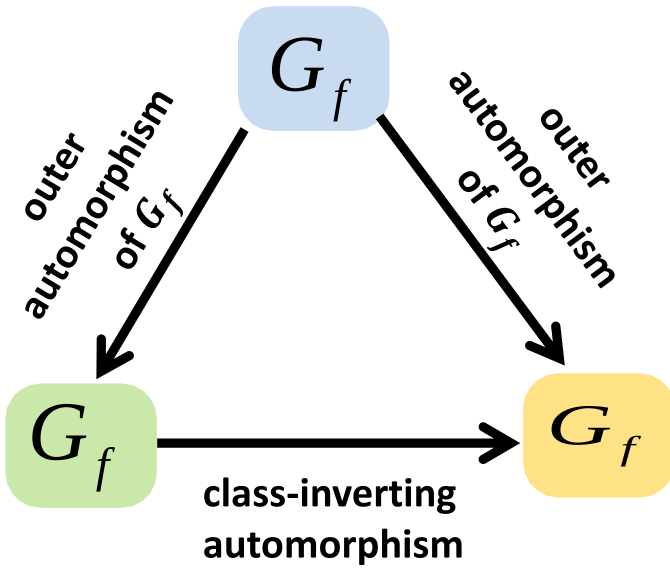
Each modular transformation $\rho(\gamma)$ corresponds an automorphism u_γ of flavor symmetry group G_f. Finite modular group Γ'_N (Γ_N) must be a subgroup of the outer automorphism group of G_f

$$\boxed{G_{\text{eclectic}} \cong G_f \rtimes \Gamma'_N \ (G_f \rtimes \Gamma_N)}$$

- ✓ Modular transformation $\rho(\gamma)$ is fixed by flavor symmetry transformation $\rho(g)$, they cannot be freely assigned as with modular symmetry alone

- ✓ **Not any** flavor symmetry G_f have eclectic extension!

- Eclectic flavor group can combine with CP: **unification of flavor, CP and modular symmetries**



CP as outer automorphism of both traditional flavor group G_f and modular group Γ'_N (Γ_N)

- Possible eclectic flavor groups: few G_f suitable to eclectic extension

flavor group $\mathcal{G}_{\mathrm{fl}}$	GAP ID	$\mathrm{Aut}(\mathcal{G}_{\mathrm{fl}})$	finite modular groups		eclectic flavor group
Q_8	[8, 4]	S_4	without $\mathcal{CP}$	S_3	$\mathrm{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\mathbb{Z}_3 \times \mathbb{Z}_3$	[9, 2]	$\mathrm{GL}(2,3)$	without $\mathcal{CP}$	S_3	$\Delta(54)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
A_4	[12, 3]	S_4	without $\mathcal{CP}$	S_3	S_4
				S_4	S_4
			with $\mathcal{CP}$	–	–
T'	[24, 3]	S_4	without $\mathcal{CP}$	S_3	$\mathrm{GL}(2,3)$
			with $\mathcal{CP}$	–	–
$\Delta(27)$	[27, 3]	[432, 734]	without $\mathcal{CP}$	S_3	$\Delta(54)$
				T'	$\Omega(1)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
				$\mathrm{GL}(2,3)$	[1296, 2891]
$\Delta(54)$	[54, 8]	[432, 734]	without $\mathcal{CP}$	T'	$\Omega(1)$
			with $\mathcal{CP}$	$\mathrm{GL}(2,3)$	[1296, 2891]

first eclectic model with $G_{eclectic} = \Delta(54) \cup T'$,
Baur, Nilles et al,2207.10677

[Nilles,Ramos-Sanchez, Vaudrevange, et al, 2001.01736]

Eclectic lepton model with $\Omega(1) \cong \Delta(27) \rtimes T'$

- Field content: flavon fields are necessary to break $\Delta(27)$ flavor symmetry
  Matter fields can only be triplet or trivial singlet of $\Delta(27)$

