Gluon contributions to DVCS, TCS and DDVCS from low to moderate x

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Low x 2023

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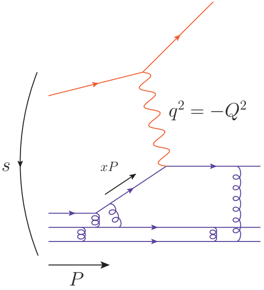
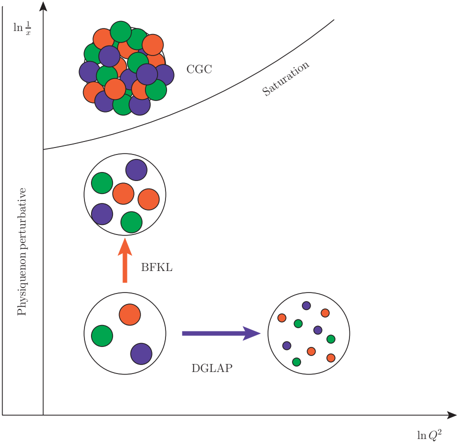
In collaboration with Y. Mehtar-Tani

Accessing the partonic content of hadrons with an electromagnetic probe

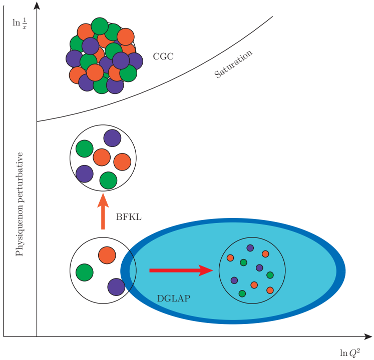


Electron-proton collision (parton model)

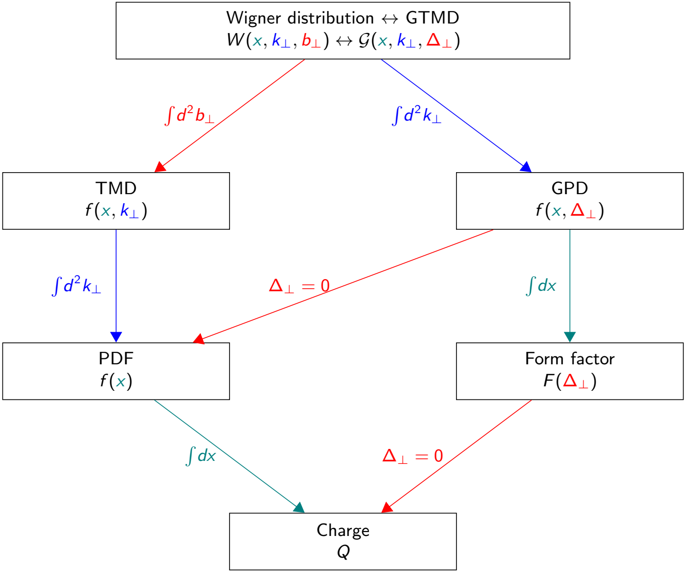


QCD at moderate $x_{\rm Bj} \sim Q^2/s$

Bjorken limit: $Q^2 \sim s$

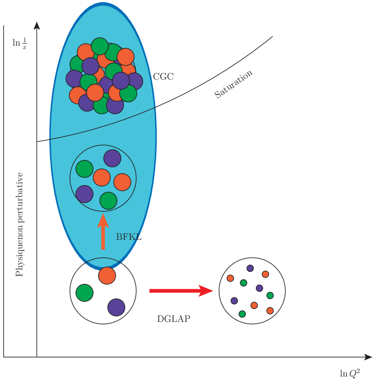


The family tree of parton distributions

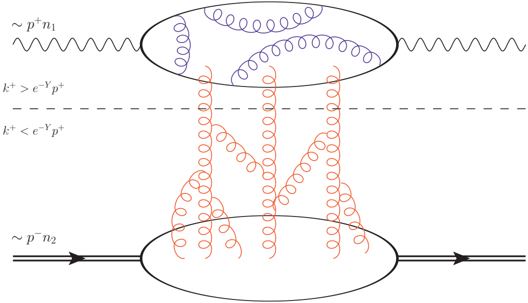


QCD at small $x_{\mathrm{Bj}} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



Rapidity separation

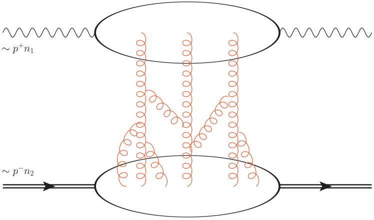


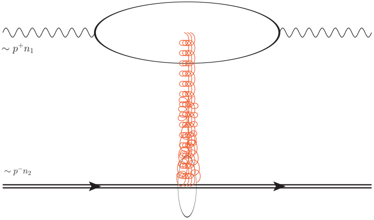
Let us split the gluonic field between "fast" and "slow" gluons

$$\mathcal{A}^{\mu a}(k^+, k^-, \boldsymbol{k}) = A^{\mu a}\left(|k^+| > e^{-Y} p^+, k^-, \boldsymbol{k}\right) + A^{\mu a}_{\mathrm{cl}}(|k^+| < e^{-Y} p^+, k^-, \boldsymbol{k})$$

