Factorisation of one-loop amplitudes in NMRK limits

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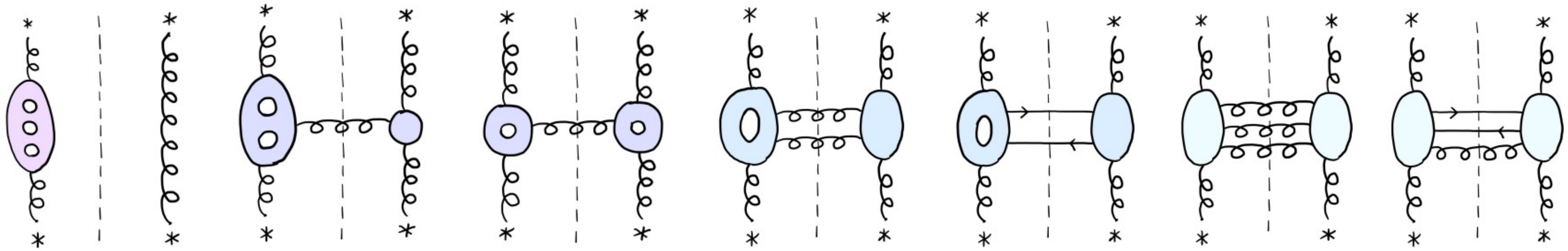
Low-x Workshop, Leros
7th September 2023

1. Overview of QCD at NNLL

Overview of the BFKL kernel at NNLL

The problem of extending the BFKL approach to NNLL accuracy has been standing for a long time. There has been much recent progress in obtaining the building blocks of the kernel:



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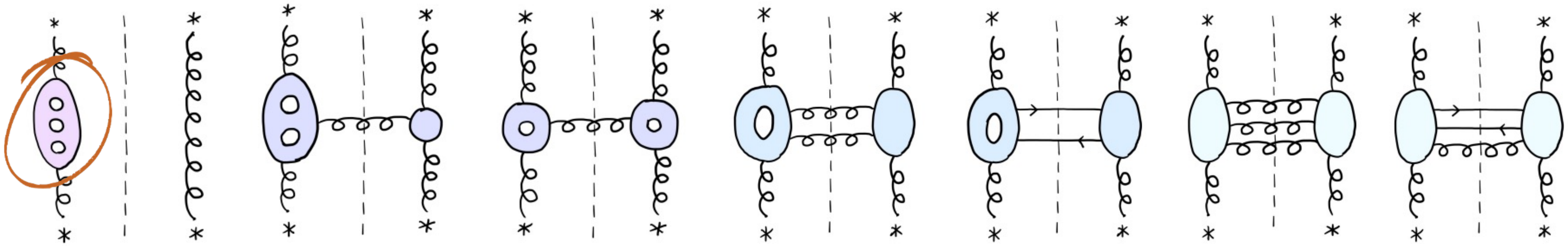

Three-loop Regge trajectory

[1] 2111.10664 Falcioni, Gardi, Maher, Milloy and Vernazza – *Regge-cut scheme*

[2] 2111.14265 Del Duca, Marzucca, Verbeek – *3-loop trajectory in planar* $N = 4$ *SYM (RCS)*

[3] 2112.11097, 2207.03503 Caola, Chakraborty, Gambuti, von Manteuffel and Tancredi – *3-loop trajectory in QCD (RCS), qq qg and gg universality*

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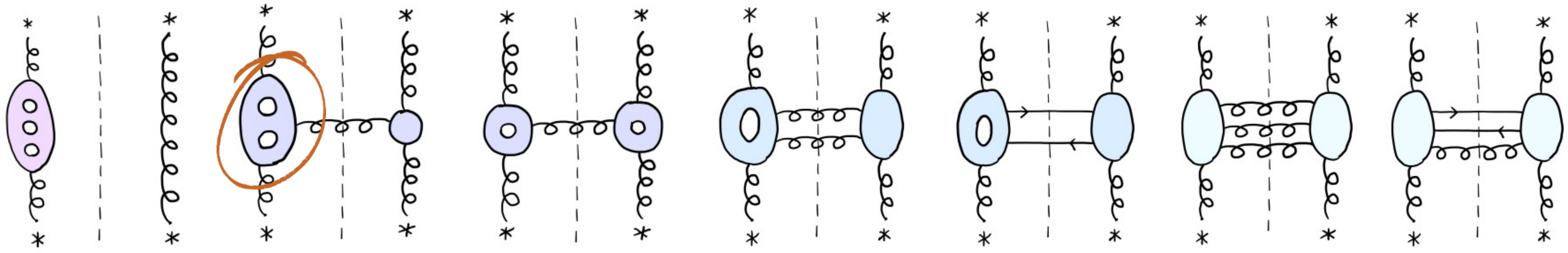

Two-loop Lipatov vertex

[4] 1812.04586 Abreu, Dormans, Cordero Ita and Page - *analytic planar two-loop five-gluon amplitudes in QCD*

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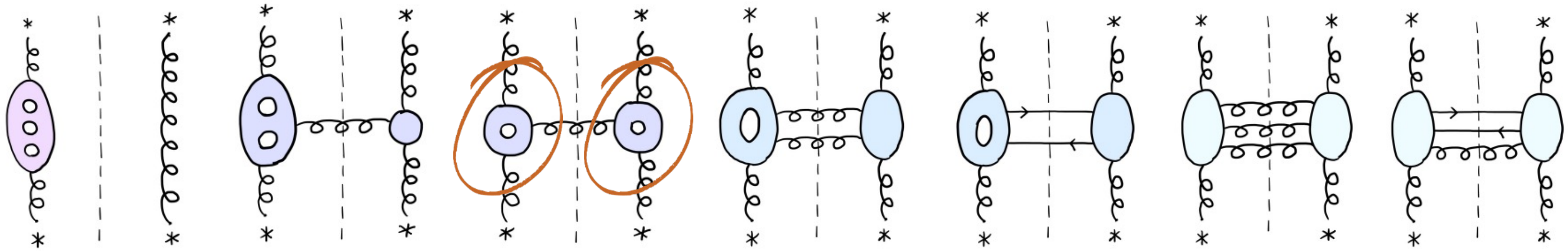

Interference of one-loop Lipatov vertex

[5] Nucl.Phys.B 406 (1993) Fadin, Lipatov
[6] Phys.Rev.D 50 (1994) Fadin, Fiore, Quartarolo
[7] 2302.098 Fadin, Fucilla, Papa - *one-loop Lipatov vertex in QCD to* ϵ^2

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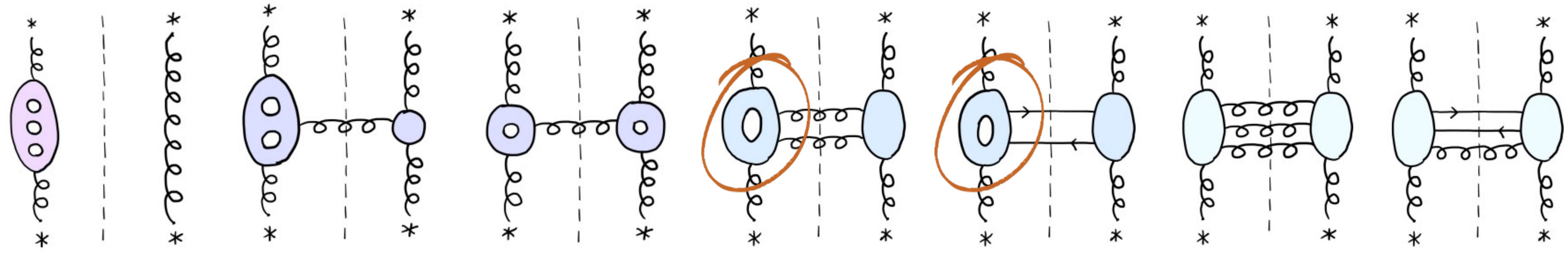

One-loop two-parton central emission vertices

[8] 2204.12459 EB, Del Duca, Dixon, Gardi – *two-gluon vertex in* $N = 4$ *SYM*

Full QCD nearing completion, with Giuseppe De Laurentis

[9] 1904.04067 De Laurentis, Maître – *analytic amplitudes from numerical sampling*

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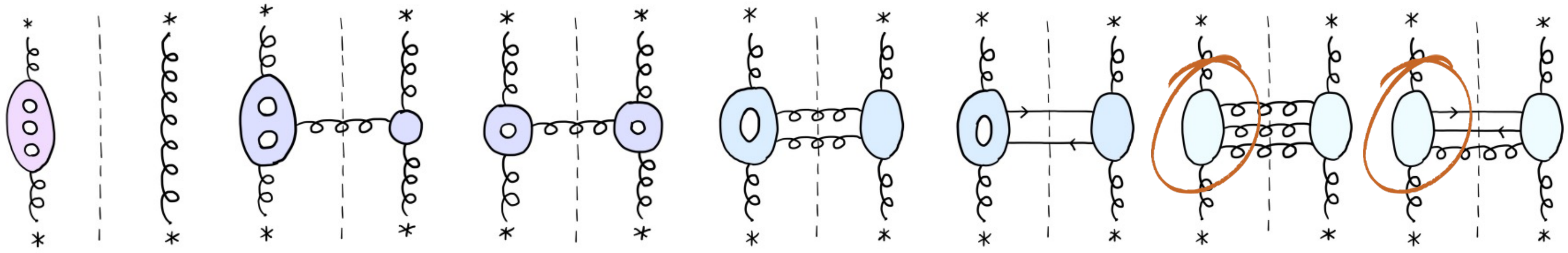

