AdS/CFT: The holographic dictionary

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Introduction

- These lectures will explore the holographic dictionary between geometry and quantum field theory.
- Holography is studied with many goals, including describing strongly coupled quantum systems; reconstructing spacetime from quantum field theory data; exploring black holes and entanglement.
- 15000+ papers.....
- Focus of these lectures: *understanding reconstruction of spacetime*.

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Introduction

- **Holography**: gravity in $(d+1)$ dimensions is described in term of a non-gravitational quantum theory in d dimensions.
- **AdS/CFT:** defining data for asymptotically (locally) negative curvature, AdS manifolds in $(d+1)$ dimensions is that of a conformal field theory in d dimensions.

Q: What is the holographic dictionary between gravity and quantum field theory?

Q: How does AdS/CFT extend to other classes of spacetimes?

Q: When will given quantum field theory data result in a regular spacetime in $(d+1)$ dimensions?

- When are black hole horizons present?
- How are black hole microstates described in the dual field theory?
- Relations between fluid hydrodynamics and perturbed black holes.
- Use of minimal surfaces to probe reconstructed spacetime.

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Introduction

Q; Holographic relations for boundaries at finite distance?

- Holography was originally proposed to understand black hole physics; spacetime asymptotics should not be essential.
- Can one set up quasi-local holography on a holographic screen at finite distance?
- Links with quantum information and quantum error correction codes.

Outline

- **What is a holographic dictionary?**
- The AdS/CFT dictionary
- Example: scalar operators and their correlation functions

Original holographic "gauge/gravity" dualities

- Maldacena 1997: dualities between string theory in Anti-de Sitter backgrounds and supersymmetric conformal field theories.
- Best understood example: string theory in ten dimensions ($AdS_5 \times S^5$) is dual to maximally supersymmetric Yang-Mills theory (SYM) in four dimensions.
- 4d gauge theory is dual to gravity in 5 *non-compact* dimensions: many dualities also involve compact dimensions (e.g. spheres S^n) whose role we will explore later.

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Asymptotically AdS

- Immediate corollary: string theory on backgrounds containing asymptotically *AdS* factors is dual to a QFT with a non-trivial fixed point in the UV.
- String theory on curved backgrounds poorly understood: most works focus on low energy limit, in which we work with *supergravity*.
- Classical supergravity corresponds to a strong coupling, large N limit of the dual quantum field theory.
- E.g. for 4d $SU(N)$ SYM the supergravity limit is $N \to \infty$ with 't Hooft coupling $\lambda = g^2 N$ large.

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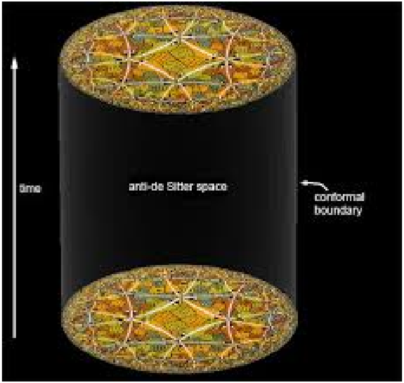
Bulk/boundary correspondence



- One can think of the gauge theory as being associated with the boundary of the spacetime (the “bulk").
- The duality is called “holographic" as the spacetime is reconstructed from information on the boundary holographic screeen.

What is a duality between physical theories?

- The term “duality" is used not just for holography, but for other relations between quantum field theories e.g. Seiberg dualities.
- Physical theories are dual if:
  1. A map between all parameters and observables of the corresponding theories exists;
  2. There is a precise computational framework for dynamical computations on both sides.
- Early critics of AdS/CFT suggested that only quantities determined by kinematics (symmetries) would match....