$e^{-Y} \ll 1$

Large longitudinal boost to the projectile frame





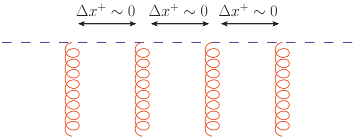
$$\begin{array}{c} A^+_{\rm cl}(x^+, x^-, \boldsymbol{x}) \\ A^-_{\rm cl}(x^+, x^-, \boldsymbol{x}) \\ A^i_{\rm cl}(x^+, x^-, \boldsymbol{x}) \end{array} \quad \xrightarrow{\Lambda \sim \sqrt{\frac{s}{m_t^2}}} \quad \begin{array}{c} \frac{1}{\Lambda} A^+_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}) \\ \Lambda A^-_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}) \\ A^i_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}) \end{array}$$

$$A^\mu_{\rm cl}(x) \to A^-_{\rm cl}(x)\, n_2^\mu = \delta(x^+)\, \mathbf{A}(\boldsymbol{x})\, n_2^\mu + O(\sqrt{\frac{m_t^2}{s}})$$

Shock wave approximation

Effective Feynman rules in the slow background field

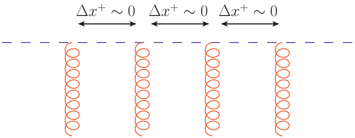
Effective fermion propagator in the external classical field



- $A^i_{\rm cl} = 0,\ A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of $+$ momentum
- $A_{\rm cl}$ is peaked around $x^+ = 0$:
  - Most external propagators get factorized out
  - Gaussians $\sim \delta$ functions: conservation of transverse position
  - Possibility to extend Wilson lines to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

Effective Feynman rules in the slow background field

The interactions with the background field can be exponentiated

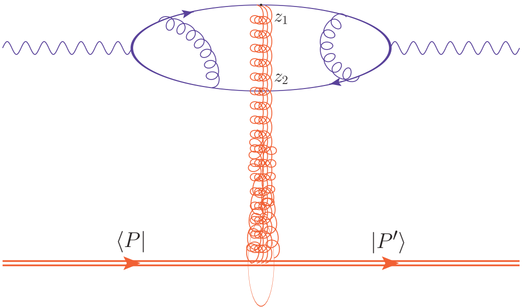


$$D_F(x_2, x_0)|_{x_2^+ > 0, x_0^+ < 0} = \int \mathrm{d}^D x_1 \, \delta(x_1^+) \, D_0(x_2, x_1) \, \gamma^+ U_{\boldsymbol{x}_1} D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$U_{\boldsymbol{x}} \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} \mathrm{d}x \cdot A_{\mathrm{cl}}(x) \right]$$

Factorized picture



Factorized amplitude

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2\, \Phi^Y(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \langle P'|[\mathrm{Tr}(U^Y_{\boldsymbol{x}_1} U^{Y\dagger}_{\boldsymbol{x}_2}) - N_c]|P\rangle$$

Written similarly for any number of Wilson lines in any color representation!

Y independence: B-JIMWLK, BK equations. Resums logarithms of s

The seemingly incompatible nature of the distributions

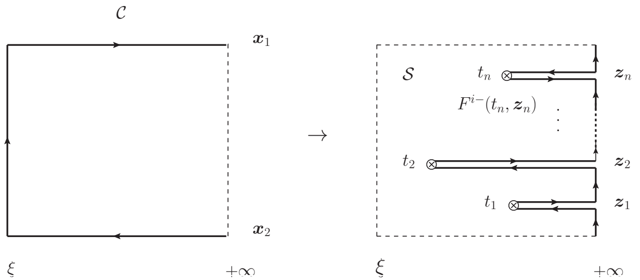
Two different kinds of gluon distributions

Moderate x distributions	Low x distributions
TMD, PDF...	Dipole scattering amplitude
$\langle P \| F^{-i} W F^{-j} W \| P \rangle$	$\langle P \| \mathrm{tr}(U_1 U_2^{\dagger}) \| P \rangle$

The Wilson line $\leftrightarrow$ parton distribution equivalence

Most general equivalence: use the Non-Abelian Stokes theorem

[RB, Mehtar-Tani]



$$\mathcal{P} \exp \left[\oint_{\mathcal{C}} dx_\mu A^\mu(x) \right] = \mathcal{P} \exp \left[\int_{\mathcal{S}} d\sigma_{\mu\nu} \, W F^{\mu\nu} W^\dagger \right]$$