Fields	L	E^c	H_u	H_d	ϕ	φ	χ	ξ	$Y_{\boldsymbol{r}}^{(k_Y)}$
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 1)$	$(\mathbf{2}, \frac{1}{2})$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Delta(27)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{\mathbf{0},\mathbf{0}}$	$\mathbf{1}_{\mathbf{0},\mathbf{0}}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{\mathbf{0},\mathbf{0}}$	$\mathbf{1}_{\mathbf{0},\mathbf{0}}$
$\Gamma'_3 \cong T'$	$\mathbf{3}_0$	$\mathbf{3}_0$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}$	$\boldsymbol{r}$
modular weight	0	0	0	0	5	5	7	-1	k_Y
Z_2	1	-1	1	1	-1	1	1	1	1
Z_3	ω	ω^2	1	1	1	ω	ω	1	1

- Superpotential for charged lepton and neutrino masses: **only 4 terms**

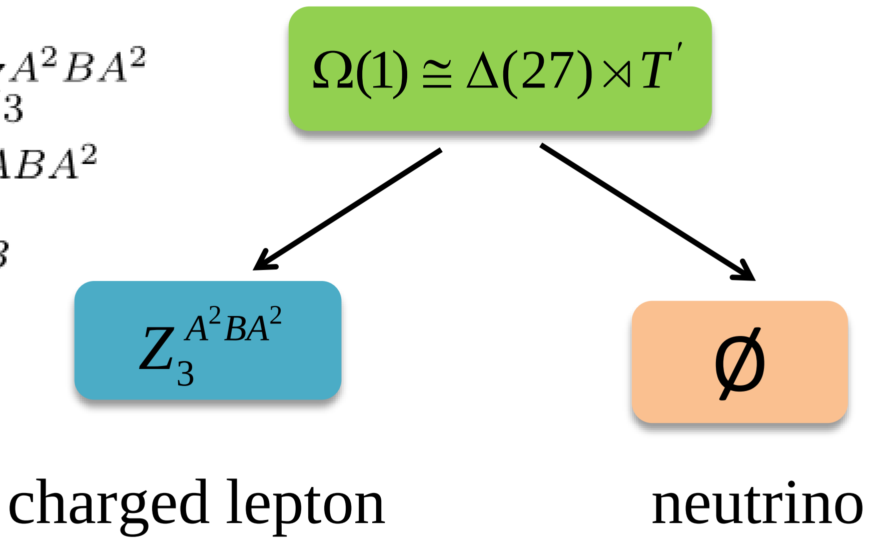
$$\begin{aligned}\mathcal{W} &= \frac{\alpha}{\Lambda}\left(E^c L \phi Y_{\mathbf{2}'}^{(5)}\right)_{(\mathbf{1}_{\mathbf{0},\mathbf{0}},\mathbf{1})} H_d + \frac{\beta}{\Lambda^2}\left(E^c L \xi \phi Y_{\mathbf{1}}^{(4)}\right)_{(\mathbf{1}_{\mathbf{0},\mathbf{0}},\mathbf{1})} H_d \\ &\quad + \frac{g_1}{2\Lambda^2}\left(L L \varphi Y_{\mathbf{2}''}^{(5)}\right)_{(\mathbf{1}_{\mathbf{0},\mathbf{0}},\mathbf{1})} H_u H_u + \frac{g_2}{2\Lambda^2}\left(L L \chi Y_{\mathbf{2}'}^{(7)}\right)_{(\mathbf{1}_{\mathbf{0},\mathbf{0}},\mathbf{1})} H_u H_u\end{aligned}$$

The Kähler corrections are suppressed by $\langle\Phi\rangle^2/\Lambda^2$ and negligible

[Ding, King, Li, Liu, Lu, 2303.02071]

➢ Symmetry breaking: vacuum alignment enforced by residual symmetry

$$\langle\phi\rangle = (\omega^2, 1, 1)^T v_\phi \;\Rightarrow\; \Delta(27) \to Z_3^{A^2BA^2}$$
$$\langle\varphi\rangle = (0, 0, 1)^T v_\varphi \;\Rightarrow\; \Delta(27) \to Z_3^{ABA^2}$$
$$\langle\chi\rangle = (1, 0, 0)^T v_\chi \;\Rightarrow\; \Delta(27) \to Z_3^{B}$$