Tree-level three-parton central emission vertices

[10] 9909464 Del Duca, Frizzo, Maltoni – *MHV case*

[11] 0411185 Antoniv, Lipatov, Kuraev – *all helicities via effective action*

[12] *New Techniques in QCD* (2005) Duhr – *all helicities via MHV rules*

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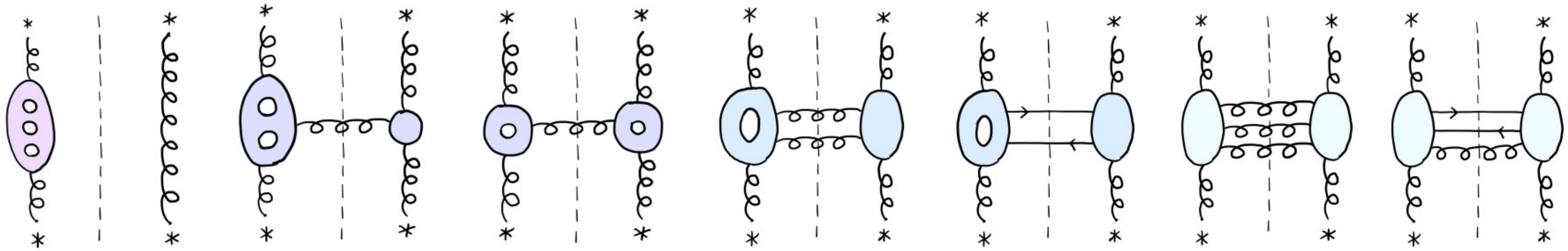
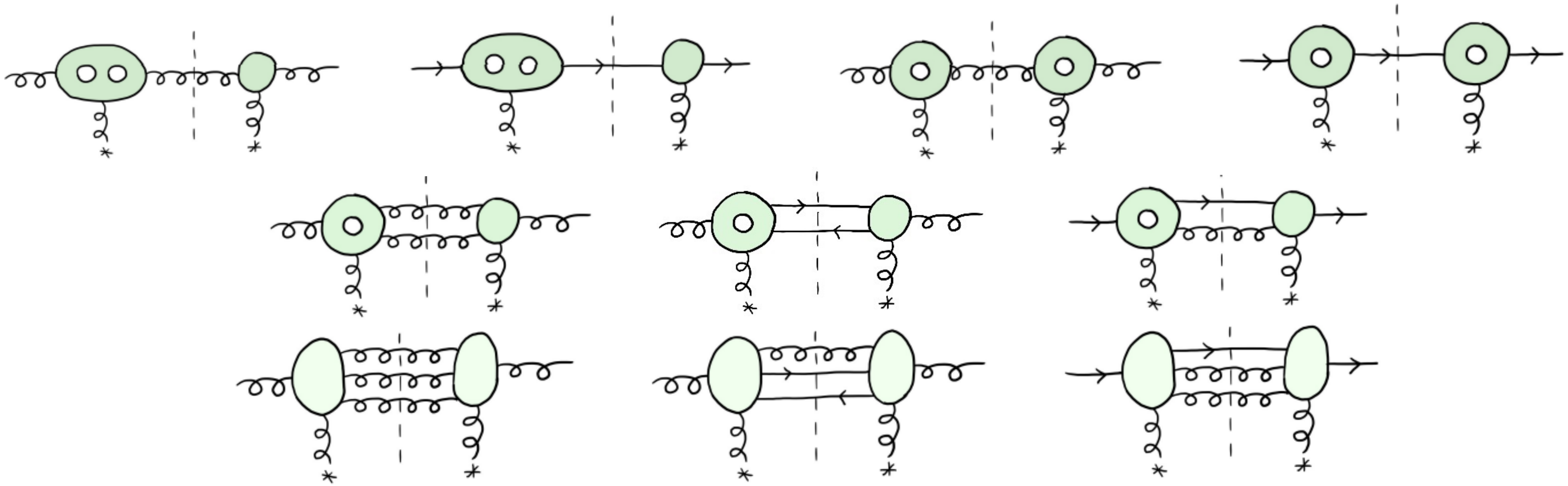


To compute jet cross sections, we also need the following building blocks for the impact factors:



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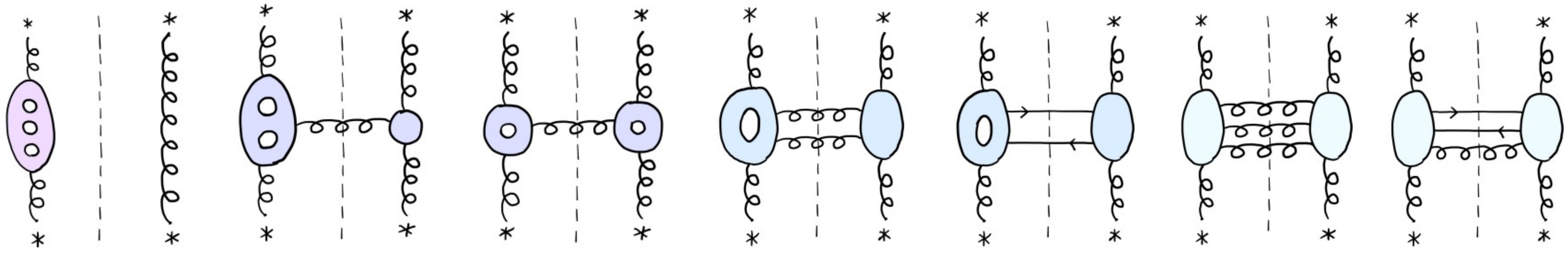

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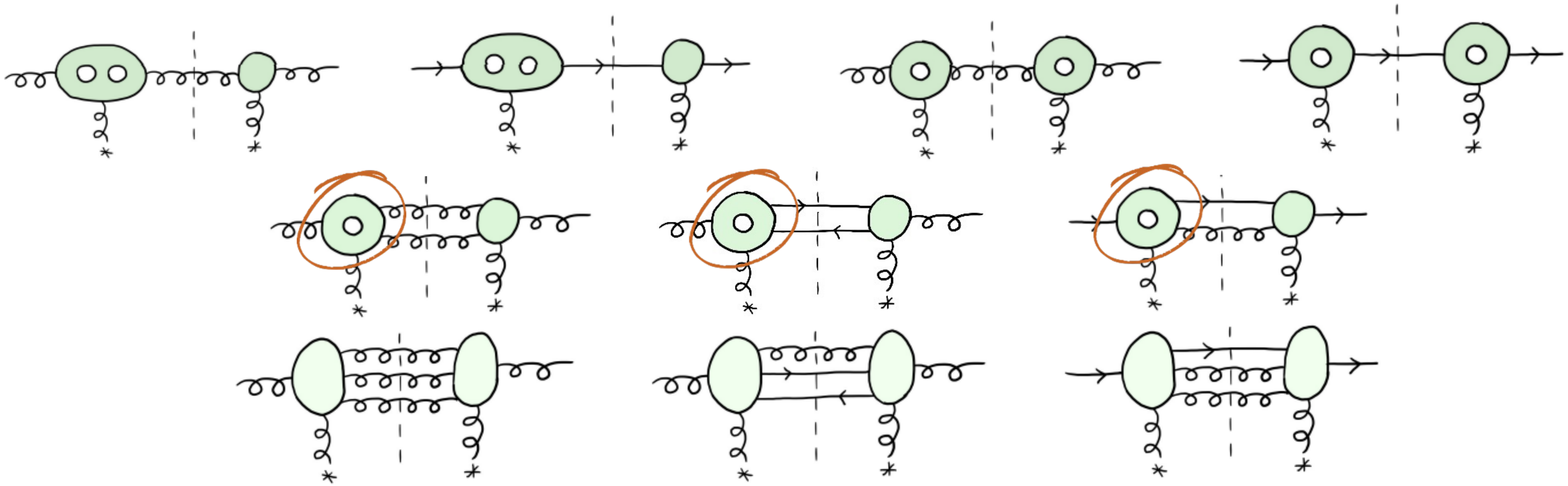
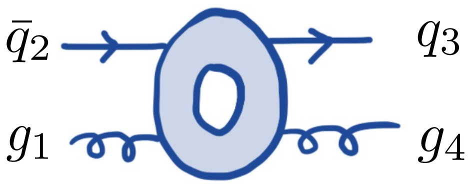


One-loop two-parton peripheral-emission vertices
[13] 2103.16593 Canay, Del Duca – pure gluon case

2. Review of one-loop $q\,g \to q\,g$ in the Regge limit

Colour-structure of $q\ g \to q\ g$ at one loop

We begin with the DDM decomposition [14] for the one-loop $q\ g \to q\ g$ amplitude:



$$
\begin{aligned}
\mathcal{A}_4^{(1,\ \mathrm{QCD})}(\bar{q}_2, q_3, g_4, g_1) = g^4 \Bigg\{ \sum_{\sigma \in S_2} \Bigg[& (T^{c_2} T^{c_1})_{\bar{\imath}_2 \imath_3} (F^{a_{\sigma_4}} F^{a_{\sigma_1}})_{c_1 c_2}\, A_4^{R(1,\ g)}(2_{\bar{q}}, \sigma_4, \sigma_1, 3_q) \\
& + (T^{c_2} T^{a_{\sigma_4}} T^{c_1})_{\bar{\imath}_2 \imath_3} (F^{a_{\sigma_1}})_{c_1 c_2}\, A_4^{R(1,\ g)}(2_{\bar{q}}, \sigma_1, 3_q, \sigma_4) \\
& + (T^{c_2} T^{a_{\sigma_4}} T^{a_{\sigma_1}} T^{c_1})_{\bar{\imath}_2 \imath_3}\, \delta_{c_1 c_2}\, A_4^{R(1,\ g)}(2_{\bar{q}}, 3_q, \sigma_4, \sigma_1) \Bigg] \\
+ \frac{n_f}{N_c} \Bigg[\sum_{\sigma \in S_2} N_c (T^{a_{\sigma_4}} T^{a_{\sigma_1}})_{\bar{\imath}_2 \imath_3}\, A_4^{L(1,\ q)}(2_{\bar{q}}, 3_q, \sigma_4, \sigma_1) & + \mathrm{tr}(T^{a_4} T^{a_1}) \delta_{\bar{\imath}_2 \imath_3}\, A_{4;3}^{(1,\ q)}(2_{\bar{q}}, 3_q; 4, 1) \Bigg] \Bigg\}
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[14] hep-th/0501052 Del Duca, Dixon, Maltoni

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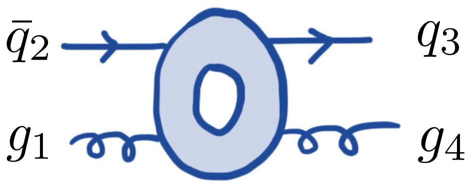

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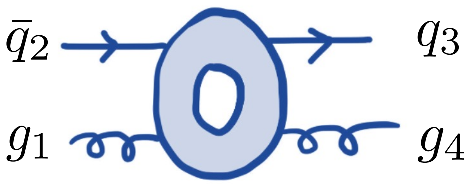

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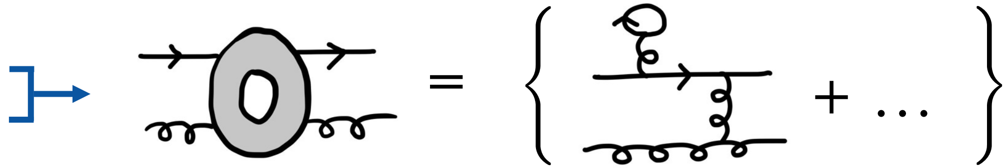

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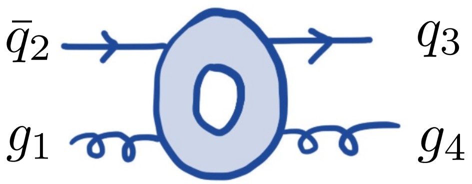