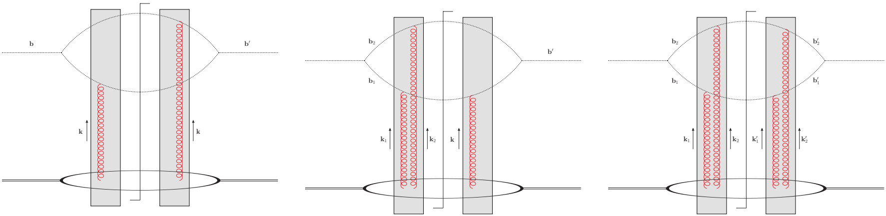
$$U_{x_{1\perp}} U^\dagger_{x_{2\perp}} = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross section

Inclusive low x cross section = TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

Generalizes [Dominguez, Marquet, Xiao, Yuan]



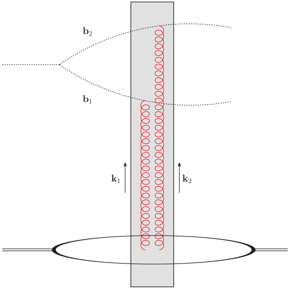
$$\sigma = \mathcal{H}_2^{ij}(\boldsymbol{k}) \otimes f_2^{ij}(x=0, \boldsymbol{k})$$
$$+ \mathcal{H}_3^{ijk}(\boldsymbol{k}, \boldsymbol{k}_1) \otimes f_3^{ijk}(x=0, x_1=0, \boldsymbol{k}, \boldsymbol{k}_1)$$
$$+ \mathcal{H}_4^{ijkl}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1') \otimes f_4^{ijkl}(x=0, x_1=0, x_1'=0, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1')$$

All distributions are evaluated in the strict $x = 0$ limit

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude

[Altinoluk, RB]

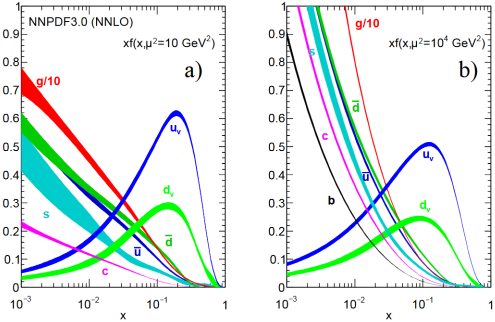


$$\mathcal{H}^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) \otimes f^{ij}(x = 0, \xi = 0; \boldsymbol{k}, \boldsymbol{\Delta})$$

Every exclusive low x process probes a Wigner distribution!

All distributions are evaluated in the strict $x = 0$ limit

All distributions are evaluated in the strict $x = 0$ limit



[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

All distributions are evaluated in the strict $x = 0$ limit

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit	Leading twist of the CGC
$s \sim Q^2$	$s \gg Q^2, Q^2 \to \infty$
$\int \mathrm{d}x f(x)\mathcal{H}(x)$	$f(0) \int \mathrm{d}x \mathcal{H}(x)$

Strong mismatch beyond LL: the PDF is not a constant in $x \simeq 0$.

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

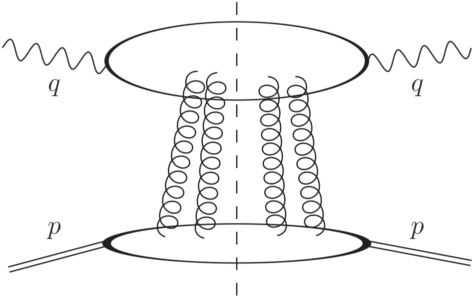
Overarching scheme?

$$f(x_1 \dots x_n; k_{\perp 1} \dots k_{\perp n})$$

Bjorken limit	Regge limit
$s \sim Q^2$	$s \gg Q^2$
$f(x; 0_\perp) + O(Q^{-2})$	$f(0 \dots 0, k_{\perp 1} \dots k_{\perp n}) + O(x_{\mathrm{Bj}})$

Look for an interpolating scheme for simple observables

An interpolating scheme for exclusive Compton scattering



Bjorken limit

$$s \sim Q^2$$

$$f(x, k_\perp = 0) + O(Q^{-2})$$

Regge limit

$$s \gg Q^2$$

$$f(x = 0, k_\perp) + O(x_{\mathrm{Bj}})$$

Interpolation?

$$s \gtrsim Q^2$$

$$f(x, k_\perp) + O(x_{\mathrm{Bj}} Q^{-2})$$

Basic observation: in both limits, $k^+ \simeq 0$ for t-channel gluons

Factorization in k^+ space is consistent

[Balitsky, Tarasov]

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, x^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, x^-, \boldsymbol{x})$$

Building a semi-classical picture

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Dependence on x^-: sub-sub-leading in twist counting

Building a semi-classical picture

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Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, 0^-, \boldsymbol{x})$$

Non-zero $A_\perp$: only two A^i contribute to DDVCS

They can be computed using Ward-Takahashi: only necessary for consistency checks, can be dropped.