➢ Lepton mass matrices

$$m_l = \frac{\alpha v_\phi v_d}{\Lambda}\begin{pmatrix} \sqrt{2}\omega^2 Y^{(5)}_{\mathbf{2'},1} & \omega Y^{(5)}_{\mathbf{2'},2} & \omega Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} & Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} \end{pmatrix} + \frac{i\beta Y^{(4)}_{\mathbf{1}} v_\xi v_\phi v_d}{\Lambda^2}\begin{pmatrix} 0 & \omega & -\omega \\ -\omega & 0 & 1 \\ \omega & -1 & 0 \end{pmatrix},$$

$$m_\nu = \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix} 0 & \omega Y^{(5)}_{\mathbf{2''},2} & 0 \\ \omega Y^{(5)}_{\mathbf{2''},2} & 0 & 0 \\ 0 & 0 & \sqrt{2} Y^{(5)}_{\mathbf{2''},1} \end{pmatrix} + \frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2} Y^{(7)}_{\mathbf{2'},1} & 0 & 0 \\ 0 & 0 & \omega Y^{(7)}_{\mathbf{2'},2} \\ 0 & \omega Y^{(7)}_{\mathbf{2'},2} & 0 \end{pmatrix}$$

- Including CP symmetry with $\mathrm{Re}\langle\tau\rangle = 0$, the neutrino mass matrix has μτ reflection symmetry [Harrison, Scott, hep-ph/0210197; Grimus, Lavoura,hep-ph/0305309]

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \to \begin{pmatrix} \nu_e^c \\ \nu_\tau^c \\ \nu_\mu^c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_e^c \\ \nu_\mu^c \\ \nu_\tau^c \end{pmatrix} \Longrightarrow \theta_{23} = \frac{\pi}{4}, \quad \delta_{CP} = \pm\frac{\pi}{2}$$

- **5 real parameters** explain all lepton masses and mixing parameters

$$\sin^2\theta_{13} = 0.02238, \quad \sin^2\theta_{12} = 0.3266, \quad \sin^2\theta_{23} = 0.5, \quad \delta_{CP} = 1.5\pi,$$

$$\alpha_{21} = \pi, \quad \alpha_{31} = \pi, \quad m_1 = 15.18\mathrm{meV}, \quad m_2 = 17.44\mathrm{meV},$$

