Supersymmetry and Higgs Physics

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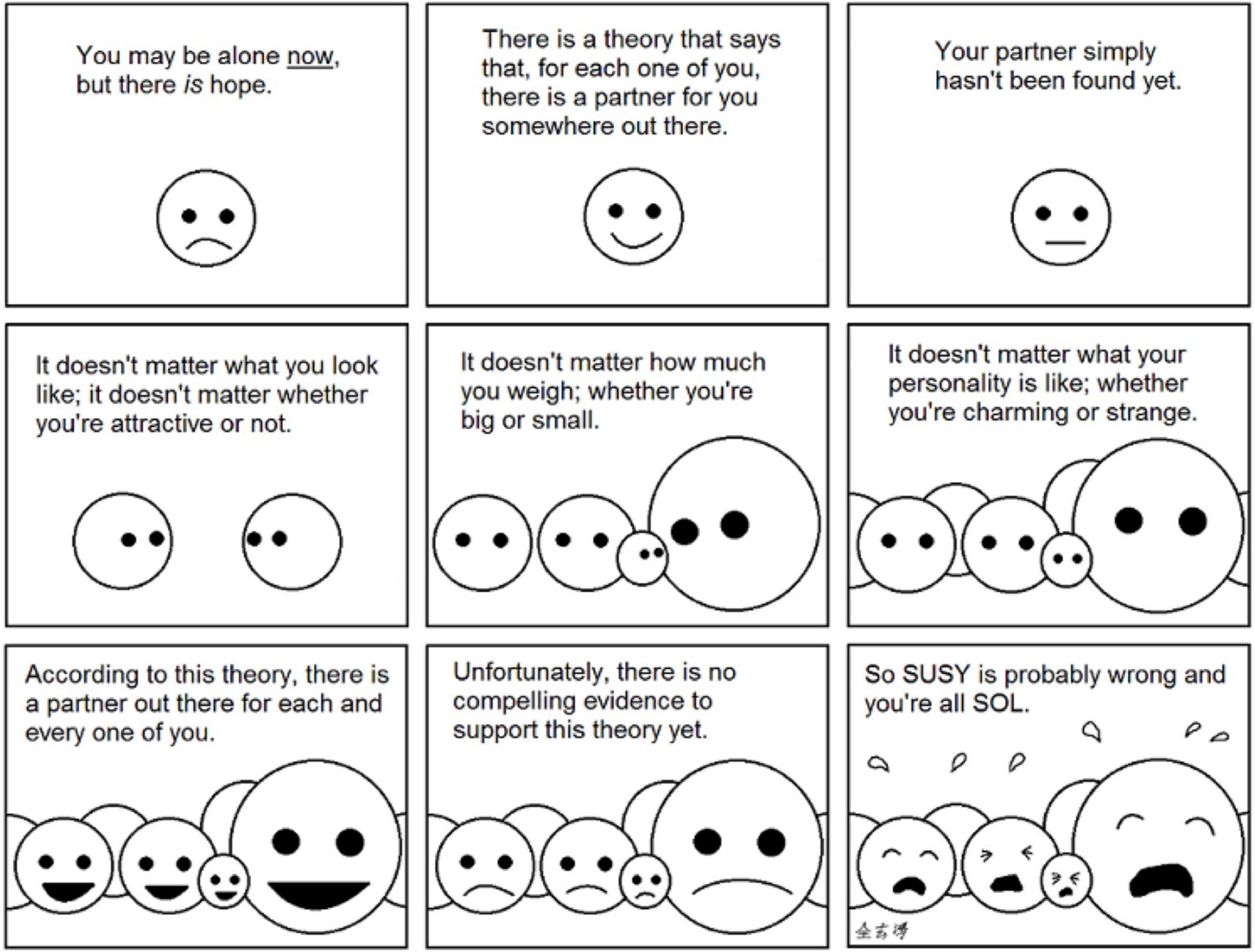
Summarizing the previous lectures on Supersymmetry


You may be alone now, but there is hope.
There is a theory that says that, for each one of you, there is a partner for you somewhere out there.
Your partner simply hasn't been found yet.
It doesn't matter what you look like; it doesn't matter whether you're attractive or not.
It doesn't matter how much you weigh; whether you're big or small.
It doesn't matter what your personality is like; whether you're charming or strange.
According to this theory, there is a partner out there for each and every one of you.
Unfortunately, there is no compelling evidence to support this theory yet.
So SUSY is probably wrong and you're all SOL.