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

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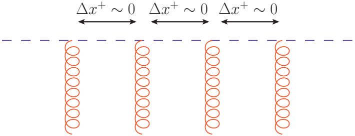
Necessary gluon fields in the Bjorken limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu$$

Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field



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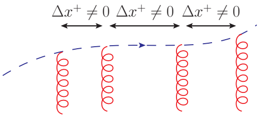

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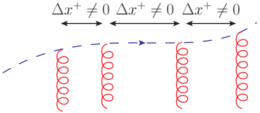
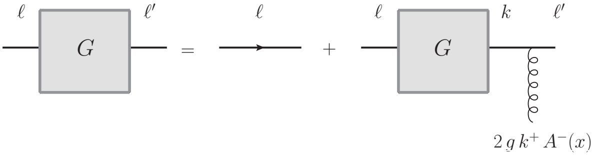


- $A^i_{\rm cl} = 0,\ A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of $+$ momentum

$$D_F(\ell', \ell) = i\frac{\gamma^+}{2\ell^+}(2\pi)^D \delta^D(\ell' - \ell) + i\frac{\not{\ell}'\gamma^+\not{\ell}}{2\ell^+} G_{\rm scal}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field

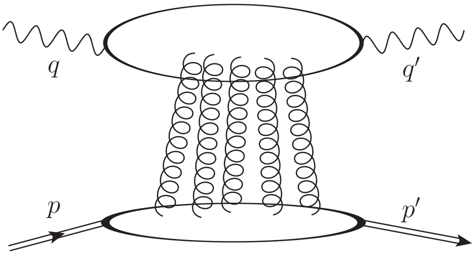


$$G_{\rm scal}(\ell',\ell) - G_0(\ell')(2\pi)^D\delta^D(\ell'-\ell)$$
$$= 2g\int \mathrm{d}^D z\int\frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{e}^{i(\ell'-k)\cdot z}G_0(\ell')\,(k\cdot A)(z)\,G_{\rm scal}(k,\ell).$$

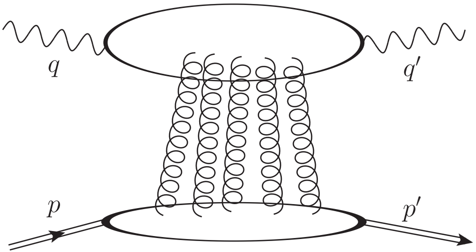
In coordinate space, it satisfies the Klein-Gordon equation in a potential

$$[-\Box_z + 2igA(z)\cdot\partial_z]\,G_{\rm scal}(z,z_0) = \delta^D(z-z_0)$$

Application to the exclusive $\gamma^{(*)}(q)P(p) \to \gamma^{(*)}(q')P(p')$ amplitude



Double, Spacelike, and Timelike exclusive Compton Scattering

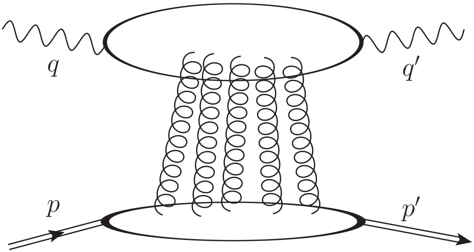


Longitudinal momentum variables:

$$x, \quad \xi \sim \frac{-q^2 + q'^2}{2q \cdot (p + p')}, \quad x_{\mathrm{Bj}} = \frac{-q^2 - q'^2}{2q \cdot (p + p')}$$

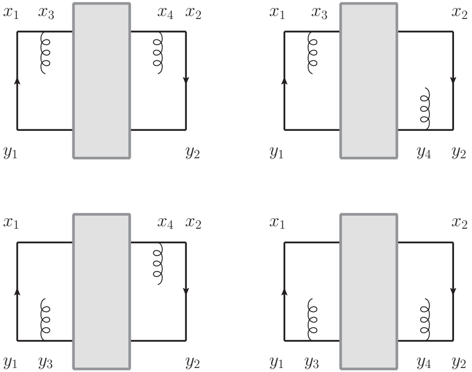
Can we restore the dependence on all 3 variables in our CGC-like scheme?

Double, Spacelike, and Timelike exclusive Compton Scattering



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^{\mu} \varepsilon_{q'}^{\nu *} \sum_f q_f^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$
$$\times \langle p' | \mathrm{tr} \left[\gamma_\nu D_F(k, \ell) \gamma_\mu D_F(-q + \ell, -q' + \ell + k) \right] | p \rangle$$

Some operator algebra



$$
\begin{aligned}
&\mathrm{tr}\, G^R_{\mathrm{scal}}(x_2, x_1)\, G^A_{\mathrm{scal}}(y_1, y_2) \\
&= 16 g^2 \int \mathrm{d}^D x_3 \int \mathrm{d}^D x_4 \int \mathrm{d}^D y_3 \int \mathrm{d}^D y_4\, \delta(y_3^+ - x_3^+)\,\delta(x_4^+ - y_4^+) \\
&\times (\partial^+_{x_3} G_0^R)(x_3, x_1)(\partial^+_{x_4} G_0^R)(x_2, x_4)(\partial^+_{y_3} G_0^A)(y_1, y_3)(\partial^+_{y_4} G_0^A)(y_4, y_2) \\
&\times \mathrm{tr} \left\{ \left[A^-(y_3) - A^-(x_3)\right] G^A_{\mathrm{scal}}(y_3, y_4) \left[A^-(y_4) - A^-(x_4)\right] G^R_{\mathrm{scal}}(x_4, x_3) \right\}
\end{aligned}
$$

