Goldstone Bosons, Convexity and Weak Gravity

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The Weak Gravity Conjecture proposes a relation for any theory with a U(1) gauge symmetry:

$$\sqrt{2}\, g\, Q\, M_p \geq m$$

[Arkani-Hamed, Motl, Nicolis, Vafa ’06]

Underlies much of the Swampland program, by appropriate generalisations

Evidence from string theory. But very few (none?) completely non-supersymmetric tests.

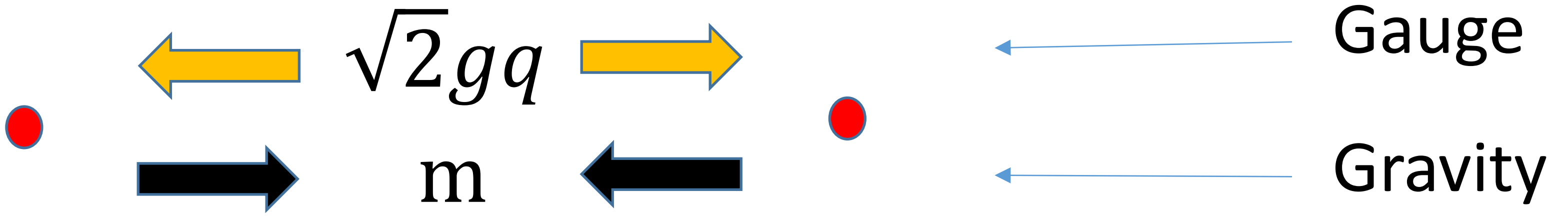
No clear underlying microscopic physics, only `motivating’ arguments

Proofs / underlying microscopic physics for Swampland conjectures are likely most accessible in AdS space through holography: CFTs sharply defined

Propose a formulation of the WGC which is cleanly translatable to a CFT statement:

There must exists a particle with a positive self-binding energy

[EP ’17] [Aharony, EP ‘21]



Positive binding:

$$\gamma = E_{Min}(2) - 2E_{Min}(1) \geq 0$$

AdS/CFT dictionary:

Particle state in bulk ϕ = Operator in CFT ϕ

Charge Q under U(1) gauge symmetry = Charge Q under U(1) Global symmetry

Energy of state = Dimension of operator in CFT

Charged operators convexity:

$$\gamma_{\phi^2} = \Delta(\phi^2) - 2\Delta(\phi) \geq 0$$

$$\gamma = \Delta(2Q) - 2\Delta(Q) \geq 0$$

Guiding example: $O(2)$ model in $4-\epsilon$ dimensions

$$\mathcal{L} = |\partial\phi|^2 - \frac{1}{4} g|\phi|^4$$

Operator charge $Q : \phi^Q$

e.g. $\Delta(2n) - n\,\Delta(2) \geq 0$

Using perturbative Feynman diagrams find:

$$\Delta(Q) = Q\left[1 + \frac{1}{2}\frac{g}{(4\pi)^2}(Q-1) - \frac{1}{2}\left(\frac{g}{(4\pi)^2}\right)^2 (Q^2 - 4Q + 5) + \cdots\right]$$

[Badel, Cuomo, Monin, Rattazzi ‘19]

In weakly-coupled theories, is a remarkably simple statement about the one-loop correction – “Peskin+Schroeder physics”

Useful to write it as convexity: $\dfrac{\partial^2 \Delta(Q)}{\partial Q^2} > 0$

Convexity in explicit examples where can be checked:

- ✓ $U(1)$ and $O(N)$ (quartic) model in $4-\epsilon$ dimensions
- ✓ $U(1)$ and $O(N)$ (sextic) model in $3-\epsilon$ dimensions:
- ✓ $O(N)$ (quartic) model in 3 dimensions (using large N)
- ✓ $U(M)\times U(N)$ (quartic) model in $4-\epsilon$ dimensions

[Antipin, Bersini, Sannino, Wang, Zhang '21]