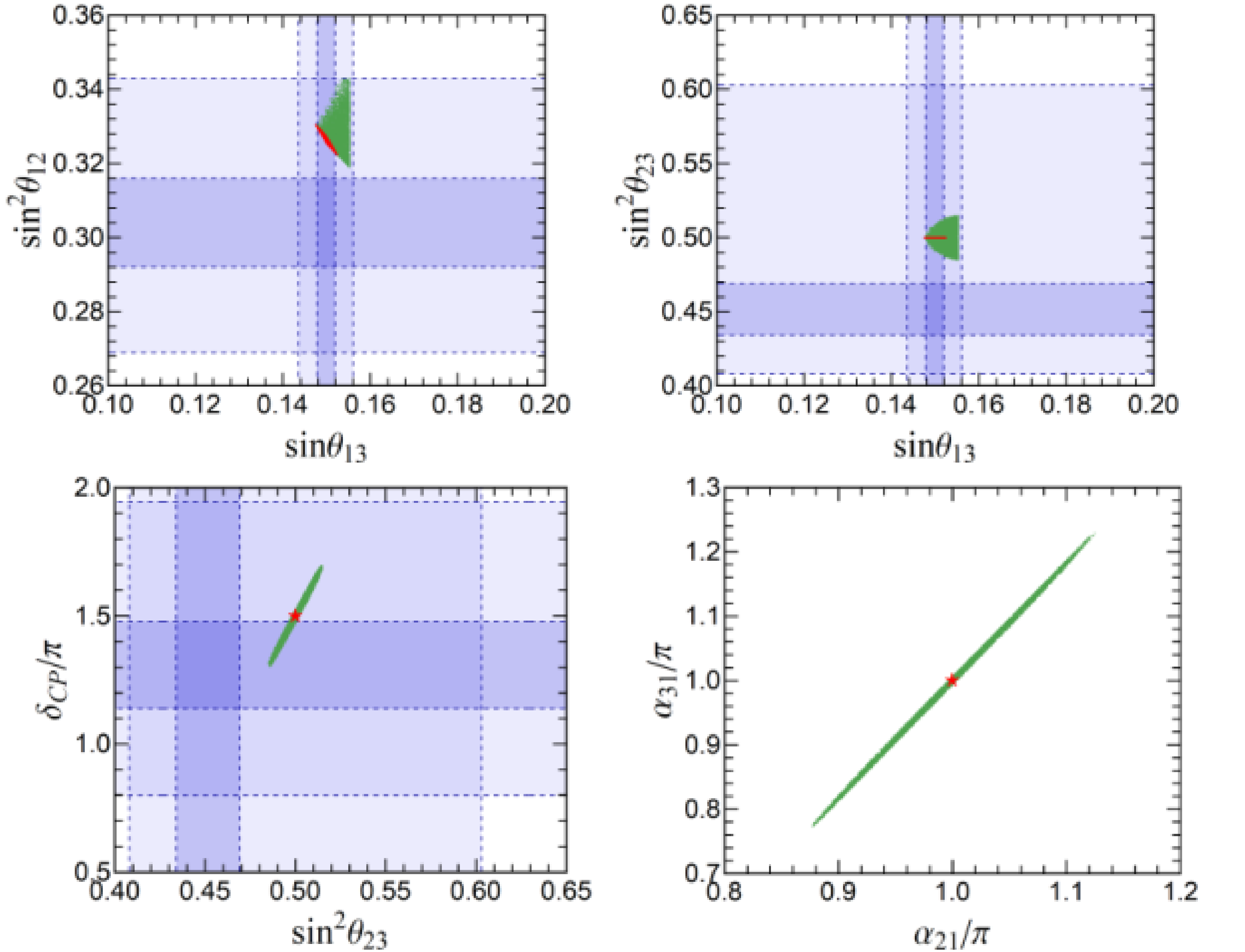
$$m_3 = 52.43\mathrm{meV}, \quad \sum_i m_i = 85.05\mathrm{meV}, \quad m_{\beta\beta} = 5.595\mathrm{meV},$$

$$m_e = 0.511\mathrm{MeV}, \quad m_\mu = 106.5\mathrm{MeV}, \quad m_\tau = 1.807\mathrm{GeV}.$$

In agreement with the experimental data from neutrino oscillation, $0\nu\beta\beta$ decay and Planck

[Ding, King, Li, Liu, Lu, 2303.02071]

close correlations between mixing parameters



[Ding, King, Li, Liu, Lu, 2303.02071]

Summary

- Flavor symmetry is a useful tool to understand the flavor structure of SM, but no compelling and unique picture have emerged so far.

- Modular flavor symmetry is a new elegant and very promising approach to the flavor puzzle.
  - Bottom-up: enhanced predictability of flavor models
  - Top-down: eclectic flavor group $G_{eclectic} = G_{flavor} \cup G_{modular}$
  - Open questions: moduli stabilization?...

- Future experimental data on neutrino mixing angles, δ_{CP} and $0\nu\beta\beta$ decay can exclude many models, will provide important hints for the underlying principle.

Thank you for your attention!

Backup

Tests of modulus couplings

Non-standard neutrino interactions

[Ding,Feruglio, 2003.13448]