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Defining a quantum field theory

Usually a definition of QFT involves:

- A set of fields $\{\phi^i\}$.
- A Lagrangian $L[m, g, ...]$, where m and g are masses and couplings of the theory
- Specification of a free field point around which the theory is perturbatively quantized. Two cases are possible:
  1. Quantization around the trivial vacuum $\langle \phi^i \rangle = 0$
  2. Quantization around a condensate $\langle \phi^i \rangle \neq 0$

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Example: SU(N) Super Yang-Mills

- Vector field A^a_μ, 4 Weyl fermions $\psi^{A,a}$ ($A = 1, ..., 4$) and 6 real scalars $\phi^{i,a}$ ($i = 1, ..., 6$).
- Lagrangian is given by

$$L[g^2_{YM}, \theta] = \frac{1}{g^2_{YM}} Tr[F^2 + (D\phi)^2 + \overline{\psi^A} D\psi^A + \overline{\psi^A}\Gamma^i \phi^i \psi^A$$

$$+ [\phi^i, \phi^j]^2 + \theta F \wedge F]$$

- Scalar potential is minimized when $[\phi^i, \phi^j] = 0$.
- We can add masses for the scalars and fermions, but these break super and conformal symmetries.

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Non-Lagrangian definition of a QFT

- Define in terms of:
  1. Basis of gauge invariant operators $\{\mathcal{O}^i(x)\}$.
  2. Correlation functions of these operators, including one point functions.
  3. Moduli space of theories i.e. different vacua of the theory, characterised by one point functions.
- In a general quantum field couplings will run with the energy scale, $g(\Lambda)$, and operators will correspondingly renormalise/depend explicitly on the scale.

Example: SU(N) Super Yang-Mills

- We can define gauge invariant local operators such as

$$\mathcal{O}_F = \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \qquad \mathcal{O}_{ijk} = \mathrm{Tr}(\phi^i\phi^j\phi^k\cdots)$$

- Some operators have dimensions Δ that are protected due to susy/conformal invariance (chiral primaries).
- Generically the dimensions of operators depend on the coupling, $\Delta(\lambda, N)$, and in fact become very large in the $\lambda \to \infty$ limit.

NB Non-local operators are also important: see later.

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Dual representation

Any dual representation of a QFT should include:

1. how parameters and vacuum expectation values (vevs) are represented;
2. how gauge invariant operators $\{\mathcal{O}(x)\}$ are represented;
3. how correlation functions $\langle O_1(x_1)...O_n(x_n)\rangle$ are computed.

Gauge *dependent* quantities are not observables.

$AdS_5 \times S^5$ and N=4 SYM

- The isometry group of $AdS_5 \times S^5$ is $SO(4,2) \times SO(6)$.
- The conformal group in $3+1$ dimensions is $SO(4,2)$ and N=4 SYM also has a global symmetry group of $SO(6)$.
- In fact the full superconformal symmetries ($SU(2,2)|4$) match.
- Curvature radius and string coupling correspond to the SYM parameters (λ, N).

Kinematics versus dynamics

- Clearly on both sides states will be classified in representations of $SU(2,2|4)$: this is *kinematics*.
- Non-trivial matching requires:
  1. The same representations arise on both sides of the correspondence;
  2. Matching of dynamical quantities such as n-point functions with $n \geq 3$.
- Easiest to show for supersymmetric states...

How does additional spacetime direction arise on the gravity side?

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Outline

- What is a holographic dictionary?
- **The AdS/CFT dictionary**
- Example: scalar operators and their correlation functions

Dictionary between $\text{AdS}_{d+1}/\text{CFT}_d$

Relation between gravity in AdS_{d+1} spacetimes and CFTs in d dimensions (Gubser, Klebanov and Polyakov; Witten).

- For every gauge invariant operator $\mathcal{O}$ of QFT there exists a corresponding bulk field Φ.
- Let $\varphi_{(0)}$ parametrize the boundary condition of Φ at infinity: $\varphi_{(0)}$ is the source of the dual field theory operator $\mathcal{O}$.
- The bulk partition function as a function of boundary condition $\varphi_{(0)}$ is identified with the generating functional of QFT correlation functions.