Source material for these lectures

These lectures (with references to the original literature) are based on material found in: Herbi K. Dreiner, Howard E. Haber and Stephen P. Martin, *From Spinors to Supersymmetry* (Cambridge University Press, 2023).

See Sections 6.2, 13.8, 19.8 and 19.9

Photo taken on July 1, 2023 at Maroon Lake [elevation: 2920 m], located 10 miles from the Aspen Center for Physics, in Colorado USA

OUTLINE OF THE LECTURES

1. The Two-Higgs Doublet Model (2HDM)
   - Theoretical Structure of the 2HDM
   - A choice of scalar field basis
   - The scalar mass eigenstate fields
   - The Higgs alignment limit
   - Higgs-fermion Yukawa interactions
   - Eliminating the tree-level Higgs-mediated FCNCs
   - The CP-conserving 2HDM

2. The MSSM Higgs Sector
   - The tree-level MSSM Higgs sector
   - The one-loop corrected MSSM Higgs masses
   - The MSSM Higgs mass in the decoupling limit
   - The MSSM wrong-Higgs couplings

The Two Higgs Doublet Model (2HDM)

Theoretical structure of the 2HDM

The 2HDM consist of two identical complex hypercharge-one,[1] SU(2)$_L$ doublet scalar fields $\Phi_i(x) \equiv (\Phi_i^+(x)\,,\,\Phi_i^0(x))$, where the "Higgs flavor" index $i \in \{1,2\}$ labels the two Higgs doublet fields. The Higgs Lagrangian is given by,

$$\mathscr{L} = \mathscr{L}_{\rm KE} + \mathcal{V}\,.$$

Explicitly, $\mathscr{L}_{\rm KE} = |D_\mu \Phi|^2$, with

$$D_\mu \Phi_i = \begin{pmatrix} \partial_\mu \Phi_i^+ + \left[\dfrac{ig}{c_W}\left(\frac{1}{2} - s_W^2\right) Z_\mu + ieA_\mu\right]\Phi_i^+ + \dfrac{ig}{\sqrt{2}} W_\mu^+ \Phi_i^0 \\ \partial_\mu \Phi_i^0 - \dfrac{ig}{2c_W} Z_\mu \Phi_i^0 + \dfrac{ig}{\sqrt{2}} W_\mu^- \Phi_i^+ \end{pmatrix},$$

and $s_W \equiv \sin\theta_W$ and $c_W \equiv \cos\theta_W$.

[1] The U(1)$_Y$ hypercharge is normalized such that the electric charge is given by $Q = T_3 + Y/2$.

The scalar potential is,

$$\mathcal{V} = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - [m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.}]$$
$$+\tfrac{1}{2}\lambda_1(\Phi_1^\dagger\Phi_1)^2 + \tfrac{1}{2}\lambda_2(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)$$
$$+\left\{\tfrac{1}{2}\lambda_5(\Phi_1^\dagger\Phi_2)^2 + \big[\lambda_6(\Phi_1^\dagger\Phi_1) + \lambda_7(\Phi_2^\dagger\Phi_2)\big]\Phi_1^\dagger\Phi_2 + \text{h.c.}\right\},$$

where m_{11}^2, m_{22}^2, and $\lambda_1, \cdots, \lambda_4$ are real and m_{12}^2, λ_5, λ_6 and λ_7 are potentially complex.

After minimizing the scalar potential,[2]

$$\langle\Phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |v_1| \end{pmatrix}, \qquad \langle\Phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ |v_2|\, e^{i\xi} \end{pmatrix},$$

where $0 \leq |\xi| < 2\pi$. In particular, $v^2 \equiv |v_1|^2 + |v_2|^2 = (246\ \text{GeV})^2$ and $\tan\beta \equiv |v_2|/|v_1|$.

[2]Without loss of generality, we have performed a $\mathrm{U}(1)_Y$ transformation to remove the phase of $v_1 = \langle\Phi_1^0\rangle$.

A choice of scalar field basis

In a general 2HDM, the parameters appearing in $\mathcal{V}$ are not physical since they depend on a particular *basis choice* of the two scalar fields (denoted as the Φ-basis).

The most general redefinition of the scalar fields that leaves $\mathscr{L}_{\rm KE}$ invariant corresponds to a global U(2) transformation,[3]

$$\Phi_i \to U_i{}^j \Phi_j \,,$$

for $i,j \in \{1,2\}$, where the 2×2 unitary matrix U satisfies $U_i{}^j (U^\dagger)_j{}^k = \delta_i^k$. The indices i and j run over the Higgs flavor indices and take on two values in the 2HDM.