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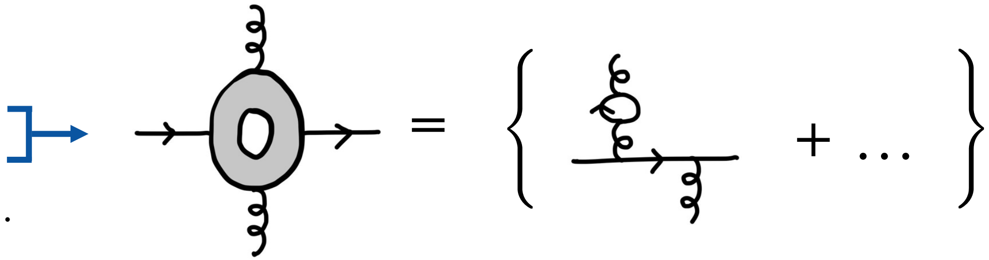

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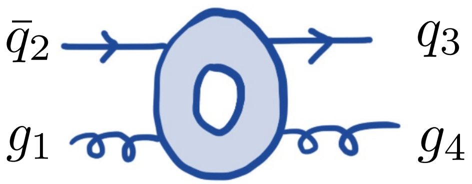

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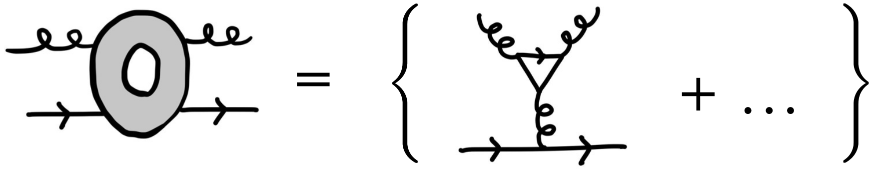

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Colour-structure of $q\ g \to q\ g$ in the Regge limit

Now we use two facts about the primitive amplitudes in the Regge limit, $s_{12} \gg -s_{41}$:

I. All primitive amplitudes with $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$ are power suppressed in this limit.
II. All primitive amplitudes with a_1 and a_4 not colour-adjacent are power suppressed.

Using these facts, we can write the one-loop amplitude as

$$\mathcal{A}_4^{(1)}(\bar{q}_2, q_3, g_4, g_1) \to g^4 \sum_{\sigma \in S_2} \Big[(T^{c_2} T^{c_1})_{\bar{\imath}_2 \imath_3} (F^{a_{\sigma_4}} F^{a_{\sigma_1}})_{c_1 c_2} \ A_4^{R(1,\ g)}(2_{\bar{q}}, \sigma_4, \sigma_1, 3_q)$$
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It is natural to consider amplitudes of definite signature in the, s_{41} channel:

$$A_4^{(1,\ m)[\pm]}(2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 1^{\lambda_1}) = \frac{1}{2} \left(A_4^{(1,\ m)}(2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 1^{\lambda_1}) \pm A_4^{(1,\ m)}(2^{\lambda_2}, 3^{\lambda_3}, 1^{\lambda_1}, 4^{\lambda_4}) \right) .$$

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The colour structure of the signature-odd part of the amplitude is particularly simple:

$$\mathcal{A}_4^{(1)[-]}(\bar{q}_2, q_3, g_4, g_1) \to g^4 T^d_{\bar{\imath}_2 \imath_3} F^d_{a_4 a_1} \left\{ N_c \; A_4^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 1, 4) - \frac{1}{N_c} \; A_4^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 1) + n_f \; A_4^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 1) \right\}$$

Kinematics of $q\ g \to q\ g$ in the Regge limit

Four-parton amplitudes are all (anti-)MHV so it is useful to normalise the one-loop amplitudes by the tree-level amplitude:

$$A_n^{(1,\ m)}(1,\ldots,n) = g^2\ c_\Gamma\ A_n^{(0)}(1,\ldots,n)\ a_n^{(1,\ m)}(1,\ldots,n), \qquad c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\,.$$

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We now use two more facts about primitive amplitudes in the Regge limit:

III. The leading tree-level partial amplitudes ($A_n^{(0)}$) are antisymmetric under $p_4^{-\lambda_1} \leftrightarrow p_1^{\lambda_1}$

IV. The real part of the one-loop corrections ($a_n^{(1)}$) are symmetric under $p_4^{-\lambda_1} \leftrightarrow p_1^{\lambda_1}$

For the real part of the amplitude (which is the part relevant for the NNLL contribution to the cross section) we find

$$\mathrm{Re}\left[A_4^{(1,\ m)[-]}(2,3,4,1)\right] \to g^2\ c_\Gamma\ A_4^{(0)}(2,3,4,1)\ \mathrm{Re}\left[a_4^{(1,\ m)}(2,3,4,1)\right]$$

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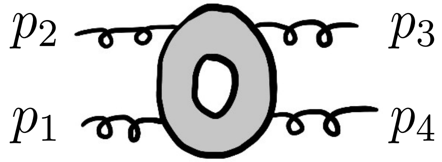
In this talk, we limit our discussion to the real part of the amplitude.

Our remaining task is to analyse the real part of the one-loop primitive amplitudes.

Aside: Regge limit of one-loop four-gluon amplitudes in $N = 4$

One-loop amplitudes in $N = 4$, [16]

$$a_4^{(1,\ \mathcal{N}=4)}(2,3,4,1) = -\frac{2}{\epsilon^2}\left[\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} + \left(\frac{\mu^2}{-s_{23}}\right)^{\epsilon}\right] + \ln^2\left(\frac{-s_{12}}{-s_{23}}\right) + \pi\,,$$



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admit an exact decomposition into one-loop building blocks, in particular, [17]

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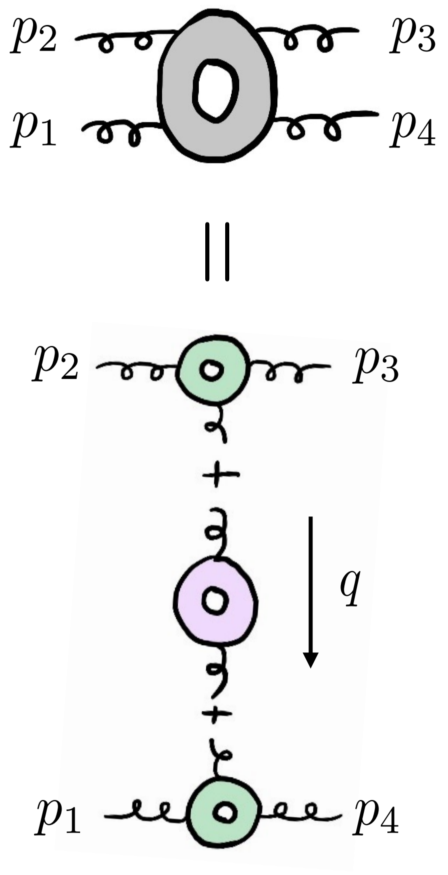

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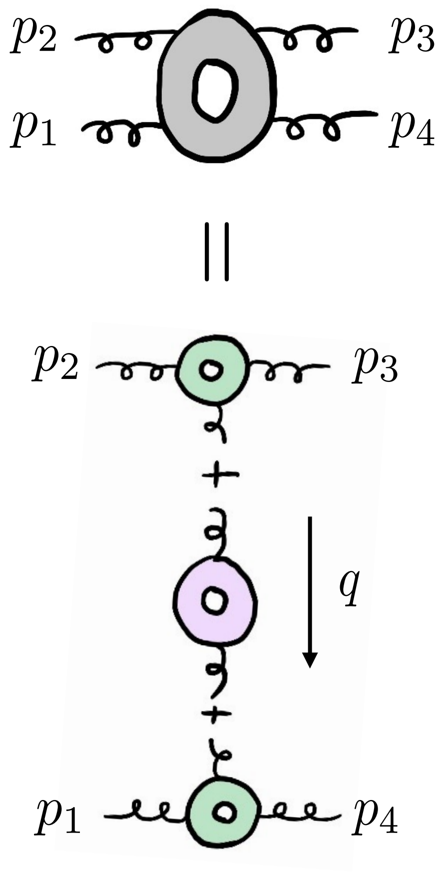

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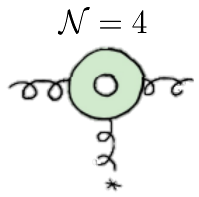

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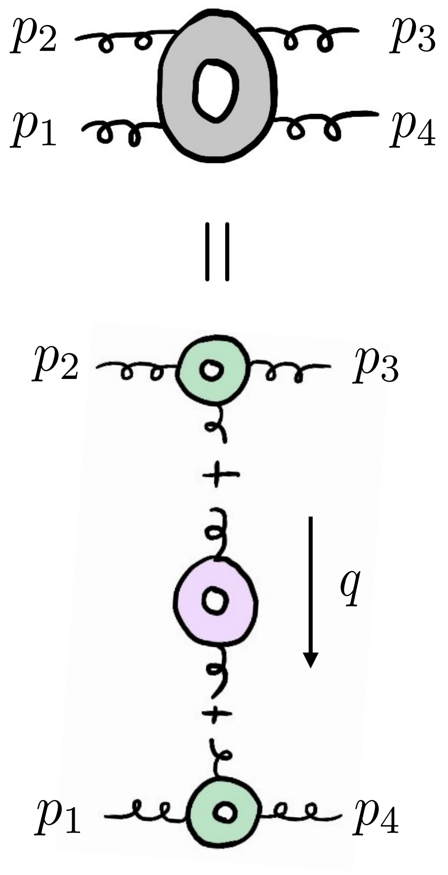

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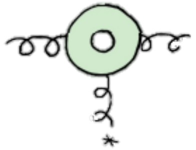

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and (normalised) one-loop correction to the Regge trajectory times logarithm

$$r_{g^*}^{(1,\ \mathcal{N}=4)}(t;s) = \frac{\alpha^{(1)}(t)}{g^2 N_c\, c_\Gamma}\log\left(\frac{s}{\tau}\right)\,, \qquad \alpha^{(1)}(t) = c_\Gamma g^2 \frac{2N_c}{\epsilon}\left(\frac{\mu^2}{t}\right)^{\epsilon}$$

Primitive amplitudes in the Regge limit: Gluon in the loop

Following ref. [15], we use a supersymmetric organisation of $0 \to \bar{q}\, q\, g\, g$ primitive amplitudes

$$A_4^{(1,\ \mathcal{N}=1_V)} = A_4^{L(1,\ g)} + A_4^{R(1,\ g)} + A_4^{L(1,\ f)} + A_4^{R(1,\ f)}.$$

[15] hep-th/9409393 Bern, Dixon, Kosower

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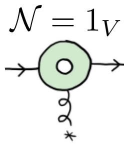
We can therefore identify the one-loop correction to the quark-emission vertex with the gluon-emission vertex:



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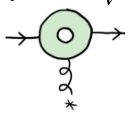

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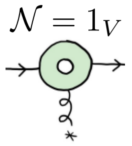