(First) final result

Fully general result

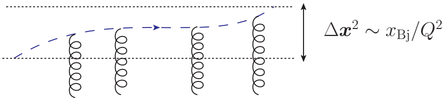
$$\mathcal{A} \propto \mathcal{U}^{ij}(z, \ell_1, \ell_2) \otimes_{z,\ell_1,\ell_2} (\partial^i \Phi)(z, \ell_1)(\partial^j \Phi^*)(z, \ell_2)$$

- Φ: standard wave functions
- $\mathcal{U}^{ij}$: generalization of the dipole operator

Contains unnecessary subleading powers of $x_{\rm Bj}, \xi$ and Q, Q'

Further simplifcations

Partial twist expansion



Typical transverse recoil of a fast parton:
$\Delta x^2 \sim 1/(q^+(p^- + p'^-)) \sim 1/s$

$1/s$: eikonally suppressed in the Regge limit

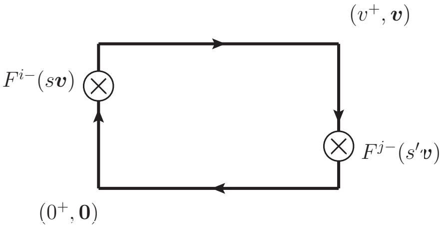
$1/s \sim 1/Q^2$: twist suppressed in in the Bjorken limit.

We can get rid of all corrections from transverse recoils without loss of accuracy

Further simplifications

Partial twist expansion

$$\frac{\langle p'|\mathcal{U}^{ij}(z,\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)|p\rangle}{\langle p|p\rangle} \simeq -i\frac{(2\pi)^d}{8z\bar{z}(q^+)^2}\int \mathrm{d}x \frac{\mathcal{G}^{ij}(x,\boldsymbol{\ell}_2-\boldsymbol{\ell}_1)}{x - x_{\mathrm{Bj}} - \frac{\left(\frac{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2}{2}\right)^2}{z\bar{z}q^+(p^- + p'^-)} + i0},$$



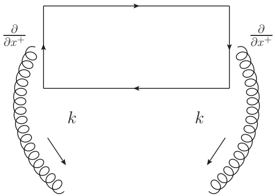
x-dependent unintegrated GPD

$$\mathcal{G}^{ij}(x,\xi,\boldsymbol{k},\boldsymbol{\Delta}) \equiv \frac{2}{p^- + p'^-}\int\frac{\mathrm{d}v^+}{2\pi}\mathrm{e}^{ix\frac{p^- + p'^-}{2}v^+}\int\frac{\mathrm{d}^d\boldsymbol{v}}{(2\pi)^d}\mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{v})}\int_0^1 \mathrm{d}s\mathrm{d}s'$$
$$\times \left\langle p'\middle|\mathrm{tr}_c\left\{\left[v^+,0^+\right]_{\boldsymbol{0}}F^{i-}\left(0^+,s\boldsymbol{v}\right)\left[0^+,v^+\right]_{\boldsymbol{v}}F^{j-}\left(v^+,s'\boldsymbol{v}\right)\right\}\middle|p\right\rangle$$

The unintegrated PDF

uGPD as a finite Wilson loop

$$\int \mathrm{d}^2\boldsymbol{k}\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\boldsymbol{r}^i\boldsymbol{r}^j\mathcal{G}^{ij}(x,\xi,\boldsymbol{k},\boldsymbol{\Delta})$$
$$= \frac{1}{\alpha_s}\int\frac{\mathrm{d}^4v_1\mathrm{d}^4v_2}{(2\pi)^4}\delta(v_1^-)\delta(v_2^-)\mathrm{e}^{-i(k-\frac{\Delta}{2})\cdot v_1+i(k+\frac{\Delta}{2})\cdot v_2}$$
$$\times\frac{\partial}{\partial v_1^+}\frac{\partial}{\partial v_2^+}\frac{\langle p'|\mathrm{tr}[v_1^+,v_2^+]_{v_1}[\boldsymbol{v}_1,\boldsymbol{v}_2]_{v_2^+}[v_2^+,v_1^+]_{v_2}[\boldsymbol{v}_2,\boldsymbol{v}_1]_{v_1^+}|p\rangle}{\langle p|p\rangle}$$



x-dependent unintegrated GPD $\Leftrightarrow$ FT of a finite Wilson loop

(Actual) final result

Final expression for the amplitude

$$\mathcal{A} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x \frac{\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Standard wave functions Φ

x-dependent unintegrated GPD $\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})$