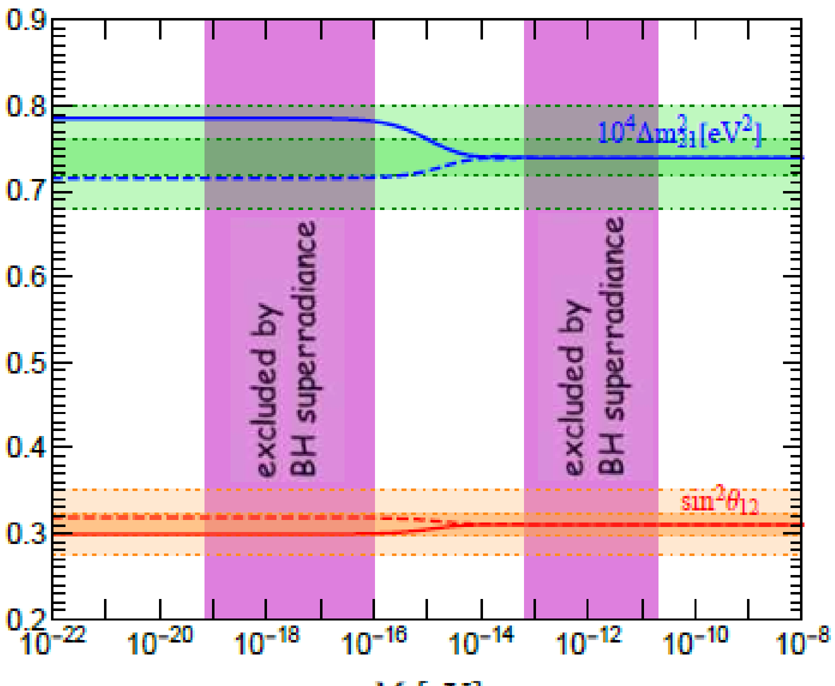
$$\mathcal{L} = i \sum_{f=e,e^c,\nu} \overline{f}\,\overline{\sigma}^\mu \partial_\mu f + \frac{1}{2}\partial_\mu \varphi_\alpha \partial^\mu \varphi_\alpha - \frac{1}{2} M_\alpha^2 \varphi_\alpha^2$$
$$- (m_e + \mathcal{Z}^e_\alpha \varphi_\alpha) e^c e - \frac{1}{2}\nu(m_\nu + \mathcal{Z}^\nu_\alpha \varphi_\alpha)\nu + h.c. + \dots .$$

$$\boxed{\delta m_\nu(0) = -n_e^0 \frac{\mathrm{Re}(\mathcal{Z}^e)\mathcal{Z}^\nu}{M^2(R)}}$$

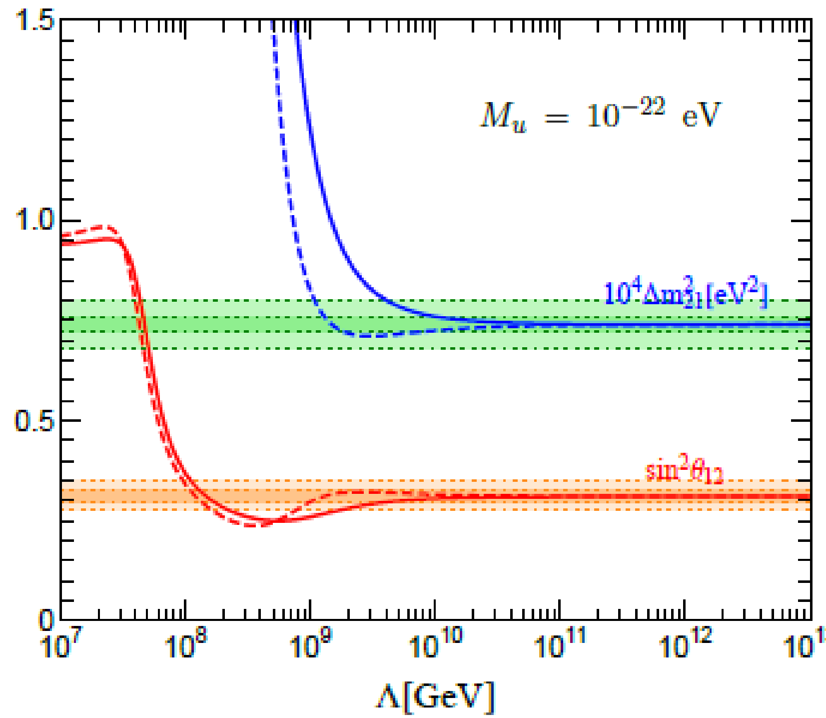
in medium with non-zero electron number density

$$\tau = <\tau> + \frac{\varphi_u + i\varphi_v}{\sqrt{2}}$$

In the sun



$\Lambda = 5 \times 10^9$ GeV



Eclectic flavor group $\Omega(1) \cong \Delta(27) \rtimes T'$

- flavor symmetry group $\Delta(27)$: smallest group with triplet and anti-triplet reps