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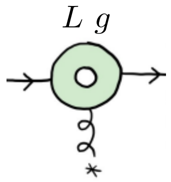
Knowledge of the (simpler) gluon-emission vertex defines the quark-emission vertex:



$$c_{\bar{q}qg^*}^{L(1,\ g)}(p_2, p_3, q) = \left(\frac{\mu^2}{-s_{23}}\right)^{\epsilon}\left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \frac{1}{\epsilon}\log\left(\frac{\tau}{-s_{23}}\right) + \frac{\pi^2}{2} + \frac{19}{18}\right)$$

Primitive amplitudes in the Regge limit: Fermion in the loop

The primitive amplitudes with an internal fermion loop are slightly more subtle. The amplitudes themselves are zero [15]:

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Let us nevertheless demand these amplitudes obey the same factorised form as the previous amplitudes, e.g.,

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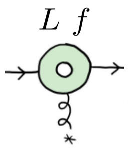

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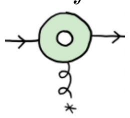

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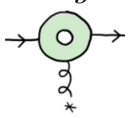

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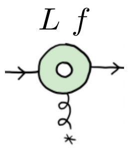

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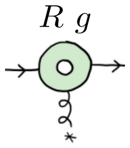
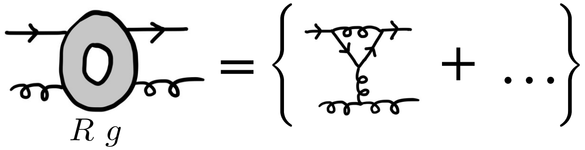


$$c_{\bar{q}qg^*}^{R(1,\ g)}(p_2, p_3, q) = c_{\bar{q}qg^*}^{(1,\ \mathcal{N}=1_V)}(p_2, p_3, q) - c_{\bar{q}qg^*}^{L(1,\ g)}(p_2, p_3, q) - c_{\bar{q}qg^*}^{L(1,\ f)}(p_2, p_3, q)$$

This makes intuitive sense if we consider the diagrams contributing to $R(1, g)$:



[15] hep-th/9409393 Bern, Dixon, Kosower

Colour-dressed amplitude for $q\ g \to q\ g$ in the Regge limit

We can now combine our study of the colour structure and primitive amplitudes of $qg \to qg$ at one loop.

Recall our result for the signature odd amplitude:

$$\mathcal{A}_4^{(1)[-]}(\bar{q}_2, q_3, g_4, g_1) \to g^4 T^d_{\bar{\imath}_2 \imath_3} F^d_{a_4 a_1} \left\{ N_c\ A_4^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 1, 4) - \frac{1}{N_c}\ A_4^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 1) + n_f\ A_4^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 1) \right\}.$$

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Inserting the factorised form of the real part of the primitive amplitudes, we obtain

$$\begin{aligned}
\mathrm{Re}\left[\mathcal{A}_4^{(1)[-]}(\bar{q}_2, q_3, g_4, g_1)\right] \to& \left[g\ T^d_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qg^*}(p_2, p_3, q)\right] \times \frac{1}{t} \times \left[g\ F^d_{a_4 a_1} C^{(0)}_{ggg^*}(p_4, p_1, -q)\right] \\
&\times c_\Gamma\ g^2 \Bigg\{ \left(N_c\ c^{L(1,\ g)}_{\bar{q}qg^*}(p_2, p_3, q) - \frac{1}{N_c} c^{R(1,\ g)}_{\bar{q}qg^*}(p_2, p_3, q) + n_f\ c^{L(1,\ q)}_{\bar{q}qg^*}(p_2, p_3, q) \right) \\
&\qquad + N_c\ r^{(1,\ g)}_{g^*}(t; s_{12}) \\
&\qquad + \left(N_c\ c^{(1,\ g)}_{ggg^*}(p_4, p_1, -q) + n_f\ c^{(1,\ q)}_{ggg^*}(p_4, p_1, -q) \right) \Bigg\}
\end{aligned}$$

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\end{aligned}$$

Our treatment of the primitive amplitudes correctly reproduces the correct n_f terms for the gluon and quark vertices, while correctly generating a $1/N_c$ factor for the quark vertex alone [18-20].

The DDM basis provided a neat (gauge invariant) way of organising these contributions.

[18] Nucl.Phys.B Proc.Suppl. 29 (1992) 93 Fadin, Lipatov; [19] Phys.Lett.B 294 (1992) 286 Fadin, Fiore; [20] hep-ph/9711309 Del Duca, Schmidt

Colour-dressed amplitude for $q\ g \to q\ g$ in the Regge limit

To all-orders, at NLL accuracy, the $qg \to qg$ amplitude factorises [5]:

$$\mathrm{Re}\left[\mathcal{A}_4^{[-]}\left(\bar{q}_2, q_3, g_4, g_1\right)\right] \to s\ \mathcal{C}_{\bar{q}qg^*}\left(p_2, p_3, q_1\right) \times \left[\frac{1}{t}\left(\left(\frac{s}{\tau}\right)^{\alpha(t)} + \left(\frac{-s}{\tau}\right)^{\alpha(t)}\right)\right] \times \mathcal{C}_{ggg^*}\left(p_2, p_3, q_1\right)$$

[5] Nucl. Phys. B 406 (1993) Fadin, Lipatov

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Here we use a calligraphic script to denote colour-dressed objects, in analogy with amplitudes. Each building block is considered to have an all-orders expansion in the coupling, e.g.

$$\mathcal{C}_{\bar{q}qg^*}(p_2, p_3, q_1) = \mathcal{C}^{(0)}_{\bar{q}qg^*}(p_2, p_3, q_1) + \mathcal{C}^{(1)}_{\bar{q}qg^*}(p_2, p_3, q_1) + \mathcal{O}\left(g_S^5\right)$$
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At tree-level we have the familiar results

$$\mathcal{C}^{(0)}_{\bar{q}qg^*}(p_2, p_3, q_1) = g\ T^d_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qg^*}(p_2, p_3, q_1)\,, \qquad \mathcal{C}^{(0)}_{ggg^*}(p_2, p_3, q_1) = g\ F^d_{a_2 a_3} C^{(0)}_{ggg^*}(p_2, p_3, q_1)\,,$$

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and from our analysis of the one-loop amplitude we can extract the colour-dressed vertices

$$\mathcal{C}^{(1)}_{\bar{q}qg^*}(p_2, p_3, q_1) = c_\Gamma\ g^3\ T^d_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qg^*}(p_2, p_3, q_1)\left(N_c\ c^{L(1,\ g)}_{\bar{q}qg^*}(p_2, p_3, q_1) + \frac{1}{N_c}\ c^{R(1,\ g)}_{\bar{q}qg^*}(p_2, p_3, q_1) - n_f\ c^{L(1,\ q)}_{\bar{q}qg^*}(p_2, p_3, q_1)\right)\,,$$
$$\mathcal{C}^{(1)}_{ggg^*}(p_2, p_3, q_1) = c_\Gamma g^3\ F^d_{a_2 a_3} C^{(0)}_{ggg^*}(p_2, p_3, q_1)\left(N_c\ c^{(1,\ g)}_{ggg^*}(p_2, p_3, q_1) + n_f\ c^{(1,\ q)}_{ggg^*}(p_2, p_3, q_1)\right)\,.$$

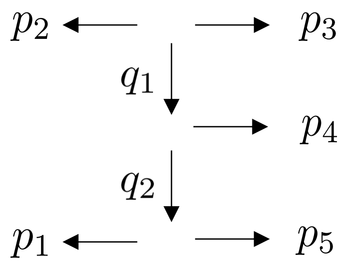
[5] Nucl. Phys. B 406 (1993) Fadin, Lipatov

2. Analysis of one-loop $q\ g \to q\ g\ g$ in the NMRK limit

Kinematic setup

We consider the physical scattering of massless partons $1\ 2 \to 3\ 4\ 5$.

We use the all-outgoing convention such that $\sum_{i=1}^{5} p_i = 0$ with $p_1^0, p_2^0 < 0$ and $p_3^0, p_4^0, p_5^0 > 0$.



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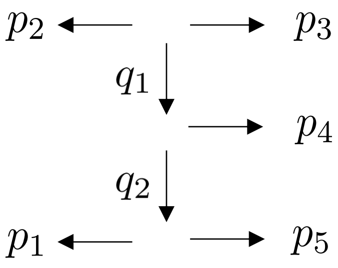

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We use lightcone coordinates and complex transverse momenta

$$p_i^{\pm} = p_i^0 + p_i^z \,, \qquad p_{i\perp} = p_i^x + i p_i^y \,.$$

We work in a frame with $p_1 = (0, p_1^-; 0)$ and $p_2 = (p_2^+, 0; 0)$. We express the remaining degrees of freedom in terms of the dimensionless variables:

$$X = \frac{p_3^+}{p_4^+} \,, \qquad Y = \frac{p_4^+}{p_5^+} \,, \qquad z = -\frac{p_{3\perp}}{p_{4\perp}}$$

In terms of these variables, the forward NMRK limit is given by $Y \to \infty$, with fixed X and transverse momenta, while the MRK limit is given by $X, Y \to \infty$ with fixed transverse momenta.

Minimal Variables: Gardi, Mo

Colour-structure of $q\ g \to q\ g\ g$ at one loop

We begin with the DDM decomposition [14] for the one-loop $q\ g \to q\ g\ g$ amplitude and perform analogous steps to the four-parton case. Again, the signature-odd (in the s_{51} channel) part of the amplitude has a particularly simple colour structure:

$$
\begin{aligned}
&\mathcal{A}_5^{(1)[-]}(\bar{q}_2, q_3, g_4, g_5, g_1) \to g^5\left(-F_{51}^d\right) \\
&\times \Big\{ \left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \left(F^{a_4}F^d\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \left(F^{d}F^{a_4}\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) \\
&+ \left(T^{c_2}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \left(F^d\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) + \left(T^{c_2}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \left(F^{a_4}\right)_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) \\
&+ \left(T^{c_2}T^{a_4}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \delta_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + \left(T^{c_2}T^{d}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3} \delta_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) \\
&+ \frac{n_f}{N_c}\Big[N_c (T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + N_c (T^{d}T^{a_4})_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \mathrm{tr}(T^{a_4}T^d)\delta_{\bar{\imath}_2 \imath_3} A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1) \Big] \Big\}
\end{aligned}
$$

We find that the overall colour structure factorises into an adjoint generator times the one-loop four-point amplitude.

[14] hep-th/0501052 Del Duca, Dixon, Maltoni

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$$\mathcal{A}_5^{(1)[-]}(\bar{q}_2, q_3, g_4, g_5, g_1) \to g^5\left(-F^d_{51}\right)$$
$$\times\Big\{\left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^{a_4}F^d\right)_{c_1c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^dF^{a_4}\right)_{c_1c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1)$$
$$+\left(T^{c_2}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^d\right)_{c_1c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) + \left(T^{c_2}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^{a_4}\right)_{c_1c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1)$$
$$+\left(T^{c_2}T^{a_4}T^dT^{c_1}\right)_{\bar{\imath}_2 \imath_3}\delta_{c_1c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + \left(T^{c_2}T^dT^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\delta_{c_1c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4)$$
$$+\frac{n_f}{N_c}\left[N_c(T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + N_c(T^dT^{a_4})_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \mathrm{tr}(T^{a_4}T^d)\delta_{\bar{\imath}_2 \imath_3} A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1)\right]\Big\}$$

We find that the overall colour structure factorises into an adjoint generator times the one-loop four-point amplitude.