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Dictionary between $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$

- Working in the supergravity limit, the bulk partition function is expressed in terms of the onshell gravity action, so

$$S^{on-shell}_{SUGRA}[\varphi_{(0)}] = -W_{QFT}[\varphi_{(0)}].$$

- The left hand side is the gravity action evaluated on a solution of the field equations with boundary conditions $\varphi_{(0)}$.
- The right hand side is the QFT partition function, in the planar ($N \to \infty$) strong 't Hooft coupling limit.

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Renormalisation and volume divergences

- In any QFT calculation, one needs to implement renormalisation for the UV (high energy) divergences.
- This is still the case in a conformal field theory, even though it is "scale invariant"!
- The left hand side of the holographic relation also has divergences, associated with the infinite volume of spacetime.
- These volume divergences exactly match the QFT divergences, structurally, and are managed through holographic renormalization.

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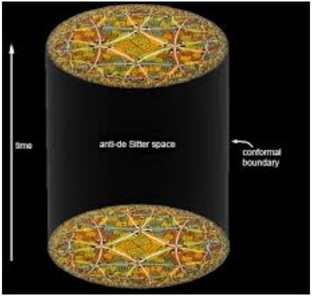
Volume divergences of anti-de Sitter

- Partition functions are typically worked out in Euclidean signature, and Euclidean AdS can be written as

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + dx^i dx_i + d\tau^2\right)$$

  where τ is Euclidean time; x^i are spatial coordinates and $z > 0$ is the radial coordinate.
- Metric describes a negative curvature manifold with curvature radius l, with $z \to 0$ at the (conformal) boundary.

Bulk/boundary correspondence



- Consider AdS_4 for definiteness: volume behaves as

$$l^4 V_x R_\tau \int_\infty^\epsilon \frac{dz}{z^4} \sim \frac{l^4 V_x R_\tau}{\epsilon^3}$$

where we regulate $z \geq \epsilon$.

- This cutoff is dual to the standard UV QFT cutoff.

Outline

- What is a holographic dictionary?
- The AdS/CFT dictionary
- **Example: scalar operators and their correlation functions**

Scalar fields in AdS

- Let us apply this relation

$$S^{on-shell}_{SUGRA}[\varphi_{(0)}] = -W_{QFT}[\varphi_{(0)}].$$

  to the case in which we have a bulk scalar field, with boundary condition $\varphi_{(0)}$, which is dual to a scalar operator.
- We will temporarily switch off dynamical gravity and put the scalar in a fixed Anti-de Sitter background.

- Consider a massive scalar Φ in AdS. Its dynamics is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} m^2 \Phi^2 + \mathcal{O}(\Phi^3).$$

- The field equation is to linear order in Φ:

$$\Box \Phi = m^2 \Phi.$$

(We will consider higher order terms later.)

Scalar fields in AdS

- Using the AdS metric shown previously we can write the scalar equation as (*exercise*)

$$z^2 \partial_z^2 \Phi + (2 - D) z \partial_z \Phi = (m^2 + k^2 z^2) \Phi$$

  where $D = (d+1)$ is the bulk dimension and d is the boundary dimension.
- Here we Fourier transform the field along the (τ, x) directions, with k being the Euclidean d momentum.
- This is a second order equation, with two separate boundary conditions: the behaviour as $z \to 0$ and the behaviour as $z \to \infty$.

Asymptotic solutions as $z \to 0$

- Analysing the equation near $z \to 0$, we find that there are two independent solutions

$$\Phi = z^{d-\Delta}\left(\varphi_{(0)}(k) + \mathcal{O}(z^2)\right) + z^{\Delta}\left(\tilde{\varphi}_{(0)}(k) + \mathcal{O}(z^2)\right)$$

where

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + 4m^2}$$

- Here Δ will turn out to be the dimension of the dual scalar operator.
- For $\Delta > d/2$ the first solution dominates as $z \to 0$.

Exercise: What happens if $\Delta = d/2 + n$ with n an integer?