Multiplication rules: $A^3 = B^3 = (AB)^3 = (AB^2)^3 = 1\,.$

$\Delta(27)$ has 9 singlet representations and 2 triplet reps $3/\bar{3}$

$$\mathbf{1}_{r,s} \quad : \quad \rho_{\mathbf{1}_{r,s}}(A) = \omega^r\,, \qquad \rho_{\mathbf{1}_{r,s}}(B) = \omega^s\,, \quad \text{with} \quad r,s = 0,1,2\,,$$

$$\mathbf{3} \quad : \quad \rho_{\mathbf{3}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\mathbf{3}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

$$\bar{\mathbf{3}} \quad : \quad \rho_{\bar{\mathbf{3}}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho_{\bar{\mathbf{3}}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

[Nilles et al, 2001.01736]

- modular symmetry and CP as automorphisms of $\Delta(27)$

Modular symmetry $\begin{cases} u_S(A) = B^2 A, & u_S(B) = B^2 A^2 \\ u_T(A) = BA, & u_T(B) = B \end{cases}$ $\Longrightarrow$ $\Gamma_3' \cong T'$

CP symmetry $u_{K_*}(A) = A^2 B, \quad u_{K_*}(B) = A^2 BA$

	E	A^2B^2	A^2B	A	AB^2	AB	A^2	B^2	B	BAB^2A^2	ABA^2B^2
	$1C_1$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3C_3^{(3)}$	$3C_3^{(4)}$	$3C_3^{(5)}$	$3C_3^{(6)}$	$3C_3^{(7)}$	$3C_3^{(8)}$	$1C_3^{(1)}$	$1C_3^{(2)}$
$\mathbf{1_{0,0}}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1_{0,1}}$	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω	1	1
$\mathbf{1_{0,2}}$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2	1	1
$\mathbf{1_{1,0}}$	1	ω^2	ω^2	ω	ω	ω	ω^2	1	1	1	1
$\mathbf{1_{1,1}}$	1	ω	1	ω	1	ω^2	ω^2	ω^2	ω	1	1
$\mathbf{1_{1,2}}$	1	1	ω	ω	ω^2	1	ω^2	ω	ω^2	1	1
$\mathbf{1_{2,0}}$	1	ω	ω	ω^2	ω^2	ω^2	ω	1	1	1	1
$\mathbf{1_{2,1}}$	1	1	ω^2	ω^2	ω	1	ω	ω^2	ω	1	1
$\mathbf{1_{2,2}}$	1	ω^2	1	ω^2	1	ω	ω	ω	ω^2	1	1
$\mathbf{3}$	3	0	0	0	0	0	0	0	0	3ω	$3\omega^2$
$\mathbf{\bar{3}}$	3	0	0	0	0	0	0	0	0	$3\omega^2$	3ω

blue: u_S
red: u_T
green: u_{K_*}

All the 8 nontrivial singlets are related by T' modular symmetry, they form a $\Delta(27)$ octet → right-handed leptons can not be $\Delta(27)$ nontrivial singlets .

- Modular transformations fixed by $\Delta(27)$ flavor symmetry, they cannot be freely assigned as with modular symmetry alone

$$\rho_{\mathbf{3}}(S)=\frac{i}{\sqrt{3}}\begin{pmatrix}\omega^2 & \omega & \omega\\ \omega & \omega^2 & \omega\\ \omega^2 & \omega^2 & 1\end{pmatrix}\,,\quad \rho_{\mathbf{3}}(T)=\omega^k\begin{pmatrix}\omega & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & 1\end{pmatrix}\,,\quad k=0,1,2$$