Furthermore, in the NMRK limit, we find that the $tr\left(T^aT^b\right)\delta_{ij}$ coefficient vanishes, as in the four-point amplitude:

$$A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1) \xrightarrow[\text{NMRK}]{} -A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) - A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) - A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4)\ .$$

[14] hep-th/0501052 Del Duca, Dixon, Maltoni

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&+ \left(T^{c_2}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^d\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) + \left(T^{c_2}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^{a_4}\right)_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) \\
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$$
A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1) \xrightarrow[\text{NMRK}]{} \underbrace{-A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1)}_{0 \text{ in general kinematics}} - A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) - A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4)\ .
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[14] hep-th/0501052 Del Duca, Dixon, Maltoni

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$$\times\Big\{\left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^{a_4}F^d\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \left(T^{c_2}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^d F^{a_4}\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1)$$
$$+\left(T^{c_2}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^d\right)_{c_1 c_2} A_5^{L(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1) + \left(T^{c_2}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\left(F^{a_4}\right)_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1)$$
$$+\left(T^{c_2}T^{a_4}T^{d}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\delta_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + \left(T^{c_2}T^{d}T^{a_4}T^{c_1}\right)_{\bar{\imath}_2 \imath_3}\delta_{c_1 c_2} A_5^{R(1,\ g)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4)$$
$$+\frac{n_f}{N_c}\left[N_c (T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + N_c (T^{d}T^{a_4})_{\bar{\imath}_2 \imath_3} A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) + \mathrm{tr}(T^{a_4}T^d)\delta_{\bar{\imath}_2 \imath_3} A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1)\right]\Big\}$$

We find that the overall colour structure factorises into an adjoint generator times the one-loop four-point amplitude.

Furthermore, in the NMRK limit, we find that the $tr\left(T^a T^b\right)\delta_{ij}$ coefficient vanishes, as in the four-point amplitude:

$$A_{5;4}^{(1,\ q)[-]}(2_{\bar{q}}, 3_q; 4, 5, 1) \xrightarrow[\text{NMRK}]{} \underbrace{-A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 4, 3_q, 5, 1)}_{0 \text{ in general kinematics}} \underbrace{- A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) - A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4)}\,.$$

$$A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 4, 5, 1) + A_5^{L(1,\ q)[-]}(2_{\bar{q}}, 3_q, 5, 1, 4) \xrightarrow[\text{NMRK}]{} 0\,.$$

"Furry's theorem" for off-shell gluon $(p_5 + p_1)$

[14] hep-th/0501052 Del Duca, Dixon, Maltoni

Simplified colour basis in the NMRK limit

While the DDM decomposition is very useful for organising the kinematic terms, it is overcomplete. We can move to a basis consisting of the two tree-level colour structures plus one new colour structure:

$$\mathcal{A}_5^{(1)[-]}\left(\bar{q}_2, q_3, g_4, g_5, g_1\right) \to g^5\left(-F_{51}^d\right)$$
$$\times\Big\{\left(T^d T^{a_4}\right)_{\bar{\imath}_2 \imath_3}\left(N_c A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) - \frac{1}{N_c} A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + n_f A_5^{L(1,\ q)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right)\right)$$
$$+\left(T^{a_4} T^d\right)_{\bar{\imath}_2 \imath_3}\left(N_c A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right) - \frac{1}{N_c} A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right) + n_f A_5^{L(1,\ q)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right)\right)$$
$$+\,\delta_{\bar{\imath}_2 \imath_3} \Delta_{d a_4}\Big(A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + A_5^{L(1,\ g)[-]}\left(4, 2_{\bar{q}}, 3_q, 5, 1\right) + A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 4, 3_q, 5, 1\right)$$
$$+\,A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + A_5^{R(1,\ g)[-]}\left(4, 2_{\bar{q}}, 3_q, 5, 1\right) + A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 4, 3_q, 5, 1\right)\Big)\Big\}$$

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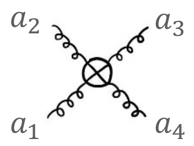
While the DDM decomposition is very useful for organising the kinematic terms, it is overcomplete. We can move to a basis consisting of the two tree-level colour structures plus one new colour structure:

$$
\begin{aligned}
\mathcal{A}_5^{(1)[-]}\left(\bar{q}_2, q_3, g_4, g_5, g_1\right) \to g^5\left(-F^d_{51}\right) \\
\times\Big\{ \left(T^d T^{a_4}\right)_{\bar{\imath}_2 \imath_3} \Big(N_c A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) - \frac{1}{N_c} A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + n_f A_5^{L(1,\ q)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right)\Big) \\
+\left(T^{a_4} T^d\right)_{\bar{\imath}_2 \imath_3} \Big(N_c A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right) - \frac{1}{N_c} A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right) + n_f A_5^{L(1,\ q)[-]}\left(2_{\bar{q}}, 3_q, 4, 5, 1\right)\Big) \\
+\,\delta_{\bar{\imath}_2 \imath_3} \Delta_{d a_4} \Big(A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + A_5^{L(1,\ g)[-]}\left(4, 2_{\bar{q}}, 3_q, 5, 1\right) + A_5^{L(1,\ g)[-]}\left(2_{\bar{q}}, 4, 3_q, 5, 1\right) \\
+\, A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 3_q, 5, 1, 4\right) + A_5^{R(1,\ g)[-]}\left(4, 2_{\bar{q}}, 3_q, 5, 1\right) + A_5^{R(1,\ g)[-]}\left(2_{\bar{q}}, 4, 3_q, 5, 1\right) \Big)\Big\}
\end{aligned}
$$

This is analogous to the pure-gluon case where the amplitude can be written in terms of the basis [8]

$$
\left\{ \left(F^{a_3} F^{a_4}\right)_{a_2 d},\, \left(F^{a_4} F^{a_3}\right)_{a_2 d},\, d^{a_2 a_3 a_4 d} \right\}
$$



$$
= d_A^{a_1 a_2 a_3 a_4} = \frac{1}{4!} \sum_{S_4} \mathrm{tr}\left(F^{a_{\sigma_1}} F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}}\right)
$$

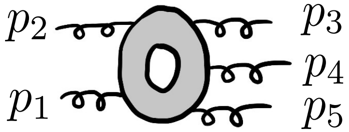
These bases are particularly convenient for demonstrating how the known MRK limit arises from the NMRK limit.

[8] 2204.12459 EB, Del Duca, Dixon, Gardi, Smillie

Aside: Regge limit of one-loop five-gluon amplitudes in $N = 4$

Just as in the four-gluon case, the one-loop five gluon amplitudes in $N = 4$,

$$a_5^{(1,\ \mathcal{N}=4)}(1,2,3,4,5) = -\frac{1}{\epsilon^2}\sum_{i=1}^{5}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} + \frac{5}{6}\pi^2 - \frac{\delta_R}{3} + \sum_{i=1}^{5}\log\left(\frac{-s_{i,i+1}}{-s_{i+1,i+2}}\right)\log\left(\frac{-s_{i+2,i-2}}{-s_{i-2,i-1}}\right)\,,$$



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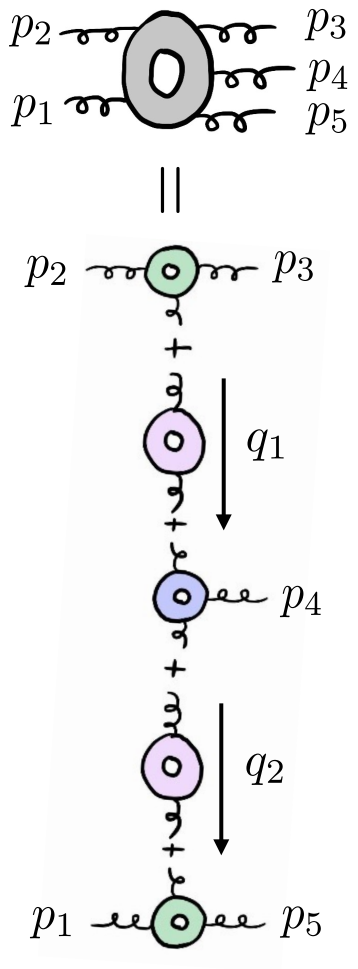

admit an exact decomposition into one-loop building blocks, in particular, [17]

$$\begin{aligned}\mathrm{Re}\left[a_5^{(1,\ \mathcal{N}=4)}(2,3,4,5,1)\right] =& c_{ggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,q_1) + r_{g^*}^{(1,\ \mathcal{N}=4)}(s_{34},t_1) + v^{(1,\ \mathcal{N}=4)}\left(t_1,\frac{s_{34}s_{45}}{s_{345}},t_2\right) \\ &+ r_{g^*}^{(1,\ \mathcal{N}=4)}(s_{45},t_2) + c_{ggg^*}^{(1,\ \mathcal{N}=4)}(p_5,p_1,-q_2)\,,\end{aligned}$$

with special function

$$v^{(1,\ \mathcal{N}=4)}(t_1,\eta,t_2) = -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{\eta}\right)^{\epsilon} + \frac{\pi^2}{3} - \frac{1}{2}\log^2\left(\frac{t_1}{t_2}\right) + \frac{1}{\epsilon}\left[\left(\frac{\mu^2}{t_1}\right)^{\epsilon} + \left(\frac{\mu^2}{t_2}\right)^{\epsilon}\right]\log\left(\frac{\tau}{\eta}\right).$$

[17] 0802.2065 Bartels, Lipatov, Sabio Vera

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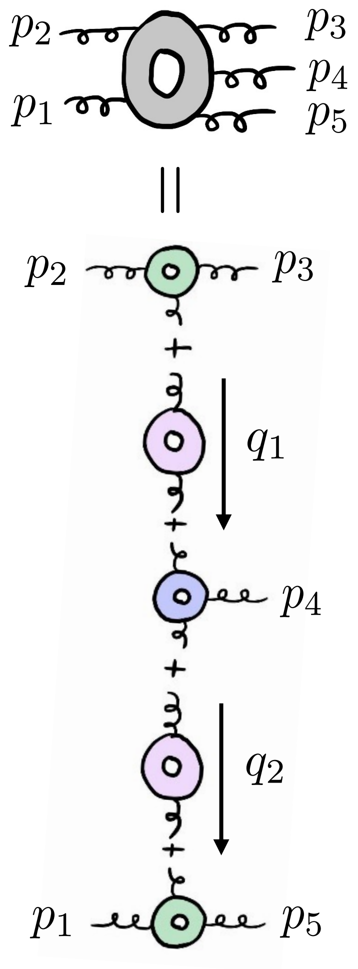