Exact solutions

- We can also solve the linear equation exactly. For definiteness let us choose $D = 5$ and $m = 0$, and

$$\Phi(k,z) = c_1(k) z^2 I_2(kz) + c_2(k) z^2 K_2(kz)$$

  where I_2 and K_2 are Bessel functions of order two.
- If we impose regularity of the field as $z \to \infty$, then we fix $c_1(k) = 0$.
- The asymptotic expansion of the remaining term as $z \to 0$ behaves as

$$\Phi(k,z) \sim \frac{2}{k^2} c_2(k) + \mathcal{O}(z^2)$$

Dual interpretation: onshell action

- The Euclidean action is

$$S = \frac{1}{2} \int d^D x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2 \right)$$

- If Φ satisfies the field equation, then the corresponding onshell action reduces to the boundary integral

$$S_{\text{onshell}} = \frac{1}{2} \int_{z \to 0} d\Sigma^\mu \Phi \partial_\mu \Phi$$

where the integral is taken a surface of constant z with $z \to 0$.

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Dual interpretation: onshell action

- Regulating at $z = \epsilon \ll 1$ and substituting the general asymptotic solution we find the following structure

$$S_{\text{onshell}} \sim \int d^d x \left(\epsilon^{d-2\Delta} \varphi_{(0)}^2 + \mathcal{O}(\epsilon^{d+2-2\Delta}) + \cdots + \varphi_{(0)} \tilde{\varphi}_{(0)} + \cdots \right)$$

  where the integral is taken a surface of constant z with $z \to 0$.
- Terms are arranged as divergent, finite and vanishing as $\epsilon \to 0$.

Dual interpretation: onshell action

- Consider first the finite piece

$$S_{\text{onshell}}(\varphi_{(0)}) \sim \int d^d x \left(\varphi_{(0)}\tilde{\varphi}_{(0)}\right)$$

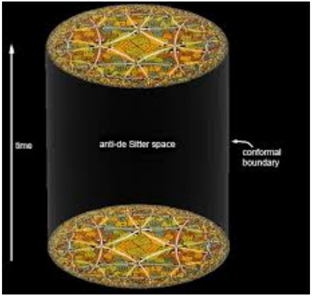
  According to the AdS/CFT relation, we interpret this as the generating functional for correlation functions for the operator $\mathcal{O}$ dual to the boundary condition (source) $\varphi_{(0)}$.

- Functionally differentiating we get

$$\langle \mathcal{O} \rangle \sim \tilde{\varphi}_{(0)}$$

  and the scaling dimension of $\mathcal{O}$ is manifestly Δ.

Bulk/boundary correspondence



- The leading term corresponds to the source; the subleading term to the operator expectation value:

$$\Phi = z^{d-\Delta}\varphi_{(0)}(x) + \cdots$$

$$+z^{\Delta}\tilde{\varphi}_{(0)}(x) + \cdots$$

- Non-normalizable (source); normalizable (expectation value).

QFT renormalization

What about the divergent terms...?

- In quantum field theory one adds to the regulated bare action

$$S^{\rm reg}_{\rm QFT}(\phi)$$

counterterms, which are expressed locally and covariantly in terms of the QFT fields i.e.

$$\int d^d x \left(c_1 (\phi)^2 + c_2 (\partial \phi)^2 + \cdots \right)$$

- Coefficients are fixed to remove the divergences.
- Cannot use non-covariant or non-local terms e.g. $(\phi \partial_\mu \phi)$; $(\phi \Box^{-1} \phi)$.

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Holographic renormalization

- The analogue is that we need to add to the regulated action

$$S^{\text{reg}}_{\text{onshell}}(\varphi_{(0)})$$

counterterms that are expressed locally and covariantly in terms of the boundary value of the field i.e.

$$S_{\text{ct}} = \int d^d x \left(c_1 (\Phi)^2 + c_2 (\partial \Phi)^2 + \cdots \right)$$

where this is evaluated at the regulated d-dimensional boundary.
- Coefficients are fixed to remove the divergences.