Quark-lepton unification based on double cover of S_4

	L	(e^c,μ^c,τ^c)	N^c	Q	(u^c,c^c,t^c)	(d^c,s^c,b^c)	$H_{u,d}$
$\Gamma'_4\equiv S'_4$	$\mathbf{3}$	$(\mathbf{1},\mathbf{1},\hat{\mathbf{1}}')$	$\mathbf{3}$	$\mathbf{3}$	$(\mathbf{1},\mathbf{1},\hat{\mathbf{1}}')$	$(\mathbf{1}',\hat{\mathbf{1}},\hat{\mathbf{1}}')$	$\mathbf{1}$
k_I	2	$(2,0,1)$	0	k_Q	$(4-k_Q,6-k_Q,3-k_Q)$	$(4-k_Q,5-k_Q,5-k_Q)$	0

We include the gCP symmetry such that all coupling constants are real

Lepton sector:

[Liu, Yao, Ding, 2006.10722]

$$\boxed{\begin{aligned}\mathcal{W}_e &= \alpha_e(E_1^cLY_{\mathbf{3}}^{(4)})_{\mathbf{1}}H_d+\beta_e(E_2^cLY_{\mathbf{3}}^{(2)})_{\mathbf{1}}H_d+\gamma_e(E_3^cLY_{\hat{\mathbf{3}}}^{(3)})_{\mathbf{1}}H_d\,,\\ \mathcal{W}_\nu &= g_1(N^cLY_{\mathbf{2}}^{(2)})_{\mathbf{1}}H_u+g_2(N^cLY_{\mathbf{3}}^{(2)})_{\mathbf{1}}H_u+\Lambda(N^cN^c)_{\mathbf{1}}\,,\end{aligned}}$$

$$M_e=\begin{pmatrix}\alpha_eY_4^{(4)} & \alpha_eY_6^{(4)} & \alpha_eY_5^{(4)}\\ \beta_eY_3^{(2)} & \beta_eY_5^{(2)} & \beta_eY_4^{(2)}\\ \gamma_eY_2^{(3)} & \gamma_eY_4^{(3)} & \gamma_eY_3^{(3)}\end{pmatrix}v_d\,,\qquad M_N=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\Lambda\,,$$

$$M_D=\begin{pmatrix}0 & g_1Y_1^{(2)}-g_2Y_5^{(2)} & g_1Y_2^{(2)}+g_2Y_4^{(2)}\\ g_1Y_1^{(2)}+g_2Y_5^{(2)} & g_1Y_2^{(2)} & -g_2Y_3^{(2)}\\ g_1Y_2^{(2)}-g_2Y_4^{(2)} & g_2Y_3^{(2)} & g_1Y_1^{(2)}\end{pmatrix}v_u\,.$$

Charged lepton masses: $\alpha,\ \beta,\ \gamma$

Light neutrino mass matrix : $g_1^2v_u^2/\Lambda,\ \ g_2/g_1,\ \tau$

Quark sector:

$$\boxed{\begin{aligned}\mathcal{W}_u &= \alpha_u(u^c QY_{\mathbf{3}}^{(4)})_{\mathbf{1}}H_u + \beta_u(c^c QY_{\mathbf{3},I}^{(6)})_{\mathbf{1}}H_u + \gamma_u(c^c QY_{\mathbf{3},II}^{(6)})_{\mathbf{1}}H_u + \delta_u(t^c QY_{\hat{\mathbf{3}}}^{(3)})_{\mathbf{1}}H_u \\ \mathcal{W}_d &= \alpha_d(d^c QY_{\mathbf{3}'}^{(4)})_{\mathbf{1}}H_d + \beta_d(s^c QY_{\hat{\mathbf{3}}',I}^{(5)})_{\mathbf{1}}H_d + \gamma_d(s^c QY_{\hat{\mathbf{3}}',II}^{(5)})_{\mathbf{1}}H_d + \delta_d(b^c QY_{\hat{\mathbf{3}}}^{(5)})_{\mathbf{1}}H_d\end{aligned}}$$