Bjorken limit and Regge limit

The Bjorken limit

Recovering the Bjorken limit

The Bjorken limit is reached by neglecting transverse momentum transfert from the target:

$$|\boldsymbol{\ell}| \sim Q \gg |\boldsymbol{k}|$$

Key observation: $\mathcal{G}^{ij}$ integrates into GPDs

$$\int \mathrm{d}^d \boldsymbol{k} (\partial^i \phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)\mathcal{G}^{ij}(x, \boldsymbol{k})$$
$$\simeq (\partial^i \phi)(z, \boldsymbol{\ell})(\partial^j \phi^*)(z, \boldsymbol{\ell}) \int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \boldsymbol{k})$$
$$\simeq (\partial^i \phi)(z, \boldsymbol{\ell})(\partial^j \phi^*)(z, \boldsymbol{\ell}) G^{ij}(x)$$

We fully recover the well-known one-loop exclusive Compton scattering amplitudes

The Bjorken limit

Unpolarized contribution

$$2\alpha_{\mathrm{em}}\alpha_s \sum_f q_f^2 \int \mathrm{d}x \frac{(\varepsilon_q \cdot \varepsilon^*_{q'})G(x,\xi,\mathbf{\Delta})}{(x+\xi-i0x_{\mathrm{Bj}})^2(x-\xi+i0x_{\mathrm{Bj}})^2}$$

$$\times \frac{1}{2\xi}\left\{\left[(x_{\mathrm{Bj}}+\xi)(x^2-\xi^2+4\xi x_{\mathrm{Bj}}+4\xi x)\ln\left(\frac{x_{\mathrm{Bj}}+x-i0}{x_{\mathrm{Bj}}+\xi-i0}\right)\right.\right.$$

$$-\frac{x_{\mathrm{Bj}}+\xi}{2}(x^2-\xi^2+2\xi x_{\mathrm{Bj}}+2\xi x)\left[\ln^2\left(\frac{x_{\mathrm{Bj}}+x-i0}{\xi}\right)-\ln^2\left(\frac{x_{\mathrm{Bj}}+\xi-i0}{\xi}\right)\right]$$

$$+\frac{x_{\mathrm{Bj}}-\xi}{2}(x^2-\xi^2-2\xi x_{\mathrm{Bj}}-2\xi x)\left[\ln^2\left(\frac{x_{\mathrm{Bj}}+x-i0}{\xi}\right)-\ln^2\left(\frac{x_{\mathrm{Bj}}-\xi-i0}{\xi}\right)\right]$$

$$\left.\left.-(x_{\mathrm{Bj}}-\xi)(x^2-\xi^2-4\xi x_{\mathrm{Bj}}+4\xi)\ln\left(\frac{x_{\mathrm{Bj}}+x-i0}{x_{\mathrm{Bj}}-\xi-i0}\right)\right]+(x\to -x)\right\}$$

The Bjorken limit

Polarized contribution

$$2\alpha_{\rm em}\alpha_s \sum_f q_f^2 \int \mathrm{d}x \frac{\epsilon^{mn} \boldsymbol{e}_h^m \boldsymbol{e}_{h'}^{\prime n*} \widetilde{G}(x,\xi,\boldsymbol{\Delta})}{(x+\xi-i0x_{\rm Bj})^2(x-\xi+i0x_{\rm Bj})^2}$$
$$\times \Bigg\{ \Bigg[2(2x^2+\xi^2)\ln\left(\frac{x_{\rm Bj}+x-i0}{\xi}\right)$$
$$+3x(x_{\rm Bj}+\xi)\ln\left(\frac{x_{\rm Bj}+x-i0}{x_{\rm Bj}+\xi-i0}\right)+3x(x_{\rm Bj}-\xi)\ln\left(\frac{x_{\rm Bj}+x-i0}{x_{\rm Bj}-\xi-i0}\right)$$
$$-\frac{1}{2}x(x_{\rm Bj}+\xi)\left[\ln^2\left(\frac{x_{\rm Bj}+x-i0}{\xi}\right)-\ln^2\left(\frac{x_{\rm Bj}+\xi-i0}{\xi}\right)\right]$$
$$-\frac{1}{2}x(x_{\rm Bj}-\xi)\left[\ln^2\left(\frac{x_{\rm Bj}+x-i0}{\xi}\right)-\ln^2\left(\frac{x_{\rm Bj}-\xi-i0}{\xi}\right)\right]$$
$$-\frac{1}{2}(x^2+\xi^2)\ln^2\left(\frac{x_{\rm Bj}+x-i0}{\xi}\right)\Bigg]-(x\to -x)\Bigg\}$$

The Bjorken limit

Transversity contribution

$$2\alpha_{\mathrm{em}}\alpha_s \sum_f q_f^2 \tau^{mn,ij} \boldsymbol{e}_h^m \boldsymbol{e}_{h'}^{\prime n*} \int \mathrm{d}x \frac{G_T^{ij}(x,\xi,\boldsymbol{\Delta})}{(x-\xi+i0x_{\mathrm{Bj}})^2(x+\xi-i0x_{\mathrm{Bj}})^2}$$
$$\times \left[(x^2-\xi^2)+(x_{\mathrm{Bj}}^2-\xi^2)\ln\frac{(x_{\mathrm{Bj}}-x-i0)(x_{\mathrm{Bj}}+x-i0)}{(x_{\mathrm{Bj}}-\xi-i0)(x_{\mathrm{Bj}}+\xi-i0)}\right]$$

The Regge limit

Recovering the Regge limit? What is x?