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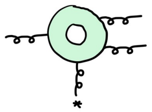
In terms of this function, we can define the two-gluon peripheral emission vertex:

$\mathcal{N} = 4$



$$c_{gggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,p_4,q_2) = c_{ggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,q_1) + r_{g^*}^{(1,\ \mathcal{N}=4)}(s_{34},q_1) + v^{(1,\ \mathcal{N}=4)}\left(t_1,\frac{s_{34}p_4^+}{(p_3^+ + p_4^+)},t_2\right)\,.$$

[17] 0802.2065 Bartels, Lipatov, Sabio Vera

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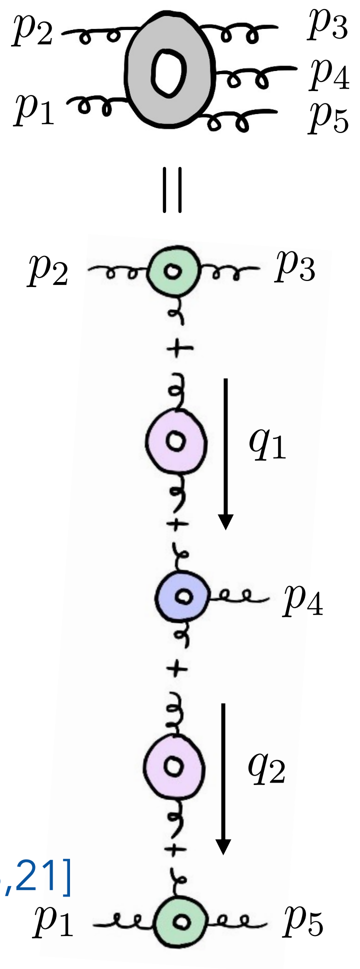

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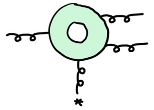

$$c_{gggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,p_4,q_2) = c_{ggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,q_1) + r_{g^*}^{(1,\ \mathcal{N}=4)}(s_{34},q_1) + v^{(1,\ \mathcal{N}=4)}(t_1,\frac{s_{34}p_4^+}{(p_3^+ + p_4^+)},t_2)\,.$$

We can easily obtain the MRK limit of this vertex, where we recognise the $N = 4$ one-loop Lipatov vertex [5,6,21]

$$c_{gggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,p_4,q_2) \xrightarrow[\text{MRK}]{} c_{ggg^*}^{(1,\ \mathcal{N}=4)}(p_2,p_3,q_1) + r_{g^*}^{(1,\ \mathcal{N}=4)}(s_{34},q_1) + v_{g^*gg^*}^{(1,\ \mathcal{N}=4)}(-q_1,p_4,q_2)\,.$$

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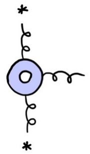

$$v_{g^*gg^*}^{(1,\ \mathcal{N}=4)}(-q_1,p_4,q_2) = v^{(1,\ \mathcal{N}=4)}(|q_{1\perp}|^2,|p_{4\perp}|^2,|q_{2\perp}|^2)$$

[5],[6], [21] 9810215 Del Duca, Schmidt

[17] 0802.2065 Bartels, Lipatov, Sabio Vera

$\mathcal{N}$=1 chiral multiplet circulating in the loop

Let us be concrete and consider the scattering $q^{\ominus}g^{\ominus} \to q^{\oplus}g^{\oplus}g^{\ominus}$ with momenta $p_2 + p_1 = p_3 + p_4 + p_5$ respectively.
The fermion and scalar contributions are simple by the fact there are no IR poles and no large logarithms in the (N)MRK.

$$\mathrm{Re}\left[a_4^{L(1,\,\mathcal{N}=1_\chi)}(2_{\bar{q}}^{\ominus},3_q^{\oplus},4^{\oplus},5^{\oplus},1^{\ominus})\right] \xrightarrow[\text{NMRK}]{} c_{\bar{q}qgg^*}^{L(1,\,\mathcal{N}=1_\chi)}(p_2,p_3,p_4,q_2)+c_{ggg^*}^{(1,\,\mathcal{N}=1_\chi)}(p_5,p_1,-q_2)\,,$$

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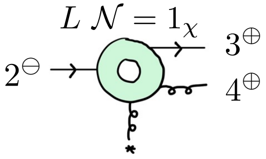
We write the 2-parton emission vertices in terms of the single parton emission vertex to make the MRK limit trivial



$$c_{\bar{q}qgg^*}^{L(1,\ \mathcal{N}=1_\chi)}(p_2^{\ominus}, p_3^{\oplus}, p_4^{\oplus}, q_2) = c_{\bar{q}qg^*}^{L(1,\ \mathcal{N}=1_\chi)}(p_2, p_3, q_2) + \frac{X(1+z-\bar{z}) - z(\bar{z}-2)}{2X|z-1|^2} L_0\left(\frac{t_1}{t_2}\right)$$

$$L_0(x) = \frac{\log(x)}{1-x}$$

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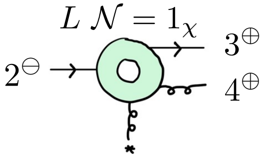

$$c_{\bar{q}qgg^*}^{L(1,\ \mathcal{N}=1_\chi)}(p_2^{\ominus}, p_3^{\oplus}, p_4^{\oplus}, q_2) = c_{\bar{q}qg^*}^{L(1,\ \mathcal{N}=1_\chi)}(p_2, p_3, q_2) + \frac{X(1+z-\bar{z}) - z(\bar{z}-2)}{2X|z-1|^2} L_0\left(\frac{t_1}{t_2}\right)$$

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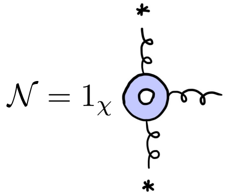
where we recognise the 1-loop Lipatov vertex in $N = 1_\chi$



$$v_{g^*gg^*}^{(1,\ \mathcal{N}=1_\chi)}(-q_1, p_4, q_2) = \frac{1}{2}\frac{\left(|q_{1\perp}|^2 + |q_{2\perp}|^2 - 2q_{1\perp}q_{2\perp}^*\right)}{|q_{2\perp}|^2} L_0\left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2}\right) = \frac{(1+z-\bar{z})}{2|z-1|^2} L_0\left(\frac{t_1}{t_2}\right)$$

Complex scalar circulating in the loop

The structure of amplitudes with a complex scalar circulating in the loop is analogous:

$$\mathrm{Re}\left[a_4^{L(1,\ s)}(2_{\bar{q}}^{\ominus}, 3_q^{\oplus}, 4^{\oplus}, 5^{\oplus}, 1^{\ominus})\right] \xrightarrow[\text{NMRK}]{} c_{\bar{q}qgg^*}^{L(1,\ s)}(p_2, p_3, p_4, q_2) + c_{ggg^*}^{(1,\ s)}(p_5, p_1, -q_2)\,,$$

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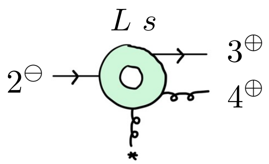


$$c_{\bar{q}qgg^*}^{L(1,\ s)}(p_2^{\ominus}, p_3^{\oplus}, p_4^{\ominus}, q_2) = c_{\bar{q}qg^*}^{L(1,\ s)}(p_2, p_3, q_2) + \frac{1}{6}\frac{X(1+z-\bar{z}) - z(\bar{z}-2)}{X|z-1|^2} L_0\left(\frac{t_1}{t_2}\right) + \frac{1}{3}\frac{z|X+z|^2(X(1+z-\bar{z})+|z|^2)}{X^3(z-1)^3(\bar{z}-1)^2} L_2\left(\frac{t_1}{t_2}\right) - \frac{|X+z|^2}{6X(1+X)(z-1)\bar{z}}$$

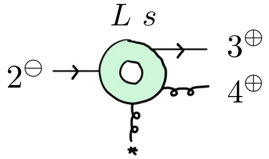
$$L_2(x) = \frac{\log(x) - \frac{1}{2}(x - \frac{1}{x})}{(1-x)^3}$$

Complex scalar circulating in the loop

The structure of amplitudes with a complex scalar circulating in the loop is analogous:

$$\mathrm{Re}\left[a_4^{L(1,\ s)}(2^{\ominus}_{\bar q}, 3^{\oplus}_{q}, 4^{\oplus}, 5^{\oplus}, 1^{\ominus})\right] \xrightarrow[\text{NMRK}]{} c^{L(1,\ s)}_{\bar q q g g^*}(p_2, p_3, p_4, q_2) + c^{(1,\ s)}_{g g g^*}(p_5, p_1, -q_2)\,,$$

We write the 2-parton emission vertices in terms of the single parton emission vertex to make the MRK limit trivial

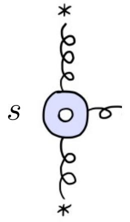


$$c^{L(1,\ s)}_{\bar q q g g^*}(p_2^{\ominus}, p_3^{\oplus}, p_4^{\ominus}, q_2) = c^{L(1,\ s)}_{\bar q q g^*}(p_2, p_3, q_2) + \frac{1}{6}\frac{X(1+z-\bar z) - z(\bar z - 2)}{X|z-1|^2} L_0\left(\frac{t_1}{t_2}\right)$$
$$+ \frac{1}{3}\frac{z|X+z|^2(X(1+z-\bar z) + |z|^2)}{X^3(z-1)^3(\bar z - 1)^2} L_2\left(\frac{t_1}{t_2}\right) - \frac{|X+z|^2}{6X(1+X)(z-1)\bar z}$$

$$c^{L(1,\ \mathcal{N}=1_\chi)}_{\bar q q g g^*}(p_2^{\ominus}, p_3^{\oplus}, p_4^{\ominus}, q_2) \xrightarrow[\text{MRK}]{} c^{L(1,\ s)}_{\bar q q g^*}(p_2, p_3, q_2) + v^{L(1,\ s)}_{g^* g g^*}(-q_1, p_4, q_2)\,,$$

$$L_2(x) = \frac{\log(x) - \frac{1}{2}(x - \frac{1}{x})}{(1-x)^3}$$

where we recognise the Lipatov vertex with a circulating complex scalar



$$v^{(1,\ s)}_{g^* g g^*}(-q_1, p_4, q_2) = \frac{1}{3} v^{(1,\ \mathcal{N}=1_\chi)}_{g^* g g^*}(q_1, p_4, q_2) - \frac{1}{6}\frac{|p_{4\perp}|^2}{q^*_{1\perp} q_{2\perp}} - \frac{1}{3}|p_{4\perp}|^2 q_{1\perp} q^*_{2\perp}(|q_{1\perp}|^2 + |q_{2\perp}|^2 - 2q_{1\perp}q^*_{2\perp})\frac{L_2\left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2}\right)}{(-|q_{2\perp}|^2)^3}$$
$$= \frac{1}{3} v^{(1,\ \mathcal{N}=1_\chi)}_{g^* g g^*}(q_1, p_4, q_2) + \frac{1}{6}\frac{1}{\bar z(z-1)} + \frac{1}{3}\frac{z(z-\bar z+1)}{(|z-1|^2)^3} L_2\left(\frac{|q_{1\perp}|^2}{|q_{2\perp}|^2}\right),$$