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Holographic renormalization

- The renormalized onshell action is by construction finite

$$S^{\rm ren} = \mathcal{L}_{\epsilon \rightarrow 0} \left(S^{\rm reg}_{\rm onshell} + S_{\rm ct} \right)$$

  and we identify this as the renormalized QFT functional.
- The renormalized one point function then generically takes the form

$$\langle \mathcal{O} \rangle = (d - 2\Delta) \tilde{\varphi}_{(0)} + \mathcal{F}(\varphi_{(0)})$$

  where $\mathcal{F}$ is a local polynomial.
- $\mathcal{F}$ is always non-zero when $\Delta = d/2 + n$
  e.g. integer dimension operators in $d = 2, 4, \cdots$.

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Holographic renormalization: comments

- Systematic matching of the analytic structure of the gravity counterterms to those of a CFT provides structural evidence of the duality, beyond supersymmetric examples.
- One obtains scheme dependence (finite counterterms) exactly where expected in a CFT.
- Logarithmic divergences (related to scale anomalies) also arise exactly where expected, see later.

Holographic duality

- One can view the renormalized one point function in the presence of sources i.e.

$$\langle \mathcal{O} \rangle = (d - 2\Delta)\tilde{\varphi}_{(0)}(\varphi_{(0)}) + \mathcal{F}(\varphi_{(0)})$$

as the defining duality relation.

- Higher correlation functions are as usual obtained by functional differentiation wrt the source e.g.

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = -\frac{\delta \mathcal{O}(x)}{\delta \varphi_{(0)}(y)}$$

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Back to the massless field

- In $D = (d+1)$ the regular solution to the linear equation is

$$\Phi(k,z) = c(k) z^{\frac{d}{2}} K_{\frac{d}{2}}(kz)$$

- Consider $d = 3$ and expand near $z = 0$

$$\Phi(k,z) = \varphi_{(0)}(k) \left(1 - \frac{1}{2} k^2 z^2 + \frac{1}{3} k^3 z^3 + \cdots \right)$$

and hence $\tilde{\varphi}_{(0)}(k) = \frac{1}{3} k^3 \varphi_{(0)}(k)$.

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Back to the massless field (ctd)

- In $d = 3$, $m = 0$ implies $\Delta = 3$ and using the holographic relation (with $\mathcal{F} = 0$) we obtain

$$\langle \mathcal{O}_3(k)\mathcal{O}_3(-k)\rangle = k^3$$

This is the two point function in momentum space.

- In position space we obtain

$$\langle \mathcal{O}_3(x)\mathcal{O}_3(0)\rangle \sim \frac{1}{x^6} \equiv \frac{1}{x^{2\Delta}},$$

the expected behaviour of a CFT two point function.

- Note that this is well defined as a distribution as $x \to 0$ in $d = 3$.

Massless field in even dimensions

- In $d = 4$, we need to be a little more careful: the asymptotic expansion has logarithmic terms; $\mathcal{F} \neq 0$ and finite counterterms are possible.
- In this case the two point function can be expressed as

$$\langle \mathcal{O}_4(k) \mathcal{O}_4(-k) \rangle = -\frac{1}{8} k^4 \log \frac{k^2}{\mu^2}$$

where μ is a reference momentum scale.

Massless field in even dimensions

- Fourier transforming back to position space

$$\langle \mathcal{O}_4(x)\mathcal{O}_4(0)\rangle \propto \Box^3 \left(\frac{1}{x^2} \log(m^2 x^2) \right)$$

which is known to be a renormalized version of $1/x^8$ i.e. well defined as $x \to 0$:

$$\langle \mathcal{O}_4(x)\mathcal{O}_4(0)\rangle \propto \mathcal{R} \left(\frac{1}{x^8} \right)$$

- In general when $\Delta = d/2 + 1$ we will obtain similar results i.e. logarithms in the asymptotic expansion, and renormalized versions of $1/x^{2\Delta}$.

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Higher point functions

- Higher point functions follow from further functional differentiation of:

$$\langle \mathcal{O} \rangle = (d - 2\Delta)\tilde{\varphi}_{(0)}(\varphi_{(0)}) + \mathcal{F}(\varphi_{(0)})$$

- For example

$$\langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z) \rangle = \frac{\delta^2 \mathcal{O}(x)}{\delta\varphi_{(0)}(y)\delta\varphi_{(0)}(z)}_{|\varphi_{(0)}=0}$$

- Three point functions require solving the scalar equation to quadratic order and so on.