- ✓ $U(1)_{\mathrm{top}}$ in 3-dimensional $U(N_c)$ gauge theory with N_f fermions
- ✓ $U(1)_{\mathrm{top}}$ in 3-dimensional $U(N_c)$ gauge theory with N_f scalars
- ✓ + quartic terms + Chern-Simons terms
- ✓ $O(2)$ model in 3 dimensions (lattice/bootstrap) – experimentally realizable!
- ✓ Banks-Zaks in 4 dimensions (for Scalar+Fermionic Mesons)
  (Aharony , Breitstein '23)

State-Operators correspondence: $E\left(\mathbb{R}\times S^{d-1}\right)R = \Delta(\mathbb{R}^d)$

$$\phi = \frac{1}{\sqrt{2}} a e^{i\chi} \qquad \mathcal{L} = \frac{1}{2}\partial_\mu a \partial^\mu a + \frac{1}{2} a^2 \partial_\mu \chi \partial^\mu \chi - \frac{1}{2}\frac{a^2}{R^2} - \frac{1}{16}\, g\, a^4$$

Evaluate the energy of the theory on a homogeneous charged background

$$\chi = mt \qquad \langle a\rangle^2 = \frac{4\left(m^2 - 1/R^2\right)}{g} \qquad \mathcal{L}_{\text{eff}}(m) = \frac{\left(m^2 - 1/R^2\right)^2}{g}$$

If there is a semi-classical approximation, then can map this to the energy through a Legendre transform

$$Q \equiv \left(2\pi^2 R^3\right)\frac{\partial \mathcal{L}_{\text{eff}}(m)}{\partial m} = \frac{8\pi^2 R^3 m\left(m^2 - 1/R^2\right)}{g} \qquad mR = \frac{3^{\frac{1}{3}} + \left(9\frac{gQ}{(4\pi)^2} - \sqrt{81\left(\frac{gQ}{(4\pi)^2}\right)^2 - 3}\right)^{\frac{2}{3}}}{3^{\frac{2}{3}}\left(9\frac{gQ}{(4\pi)^2} - \sqrt{81\left(\frac{gQ}{(4\pi)^2}\right)^2 - 3}\right)^{\frac{1}{3}}}\,.$$

$g\,Q \ll 1:$

[Badel, Cuomo, Monin, Rattazzi '19]

$$\Delta(Q) = Q\left[1 + \frac{1}{2}\frac{gQ}{(4\pi)^2} - \frac{1}{2}\left(\frac{gQ}{(4\pi)^2}\right)^2 + \cdots\right]$$

$$\Delta(Q) = Q\left[1 + \frac{1}{2}\frac{gQ}{(4\pi)^2}\left(1 - \frac{1}{Q}\right) - \frac{1}{2}\left(\frac{gQ}{(4\pi)^2}\right)^2\left(1 - \frac{4}{Q} + \frac{5}{Q^2}\right) + \cdots\right]$$

Semi-classical description works as long as $Q \gg 1$. Why?!

Allows for a semi-classical handle on the convexity sign!

Understand generally in the language of **Goldstone bosons**

Consider charged scalar operators, dual to homogenous charged states, which break the symmetry group as (superfluid)

$$SO(1,d+1) \times U(1) \to SO(d) \times D'$$

(True if lowest energy state in a charge super-selection sector)

Combination of the symmetries left over is denoted by m (chemical potential)

The symmetry implies that if the U(1) acts on a field (operator) as

$$U(1)\ :\ \Pi \to \Pi + \xi\ ,$$

then it must appear in the combination

$$\chi \equiv mt + \Pi$$

Convenient to work with the normalized field

$$\chi = mt + \frac{\pi}{f}$$

f is the scale of the symmetry breaking

For large f, we can think of π as a Goldstone boson

Large f regime of CFTS: $f\, R^{\frac{d-2}{2}} \gg 1$

- Global effects (instantons) associated to the compact sphere are suppressed exponentially by f
- There is an expansion in powers of π to a leading quadratic (semi-classical) theory

$$Y_\mu = m\delta_{\mu t} + \frac{\partial_\mu \pi}{f} \qquad\qquad Y^2 \equiv Y^\mu Y_\mu \quad = \quad m^2\left(1 + 2\frac{\dot{\pi}}{mf} + \frac{\partial_\mu \pi \partial^\mu \pi}{m^2 f^2}\right)$$