Gluon circulating in the loop I

As expected, the $L(1,g)$ piece has large logarithmic terms in the NMRK. We find

$$\mathrm{Re}\left[a_5^{L(1,\ g)}(2_{\bar{q}},3_q,4,5,1)\right] \xrightarrow[\text{NMRK}]{} c_{\bar{q}qgg^*}^{L(1,\ g)}(p_2,p_3,p_4,q_2)+r_{g^*}^{(1,\ g)}(t_2;s_{45})+c_{ggg^*}^{(1,\ g)}(p_5,p_1,-q_2)\,.$$

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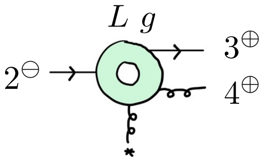
Ref. [15] writes the $L(1,g)$ amplitudes as the pure-gluon $N=4$ amplitude plus the $L(1,s)$ amplitude, plus a remainder.



$$\begin{aligned} c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) &= c^{L(1,\ g)}_{\bar{q}qg^*}(p_2,p_3,q_1)+r^{(1,\ g)}_{g^*}(t_1;s_{34}) \\ &\quad + v^{(1,\mathcal{N}=4)}\left(t_1,\frac{s_{34}p_4^+}{(p_3^++p_4^+)},t_2\right) - 4\frac{X(1+z-\bar{z})+z}{2X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right) + \left(c^{L(1,\ s)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) - c^{L(1,\ s)}_{\bar{q}qg^*}(p_2,p_3,q_1)\right) \\ &\quad - \frac{z}{X}L_{s_{-1}}\left(\frac{t_1}{t_2},\frac{t_1'}{t_2}\right) - \frac{z}{2X(z-1)^2}L_1\left(\frac{t_1'}{t_2}\right) \end{aligned}$$

$$L_{s_{-1}}(x,y) = \mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-y)+\log(x)\log(y)-\frac{\pi^2}{6}$$

[15] hep-th/9409393 Bern, Dixon, Kosower

Gluon circulating in the loop I

As expected, the $L(1,g)$ piece has large logarithmic terms in the NMRK. We find

$$\mathrm{Re}\left[a_5^{L(1,\,g)}(2_{\bar q},3_q,4,5,1)\right] \xrightarrow[\text{NMRK}]{} c^{L(1,\,g)}_{\bar q q g g^*}(p_2,p_3,p_4,q_2)+r^{(1,\,g)}_{g^*}(t_2;s_{45})+c^{(1,\,g)}_{g g g^*}(p_5,p_1,-q_2)\,.$$

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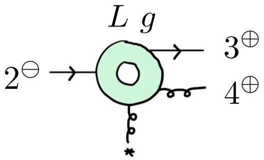


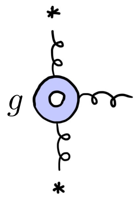
$$\begin{aligned}c^{L(1,\,g)}_{\bar q q g g^*}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) &= c^{L(1,\,g)}_{\bar q g^*}(p_2,p_3,q_1)+r^{(1,\,g)}_{g^*}(t_1;s_{34})\\ &+v^{(1,\mathcal{N}=4)}\left(t_1,\frac{s_{34}p_4^+}{(p_3^+ + p_4^+)},t_2\right)-4\frac{X(1+z-\bar z)+z}{2X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right)+\left(c^{L(1,\,s)}_{\bar q q g g^*}(p_2,p_3,p_4,q_2)-c^{L(1,\,s)}_{\bar q g g^*}(p_2,p_3,q_1)\right)\\ &-\frac{z}{X}L_{s_{-1}}\left(\frac{t_1}{t_2},\frac{t_1'}{t_2}\right)-\frac{z}{2X(z-1)^2}L_1\left(\frac{t_1'}{t_2}\right)\end{aligned}$$

$$L_{s_{-1}}(x,y)=\mathrm{Li}_2(1-x)+\mathrm{Li}_2(1-y)+\log(x)\log(y)-\frac{\pi^2}{6}$$

Once again, it is straightforward to obtain the known MRK limit

$$c^{L(1,\,g)}_{\bar q q g g^*}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) \xrightarrow[\text{MRK}]{} c^{L(1,\,g)}_{\bar q q g^*}(p_2,p_3,q_1)+r^{(1,\,g)}_{g^*}(t_1;s_{34})+v^{(1,\,g)}_{g^* g g^*}(-q_1,p_4,q_2)$$

with the gluon contribution to the Lipatov vertex



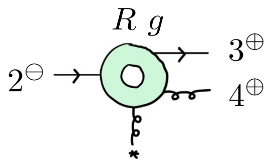
$$v^{(1,\,g)}_{g^* g g^*}(q_1,p_4,q_2)=v^{(1,\,\mathcal{N}=4)}_{g^* g g^*}(-q_1,p_4,q_2)-4\,v^{(1,\,\mathcal{N}=1_\chi)}_{g^* g g^*}(-q_1,p_4,q_2)+v^{(1,\,s)}_{g^* g g^*}(-q_1,p_4,q_2)$$

[15] hep-th/9409393 Bern, Dixon, Kosower

Gluon circulating in the loop II

We can now find the $R(1,g)$ contribution via the $N=1_V$ SUSY decomposition. All Regge trajectories and gluon peripheral-emission vertices cancel, such that these amplitudes only contribute to the qg emission vertex:

$$\mathrm{Re}\left[a_5^{R(1,\ g)}(2_{\bar{q}},3_q,4,5,1)\right] \xrightarrow[\text{NMRK}]{} c_{\bar{q}qgg^*}^{R(1,\ g)}(p_2,p_3,p_4,q_2)\,,$$



$$c_{\bar{q}qgg^*}^{R(1,\ g)}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) = c_{\bar{q}qg^*}^{R(1,\ g)}(p_2,p_3,q_1) + \frac{z}{X}L_{s_{-1}}\left(\frac{t_1}{t_2},\frac{t_1'}{t_2}\right) + \frac{z}{2X(z-1)^2}L_1\left(\frac{t_1'}{t_2}\right) + \frac{2(z-1)-|z|^2}{X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right)$$

Unlike the previous pieces, no central physics survives in the MRK, that is,

$$c_{\bar{q}qgg^*}^{R(1,\ g)}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) \xrightarrow[\text{MRK}]{} c_{\bar{q}qg^*}^{R(1,\ g)}(p_2,p_3,q_1)\,.$$

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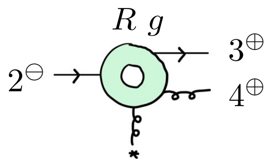

$$c_{\bar{q}qgg^*}^{R(1,\ g)}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) = c_{\bar{q}qg^*}^{R(1,\ g)}(p_2,p_3,q_1) + \frac{z}{X}L_{s_{-1}}\left(\frac{t_1}{t_2},\frac{t_1'}{t_2}\right) + \frac{z}{2X(z-1)^2}L_1\left(\frac{t_1'}{t_2}\right) + \frac{2(z-1)-|z|^2}{X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right)$$

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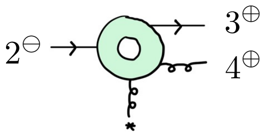
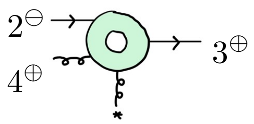
So far, we have only considered the $\{2,3,4,q_2\}$ colour ordering. The $\{4,2,3,q_2\}$ colour orderings can be obtained by discrete symmetries. They are very similar to the $\{2,3,4,q_2\}$ vertices. For example, compare



$$c_{\bar{q}qgg^*}^{L(1,\ \mathcal{N}=1_X)}(p_2^{\ominus},p_3^{\oplus},p_4^{\oplus},q_2) = c_{\bar{q}qg^*}^{L(1,\ \mathcal{N}=1_X)}(p_2,p_3,q_2)+\frac{X(1+z-\bar{z})-z(\bar{z}-2)}{2X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right)$$

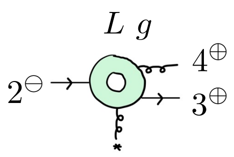


$$c_{\bar{q}qgg^*}^{L(1,\ \mathcal{N}=1_X)}(p_4^{\oplus},p_2^{\ominus},p_3^{\oplus},q_2) = c_{\bar{q}qg^*}^{L(1,\ \mathcal{N}=1_X)}(p_2,p_3,q_2)+\frac{X(1+z-\bar{z})+|z|^2}{2X|z-1|^2}L_0\left(\frac{t_1}{t_2}\right)$$

Gluon circulating in the loop III

We finally need to consider the $\{2,4,3,q_2\}$ colour ordering.

$$\mathrm{Re}\left[a_5^{L(1,\ g)}(2_{\bar{q}},4,3_q,5,1)\right] \xrightarrow[\text{NMRK}]{} c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2,p_4,p_3,q_2) + r^{(1,\ g)}_{g^*}(t_2;p_3^+p_5^-) + c^{(1,\ g)}_{ggg^*}(p_5,p_1,-q_2)\,,$$



$$c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2^{\ominus},p_4^{\oplus},p_3^{\oplus},q_2) = c^{L(1,\ g)}_{\bar{q}qg^*}(p_2,p_3,q_1) + r^{(1,\ g)}_{g^*}(s_{24};s_{43}) + v^{(1,\ \mathcal{N}=4)}\left(s_{24},\frac{s_{34}p_3^+}{(p_3^++p_4^+)},t_2\right)$$
$$-\frac{1}{2} + L_{s_{-1}}\left(\frac{t_1}{t_2},\frac{t_1'}{t_2}\right) - \frac{1}{3}\log\left(\frac{t_1}{t_2}\right) + \frac{z}{2(z-1)^2}L_1\left(\frac{t_1'}{t_2}\right) - \frac{2z}{(z-1)}L_0\left(\frac{t_1'}{t_2}\right) + \frac{2}{3}\log\left(\frac{t_1'}{t_2}\right)$$

Unlike the other colour-orderings, this does not obey a further factorisation in the MRK. For example, note the weight-2 terms are not suppressed in the MRK limit.

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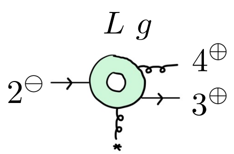

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Unlike the other colour-orderings, this does not obey a further factorisation in the MRK. For example, note the weight-2 terms are not suppressed in the MRK limit.