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Three point function

- For the quadratic equation

$$(\Box - m^2)\Phi = g\Phi^2$$

we look for a solution of the form

$$\Phi(z, x) = \Phi_1(z, x) + g\Phi_2(z, x).$$

with the latter considered a small correction.

- The overall amplitude of the field is small, with interactions controlled by $\varphi_{(0)}$.
- Φ_1 satisfies the linear eqation solved previously.

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Three point function

- The linear solution can be expressed in Green function form as

$$\Phi_1(z,\vec{x}) = \int d^d y \; K_\Delta(z,\vec{x}-\vec{y})\varphi_{(0)}(\vec{y}).$$

- $K_\Delta(z,\vec{x}-\vec{y})$ is the bulk-boundary propagator, passing boundary source $\varphi_{(0)}(\vec{y})$ to bulk solution $\Phi_1(z,\vec{x})$.
- The propagator is

$$K_\Delta(z,\vec{x}-\vec{y}) = c_\Delta \left(\frac{z}{z^2 + (\vec{x}-\vec{y})^2} \right)^\Delta .$$

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Three point function

- For the next order

$$(\Box - m^2)\Phi_2 = \Phi_1^2$$

which can be solved as

$$\Phi_2(z) = \int d^{d+1}x \sqrt{g} G_\Delta(z, y)\Phi_1^2(y).$$

$G_\Delta(z, y)$ is the bulk to bulk propagator

$$G_\Delta(x, x') \propto \zeta^\Delta F(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}, \Delta + 1 - \frac{d}{2}, \zeta^2),$$

where $\zeta = \frac{2zz'}{z^2 + z'^2 + (\vec{x} - \vec{x'})^2}$.

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Three point function

- Using these expressions we can show that

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle = g \int d^{d+1}x\sqrt{g}\prod_{k=1}^{3} K_\Delta(z,\vec{x}_k,\vec{x})$$

- This can be computed using conformal symmetry.
- Setting $\vec{x}_3 = 0$ the result can be written in the expected 3-point function form:

$$\frac{1}{|\vec{x}_1'|^{2\Delta_1}\,|\vec{x}_2'|^{2\Delta_2}}\,\frac{1}{|\vec{x}_1' - \vec{x}_2'|^{\Delta_1+\Delta_2-\Delta_3}},$$

where $\vec{x}_i'$ is related to $\vec{x}$ by an inversion transformation.

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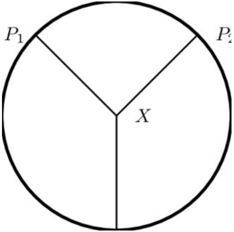
Further Witten diagrams

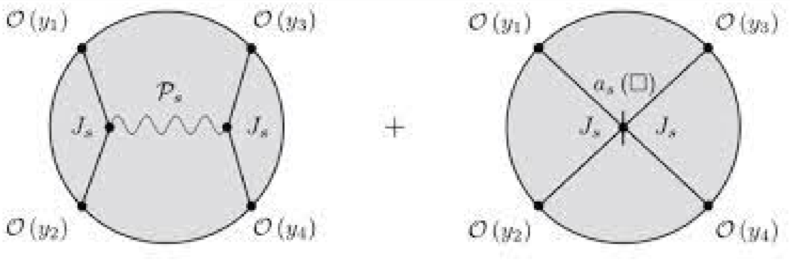


- Witten diagram to visualise:

$$\int d^{d+1}x\sqrt{g}\prod_{k=1}^{3}K_{\Delta}(z,\vec{x}_k,\vec{x})$$

- Bulk boundary progagator is shown as line from boundary into bulk point.
- Three propagators meet at single point.

Witten diagram



- Four point contributions: contact interactions, plus bulk to bulk exchange.

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