Everything must be a function of Y^2 (and higher derivatives ∂Y)

In the O(2) model we have

	Small charge $gQ \ll 1$	Large charge $gQ \gg 1$
$fR^{\frac{d-2}{2}}$	$\sqrt{Q}$	$\frac{1}{\sqrt{g}}\,(gQ)^{\frac{1}{3}}$
mR	$1+\mathcal{O}(gQ)$	$(gQ)^{\frac{1}{3}}$

Two large f regimes in the model.

This is why we can semi-classically calculate the operator dimensions

There are two scenarios: Goldstone with gap (from sphere scale), or without gap, to other fields

If there is a gap, can integrate out all the other fields, and have an expansion both in derivatives and in powers of π

$$\begin{aligned}\mathcal{L}_{\text{eff}}\left(m,\pi\right) &= \mathcal{L}_{\text{eff}}\left(m,\pi\right)\big|_{\pi=0} + \left.\frac{\partial\mathcal{L}_{\text{eff}}\left(m,\pi\right)}{\partial\left(\partial_\mu\pi\right)}\right|_{\pi=0}\left(\partial_\mu\pi\right) \\ &+ \frac{1}{2}\left.\frac{\partial^2\mathcal{L}_{\text{eff}}\left(m,\pi\right)}{\partial\left(\partial_\mu\pi\right)\partial\left(\partial_\nu\pi\right)}\right|_{\pi=0}\left(\partial_\mu\pi\right)\left(\partial_\nu\pi\right) + \ldots\ .\end{aligned}$$

Yields:

$$\mathcal{L}_{\text{eff}}\left(m,\pi\right) = \mathcal{L}_{\text{eff}}\left(m\right) + \frac{1}{2}\frac{1}{f^2}\left[\frac{\partial^2\mathcal{L}_{\text{eff}}\left(m\right)}{\partial m^2}\dot{\pi}^2 - \frac{Q}{m}\left(\partial_i\pi\right)\left(\partial_j\pi\right)g^{ij}\right], \qquad Q = \frac{\partial\mathcal{L}_{\text{eff}}\left(m\right)}{\partial m}$$

Positive kinetic terms and sub-luminality imply

$$\frac{\partial^2 \mathcal{L}_{\text{eff}}(m)}{\partial m^2} > 0 \qquad\qquad \frac{\partial^2 L_{\text{eff}}(m)}{\partial m^2} \geq \frac{Q}{m}$$

We therefore find that Goldstone boson + Gap implies convexity of the Lagrangian in the chemical potential

Semi-classically, the chemical potential is the conjugate to the charge

$$\text{Classical } : H(Q) = m\, Q - L(m)\ , \quad Q = \frac{\partial L(m)}{\partial m}\ .$$

A Legendre transform preserves convexity (convex conjugate), and therefore

$$\frac{\partial^2 H(Q)}{\partial Q^2} > 0 \qquad \frac{\partial^2 \Delta(Q)}{\partial Q^2} > 0 \qquad \Delta\big((n+m)Q\big) > \Delta(nQ) + \Delta(mQ)$$

Summary

- The WGC can be mapped to certain convexity properties of CFTs
- Have shown that convexity of charged operators follows if:

  1. The theory is in a large f regime, so there is a Goldstone boson

  2. There is a set of operators/states, of lowest dimension for their charge, which share the same Goldstone boson (embedding into the CFT degrees of freedom). e.g. ϕ^n

- The CFT result matches very nicely the gravity side, in the sense that ϕ^n can be thought of as an n-particle state

- There exist examples where ϕ can have parametrically large charge

[Sharon, EP ‘22][Sharon, Watanabe ‘23]

- Bulk dual of the Goldstone boson?

Thank You