Naive argument

- In the Regge limit, the amplitude is dominated by its imaginary part
- Leading order amplitude:

$$\mathrm{Im}\mathcal{A}_{LO} \propto \mathrm{Im} \int \mathrm{d}x H^q(x,\xi,t) \frac{1}{x - x_{\mathrm{Bj}} + i\epsilon} = -\pi H^q(x_{\mathrm{Bj}},\xi,t)$$

- Hence take $x = x_{\mathrm{Bj}}$

Problems

- At NLO, the x cut is way more complicated
- For DDVCS and for TCS, s-channel cuts also contribute to the imaginary part

The Bjorken limit

Recovering the Regge limit

The Regge limit is reached by neglecting x_{Bj} and setting $\frac{\ell^2}{z\bar{z}} \ll q \cdot P$, then taking the x cut:

$$\frac{1}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \to \frac{1}{x + i0} \to -i\pi\delta(x),$$

then taking $x_{\mathrm{Bj}}, \xi \ll 1$.

Key observation:

$$\int \frac{\mathrm{d}^d \boldsymbol{\ell}_1}{(2\pi)^d} \int \frac{\mathrm{d}^d \boldsymbol{\ell}_2}{(2\pi)^d} \mathrm{e}^{-i(\boldsymbol{\ell}_1 \cdot \boldsymbol{r}_1) + i(\boldsymbol{\ell}_2 \cdot \boldsymbol{r}_2)} \boldsymbol{r}_1^i \boldsymbol{r}_2^j \left[x G^{ij}(x, \boldsymbol{\ell}_2 - \boldsymbol{\ell}_1) \right]_{x=0}$$

$$= \frac{N_c}{2\pi^2 \alpha_s} \delta^d(\boldsymbol{r}_1 - \boldsymbol{r}_2) \int \frac{\mathrm{d}^d \boldsymbol{v}_2}{(2\pi)^d} \mathrm{Re} \frac{\left\langle P \middle| 1 - \frac{1}{N_c} \mathrm{tr}_c \left(U_{\boldsymbol{v}_2 + \boldsymbol{r}_1} U^\dagger_{\boldsymbol{v}_2} \right) \middle| P \right\rangle}{\left\langle P \middle| P \right\rangle}$$

The Regge limit

Recovering the Regge limit

$$(\partial^i \Phi)(z, \ell - \frac{\boldsymbol{k}}{2})(\partial^j \Phi^*)(z, \ell + \frac{\boldsymbol{k}}{2}) \otimes_{\ell, \boldsymbol{k}} xG^{ij}(x, \boldsymbol{k})\delta(x)$$
$$\to \Psi(z, \boldsymbol{r}_1)\Psi^*(z, \boldsymbol{r}_2) \otimes_{\boldsymbol{r}_1, \boldsymbol{r}_2} \boldsymbol{r}_1^i \boldsymbol{r}_2^j \left[xG^{ij}(x, \boldsymbol{k})\right]_{x=0}$$
$$\to \Psi(z, \boldsymbol{r}_1)\Psi^*(z, \boldsymbol{r}_2) \otimes_{\boldsymbol{r}_1, \boldsymbol{r}_2} \delta^d(\boldsymbol{r}_1 - \boldsymbol{r}_2)UU$$
$$\to |\Psi(z, \boldsymbol{r})|^2 \otimes_{\boldsymbol{r}} D(\boldsymbol{r})$$

We fully recover the small-x description of exclusive Compton scattering e.g. [Hatta, Xiao, Yuan].

Rq: $x = 0$ is the reason why wave functions involve the same dipole size in the wave functions

Non-commutativity of the limits

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int \mathrm{d}x \int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta}) H^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})$$

Bjorken limit

$$\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{0}, \boldsymbol{\Delta}) \times \left[\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})\right]$$

Regge limit

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(0, \xi, \boldsymbol{k}, \boldsymbol{\Delta}) \times \left[\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{k}, \boldsymbol{\Delta})\right]$$

We found an interpolating scheme

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\mathrm{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d}$$
$$\times (-i\pi) G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \mathcal{A}_{\mathrm{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d}$$
$$\times \lim_{x_{\mathrm{Bj}},\xi\to 0} \int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Double limit

Do the two limits commute?