$$\Rightarrow$$

$$M_u = \begin{pmatrix} \alpha_u Y_4^{(4)} & \alpha_u Y_6^{(4)} & \alpha_u Y_5^{(4)} \\ \beta_u Y_5^{(6)} + \gamma_u Y_8^{(6)} & \beta_u Y_7^{(6)} + \gamma_u Y_{10}^{(6)} & \beta_u Y_6^{(6)} + \gamma_u Y_9^{(6)} \\ \delta_u Y_2^{(3)} & \delta_u Y_4^{(3)} & \delta_u Y_3^{(3)} \end{pmatrix} v_u$$

$$M_d = \begin{pmatrix} \alpha_d Y_7^{(4)} & \alpha_d Y_9^{(4)} & \alpha_d Y_8^{(4)} \\ \beta_d Y_6^{(5)} + \gamma_d Y_9^{(5)} & \beta_d Y_8^{(5)} + \gamma_d Y_{11}^{(5)} & \beta_d Y_7^{(5)} + \gamma_d Y_{10}^{(5)} \\ \delta_d Y_3^{(5)} & \delta_d Y_5^{(5)} & \delta_d Y_4^{(5)} \end{pmatrix} v_d$$

8 real coupling constants: $\alpha_{u,d},\ \beta_{u,d},\ \gamma_{u,d},\ \delta_{u,d}$

The complex modulus τ is common in both quark and lepton sectors, and it is the unique source breaking modular symmetry and CP

$$\langle\tau\rangle = -0.2123 + 1.5201i$$

Best fit values of input parameters:

$$\begin{aligned}
&\beta_u/\alpha_u = 325.6502\,, \quad \gamma_u/\alpha_u = 2427.3101\,, \quad \delta_u/\alpha_u = 219.3019\,, \\
&\alpha_u v_u = 2.7758\times 10^{-5}\ \text{GeV}\,, \quad \beta_d/\alpha_d = 466.6990\,, \quad \gamma_d/\alpha_d = -234.0473\,, \\
&\delta_d/\alpha_d = 2.3388\,, \quad \alpha_d v_d = 1.72111\times 10^{-5}\ \text{GeV}\,, \quad \beta_e/\alpha_e = 0.0187\,, \\
&\gamma_e/\alpha_e = 0.1466\,, \quad g_2/g_1 = 0.6834\,, \quad \alpha_e v_d = 16.8880\text{MeV}\,, \quad g_1^2 v_u^2/\Lambda = 0.3043\ \text{meV}\,.
\end{aligned}$$

Predictions: almost all observables are within the 1σ regions

$$\theta^q_{12} = 0.22752\,, \quad \theta^q_{13} = 0.003379\,, \quad \theta^q_{23} = 0.038886\,, \quad \delta^q_{CP} = 75.9958^\circ\,,$$
$$m_u/m_c = 0.001929\,, \quad m_c/m_t = 0.002725\,, \quad m_d/m_s = 0.050345\,, \quad m_s/m_b = 0.017726\,,$$
$$\sin^2\theta^l_{12} = 0.34981\,, \quad \sin^2\theta^l_{13} = 0.02193\,, \quad \sin^2\theta^l_{23} = 0.56393\,,$$
$$\delta^l_{CP} = 266.1824^\circ\,, \quad \alpha_{21} = 1.1482\pi\,, \quad \alpha_{31} = 0.1522\pi\,,$$
$$m_1 = 3.5269\ \mathrm{meV}\,, \quad m_2 = 9.2919\ \mathrm{meV}\,, \quad m_3 = 50.2404\ \mathrm{meV}\,,$$
$$\sum_i m_i = 63.0592\ \mathrm{meV}\,, \quad m_{\beta\beta} = 2.5480\ \mathrm{meV}\,.$$

✓ The model uses **15** parameters including to describe the masses and mixing of both quark and lepton sectors: **12** masses+**6** mixing angles+**3** CP phases.

✓ The predictions for neutrino masses, mixing angles and CP violation phases are compatible with the experimental data of neutrino oscillation and cosmology. Precise measurements of θ_{23} , δ_{CP} and the effective mass $m_{\beta\beta}$ in $0\nu2\beta$ decay can exclude this model.