However, the tree-level prefactor is supressed in the MRK, which we can view as the MRK limit of a $U(1)$ photon decoupling relation

$$\begin{aligned} A_5^{(0)}(2_{\bar{q}},4,3_q,5,1) &= -A_5^{(0)}(2_{\bar{q}},3_q,4,5,1) - A_5^{(0)}(2_{\bar{q}},3_q,5,4,1) - A_5^{(0)}(4,2_{\bar{q}},3_q,5,1) \\ &\xrightarrow[\text{NMRK}]{} -A_5^{(0)}(2_{\bar{q}},3_q,4,5,1) - A_5^{(0)}(4,2_{\bar{q}},3_q,5,1) \\ &\xrightarrow[\text{MRK}]{} 0\,, \end{aligned}$$

Gluon circulating in the loop III

We finally need to consider the $\{2,4,3,q_2\}$ colour ordering.

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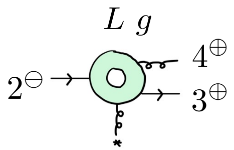

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We won't list the $R(1,g)$ vertices explicitly, but we note that through the SUSY decomposition they have no large logarithms in the NMRK.

Assembling the colour dressed amplitude

Finally, we can assemble the real part of the signature-odd, one-loop $qg \to qgg$ amplitude in the NMRK limit:

$$
\begin{aligned}
&\mathrm{Re}\left[\mathcal{A}_5^{(1)[-]}(\bar{q}_2, q_3, g_4, g_5, g_1)\right] \xrightarrow[\text{NMRK}]{} c_\Gamma\ g^5 \left(-F_{51}^d\right) C^{(0)}_{ggg^*}(p_5, p_1, -q_2) \times \frac{1}{t_2} \\
&\times \Bigg\{ \left(T^d T^{a_4}\right)_{\bar{\imath}_2 \imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) \Bigg[\Bigg(N_c\ c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - \frac{1}{N_c}\ c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) + n_f\ c^{L(1,\ q)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) \Bigg) \\
&\qquad + N_c\ r^{(1,\ g)}_{g^*}(t_2; p_4^+ p_5^-) + \left(N_c\ c^{(1,\ g)}_{ggg^*}(p_5, p_1, -q_2) + n_f\ c^{(1,\ q)}_{ggg^*}(p_5, p_1, -q_2) \right) \Bigg] \\
&+ \left(T^{a_4} T^d\right)_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \Bigg[\Bigg(N_c\ c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) - \frac{1}{N_c}\ c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) + n_f\ c^{L(1,\ q)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \Bigg) \\
&\qquad + N_c\ r^{(1,\ g)}_{g^*}(t_2; p_4^+ p_5^-) + \left(N_c\ c^{(1,\ g)}_{ggg^*}(p_5, p_1, -q_2) + n_f\ c^{(1,\ q)}_{ggg^*}(p_5, p_1, -q_2) \right) \Bigg] \\
&+ \delta_{\bar{\imath}_2 \imath_3} \Delta_{d a_4} \Bigg[C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \left(c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) - c^{L(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) + c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) - c^{R(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) \right) \\
&\qquad + C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) \left(c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - c^{L(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) + c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - c^{R(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) \right) \Bigg] \Bigg\}
\end{aligned}
$$

Colour-dressed factorisation for $q\ g \to q\ g\ g$ in NMRK

The most inclusive possibility is

$$\text{Re}\left[\mathcal{A}_4^{[-]}\left(\bar{q}_2, q_3, g_4, g_5, g_1\right)\right] \to s\ \mathcal{C}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \times \left[\frac{1}{t}\left(\left(\frac{s_{45}}{\tau}\right)^{\alpha(t)} + \left(\frac{-s_{45}}{\tau}\right)^{\alpha(t)}\right)\right] \times \mathcal{C}_{ggg^*}(p_5, p_1, -q_2)$$

where we have defined the colour-dressed, all-order object

$$\mathcal{C}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) = \mathcal{C}^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) + \mathcal{C}^{(1)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) + \mathcal{O}\left(g^5\right)$$

Colour-dressed factorisation for $q\ g \to q\ g\ g$ in NMRK

The most inclusive possibility is

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with two colour structures at tree-level

$$\mathcal{C}^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) = g\ \left[(T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) + (T^d T^{a_4})_{\bar{\imath}_2 \imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_1)\right]$$

Colour-dressed factorisation for $q\ g \to q\ g\ g$ in NMRK

The most inclusive possibility is

$$\mathrm{Re}\left[\mathcal{A}_4^{[-]}\left(\bar{q}_2, q_3, g_4, g_5, g_1\right)\right] \to s\ \mathcal{C}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \times \left[\frac{1}{t}\left(\left(\frac{s_{45}}{\tau}\right)^{\alpha(t)} + \left(\frac{-s_{45}}{\tau}\right)^{\alpha(t)}\right)\right] \times \mathcal{C}_{ggg^*}(p_5, p_1, -q_2)$$

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with two colour structures at tree-level

$$\mathcal{C}^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) = g\ \left[(T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) + (T^d T^{a_4})_{\bar{\imath}_2 \imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_1)\right]$$

and three colour structures at one-loop

$$\begin{aligned}
\mathcal{C}^{(1)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) = c_\Gamma g^3 \Big\{ & (T^{a_4}T^d)_{\bar{\imath}_2 \imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) \left(N_c\ c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_1) - \frac{1}{N_c}\ c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) + n_f\ c^{L(1,\ q)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2)\right) \\
& + (T^d T^{a_4})_{\bar{\imath}_2 \imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) \left(N_c\ c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - \frac{1}{N_c}\ c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) + n_f\ c^{L(1,\ q)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2)\right) \\
& + \delta_{\bar{\imath}_2 \imath_3}\delta_{da_4}\Big[C^{(0)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2)\left(c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) - c^{L(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) + c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2, p_3, p_4, q_2) - c^{R(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2)\right) \\
& + C^{(0)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2)\left(c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - c^{L(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2) + c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4, p_2, p_3, q_2) - c^{R(1,\ g)}_{\bar{q}gqg^*}(p_2, p_4, p_3, q_2)\right)\Big]\Big\}
\end{aligned}$$

Colour-dressed factorisation for $q\ g \to q\ g\ g$ in NMRK

The most inclusive possibility is

$$\mathrm{Re}\left[\mathcal{A}_4^{[-]}\left(\bar{q}_2, q_3, g_4, g_5, g_1\right)\right] \to s\ \mathcal{C}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) \times \left[\frac{1}{t}\left(\left(\frac{s_{45}}{\tau}\right)^{\alpha(t)} + \left(\frac{-s_{45}}{\tau}\right)^{\alpha(t)}\right)\right] \times \mathcal{C}_{ggg^*}(p_5,p_1,-q_2)$$

where we have defined the colour-dressed, all-order object

$$\mathcal{C}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) = \mathcal{C}^{(0)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) + \mathcal{C}^{(1)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) + \mathcal{O}\left(g^5\right)$$

with two colour structures at tree-level

$$\mathcal{C}^{(0)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_1) = g\ \left[(T^{a_4}T^d)_{\bar{\imath}_2\imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_1) + (T^dT^{a_4})_{\bar{\imath}_2\imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_1)\right]$$

and three colour structures at one-loop

$$\begin{aligned}\mathcal{C}^{(1)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_1) = c_\Gamma g^3 \Big\{ & (T^{a_4}T^d)_{\bar{\imath}_2\imath_3} C^{(0)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2)\left(N_c\ c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_1) - \frac{1}{N_c}\ c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) + n_f\ c^{L(1,\ q)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2)\right) \\ & + (T^dT^{a_4})_{\bar{\imath}_2\imath_3} C^{(0)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2)\left(N_c\ c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2) - \frac{1}{N_c}\ c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2) + n_f\ c^{L(1,\ q)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2)\right) \\ + \delta_{\bar{\imath}_2\imath_3}\delta_{da_4}\Big[& C^{(0)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2)\left(c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) - c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2,p_4,p_3,q_2) + c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2,p_3,p_4,q_2) - c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2,p_4,p_3,q_2)\right) \\ & + C^{(0)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2)\left(c^{L(1,\ g)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2) - c^{L(1,\ g)}_{\bar{q}qgg^*}(p_2,p_4,p_3,q_2) + c^{R(1,\ g)}_{g\bar{q}qg^*}(p_4,p_2,p_3,q_2) - c^{R(1,\ g)}_{\bar{q}qgg^*}(p_2,p_4,p_3,q_2)\right)\Big]\Big\}\end{aligned}$$

This conjecture can be tested using the two-loop amplitudes.
This vertex is unlike the pure-gluon case, which does not admit a colour-dressed factorisation [8,13]

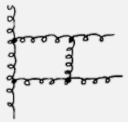
Summary

- In this talk I have given a brief overview on the status of the BFKL approach at NNLL.
- There has been much progress in recent years and the remaining building blocks are within reach.
- I have focused on the extraction of one of the remaining pieces: the one-loop qg peripheral-emission vertex:
    - Like the gg peripheral-emission vertex there is one new colour structure at one-loop. It will be interesting to investigate whether this colour structure receives large logarithmic enhancement at two-loops.
    - Unlike $gg \to ggg$ in the NMRK, $qg \to qgg$ admits a factorisation *at the colour-summed level*.
- We still need to obtain the one-loop two-parton emission vertices up to order $\mathcal{O}(\epsilon^3)$ in IR limits.

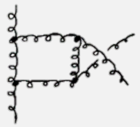
Thanks for your attention!

Backup I: Changing basis of colour structures

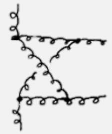
In order to connect to the tree level CEV, we move to a basis that includes the tree-level structures:



T_A

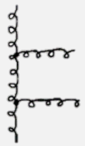


$T_{A'}$

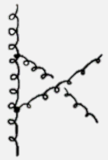


T_B

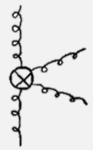
$$T_A M_A + T_{A'} M_{A'} + T_B M_B = \frac{1}{3}(T_A - T_B)(2M_A - M_{A'} - M_B) + \frac{1}{3}(T_{A'} - T_B)(2M_{A'} - M_A - M_B) + \frac{1}{3}(T_A + T_{A'} + T_B)(M_A + M_{A'} + M_B)$$

N_c 

$(T_A - T_B)$

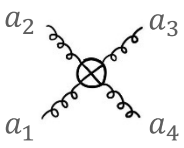
N_c 

$(T_{A'} - T_B)$

3 

$(T_A + T_{A'} + T_B)$

We find that in addition to the tree-level structures, we have a totally symmetric colour structure:



$$= d_A^{a_1 a_2 a_3 a_4} = \frac{1}{4!} \sum_{S_4} \mathrm{tr}\left(F^{a_{\sigma_1}} F^{a_{\sigma_2}} F^{a_{\sigma_3}} F^{a_{\sigma_4}}\right)$$