If $G^{ij}(x,\xi,t)$ is a constant at $x = 0$:

$$\int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq G^{ij}(0,\xi,t) \int \mathrm{d}x \, \frac{(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$= G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)$$

$$\times \ln\left(\frac{1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}{-1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}\right)$$

and thus

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq -i\pi G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)$$

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\rm Regge} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit **provided the GPDs are constant at $x = 0$**

$$\lim_{x_{\rm Bj},\xi\to 0} \mathcal{A}_{\rm Bjorken} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})$$

Checked with explicit final expressions for both double limits

Conclusion

Where do we stand?

Bad news

- Semi-classical small x physics has, at its core, issues with collinear logarithms
- The problem can be traced down to the very starting point

Good news

- We now have a minimal correction of semi-classical small x which solves the problem from first principles
- Wave functions, and thus hard parts, are not modified by the scheme
- All we need is the right evolution equation...

BACKUP

The energy denominators

$$
\begin{aligned}
&\operatorname{tr} G^R_{\rm scal}(x_2, x_1) G^A_{\rm scal}(y_1, y_2) \\
&= 16 g^2 \int \mathrm{d}^D x_3 \int \mathrm{d}^D x_4 \int \mathrm{d}^D y_3 \int \mathrm{d}^D y_4 \delta(y_3^+ - x_3^+)\delta(x_4^+ - y_4^+) \\
&\times (\partial^+_{x_3} G_0^R)(x_3, x_1)(\partial^+_{x_4} G_0^R)(x_2, x_4)(\partial^+_{y_3} G_0^A)(y_1, y_3)(\partial^+_{y_4} G_0^A)(y_4, y_2) \\
&\times \operatorname{tr}\left\{ \left[A^-(y_3) - A^-(x_3)\right] G^A_{\rm scal}(y_3, y_4) \left[A^-(y_4) - A^-(x_4)\right] G^R_{\rm scal}(x_4, x_3) \right\}
\end{aligned}
$$

Can be proven via the repeated use of Klein-Gordon in a potential, or by proving the generalization to $G_{\rm scal}$ of the relation

$$
\frac{\partial}{\partial x^+}[y^+, x^+]_{x_1}[x^+, z^+]_{x_2} = -ig[y^+, x^+]_{x_1}\left[A^-(x^+, \boldsymbol{x}_1) - A^-(x^+, \boldsymbol{x}_2)\right][x^+, z^+]_{x_2}
$$

Structurally ready for a so-called dilute (perturbative) expansion

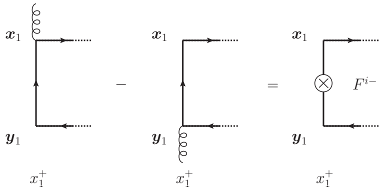
The free propagators G_0 provide the energy denominators.

Two useful technical details

- The classical field does not depend on x^- so $G_{\rm scal}(x, x_0)$ only depends on $(x^- - x_0^-)$, not on each separately: we can define

$$G_{\rm scal}(x, x_0) \equiv \int \frac{\mathrm{d}p^+}{2\pi} \frac{\mathrm{e}^{-ip^+(x^- - x_0^-)}}{2ip^+} (\boldsymbol{x}|\mathcal{G}_{p^+}(x^+, x_0^+)|\boldsymbol{x}_0)$$

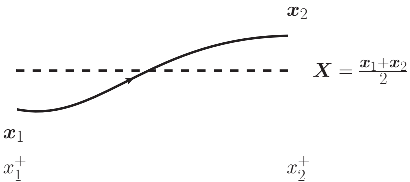
$\mathcal{G}$ satisfies the Schrödinger equation instead of Klein-Gordon



- Since $A^i = 0$, we have

$$A^-(x^+, \boldsymbol{x}) - A^-(x^+, \boldsymbol{y}) = -(\boldsymbol{x}^i - \boldsymbol{y}^i) \int_0^1 \mathrm{d}s\, F^{i-}(x^+, s\boldsymbol{x} + (1-s)\boldsymbol{y})$$

Partial twist expansion



$$(\boldsymbol{x}_1|\mathcal{G}^R_{p^+}(x_1^+,x_2^+)|\boldsymbol{x}_2) \simeq \theta(p^+)\,(\boldsymbol{x}_1|\mathcal{G}^{(0)R}_{p^+}(x_1^+,x_2^+)|\boldsymbol{x}_2)\,\left[x_1^+,x_2^+\right]_{\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}}$$

$$(\boldsymbol{y}_2|\mathcal{G}^A_{p^+}(x_2^+,x_1^+)|\boldsymbol{y}_1) \simeq \theta(-p^+)\,(\boldsymbol{y}_2|\mathcal{G}^{(0)A}_{p^+}(x_2^+,x_1^+)|\boldsymbol{y}_1)\,\left[x_2^+,x_1^+\right]_{\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2}}$$

[Altinoluk, Armesto, Beuf, Martinez, Salgado]

$$F^{i-}(x_1^+, s\boldsymbol{x}_1+\bar{s}\boldsymbol{y}_1) \simeq F^{i-}\left(x_1^+, s\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}+\bar{s}\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2}\right)$$