[3]Note that $\mathscr{L}$ is invariant under the hypercharge $\mathrm{U}(1)_Y$ group, which is a subgroup of U(2).

It is convenient to introduce a notation for the Higgs flavor indices such that

$$(U^\dagger)_j{}^k = (U_k{}^j)^* = U^k{}_j \,.$$

In this notation, we can write $U_i{}^j U^k{}_j = \delta_i^k$. Complex conjugation has the effect of raising a lowered flavor index and lowering a raised flavor index.

We shall also define a complex vector, $\widehat{v} = (\widehat{v}_1, \widehat{v}_2)$, of unit norm such that

$$\langle \Phi_i \rangle = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ \widehat{v}_i \end{pmatrix} \,, \qquad v \simeq 246~\text{GeV}, \quad \text{for } i = 1, 2,$$

in the Φ-basis.

The complex conjugate of $\widehat{v}_i$ will be denoted with a raised index, $\widehat{v}^{\,i} \equiv (\widehat{v}_i)^*$. A second unit vector $\widehat{w}$ can be defined that is orthogonal to $\widehat{v}$:[4]

$$\widehat{w}_j \equiv \widehat{v}^{\,i}\epsilon_{ij}\,,$$

where $\epsilon_{12} = -\epsilon_{21} = +1$ and $\epsilon_{11} = \epsilon_{22} = 0$. The complex conjugate of $\widehat{w}_i$ will be denoted with a raised index, $\widehat{w}^{\,i} \equiv (\widehat{w}_i)^*$. Under a unitary basis transformation $\Phi_i \to U_i{}^j\Phi_j$, the unit vectors $\widehat{v}$ and $\widehat{w}$ transform as

$$\widehat{v}_i \to U_i{}^j\,\widehat{v}_j, \quad \text{which implies that} \quad \widehat{w}_i \to (\det U)^{-1}U_i{}^j\,\widehat{w}_j.$$

$\boxed{\text{Physical quantities must be basis-independent.}}$

[4]Note that $\widehat{v}$ and $\widehat{w}$ are orthogonal due to the vanishing of the complex dot product, $\widehat{v}^{\,j}\widehat{w}_j = \widehat{v}^{\,j}\widehat{v}^{\,i}\epsilon_{ij} = 0$.

The Higgs basis

Starting from a generic Φ-basis, the Higgs basis fields $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by the linear combinations of Φ_1 and Φ_2 such that $\langle\mathcal{H}_1^0\rangle = v/\sqrt{2}$ and $\langle\mathcal{H}_2^0\rangle = 0$. That is,

$$\mathcal{H}_1 = \begin{pmatrix} \mathcal{H}_1^+ \\ \mathcal{H}_1^0 \end{pmatrix} \equiv c_\beta \Phi_1 + s_\beta e^{-i\xi} \Phi_2 \,,$$

$$\mathcal{H}_2 = \begin{pmatrix} \mathcal{H}_2^+ \\ \mathcal{H}_2^0 \end{pmatrix} = e^{i\eta}\bigl(-s_\beta e^{i\xi}\Phi_1 + c_\beta \Phi_2\bigr) \,,$$

where $c_\beta \equiv \cos\beta$, $s_\beta \equiv \sin\beta$, and the complex phase factor $e^{i\eta}$ accounts for the nonuniqueness of the Higgs basis.

In particular, $e^{i\eta}$ is a pseudoinvariant quantity that is rephased under a unitary basis transformation, $\Phi_i \to U_i{}^j\Phi_j$, as

$$e^{i\eta} \to (\det\ U)^{-1} e^{i\eta}\,,$$

where $\det U \equiv e^{i\phi}$ (such that $\phi \in \mathbb{R}$) is a complex number of unit modulus.

Note that the Higgs basis fields are *invariant* fields,

$$\mathcal{H}_1 \equiv \widehat{v}^{\,i}\Phi_i\,, \qquad\qquad \mathcal{H}_2 \equiv e^{i\eta}\widehat{w}^{\,i}\Phi_i\,.$$

It follows that

$$\Phi_i = \mathcal{H}_1\widehat{v}_i + e^{-i\eta}\mathcal{H}_2\widehat{w}_i\,, \qquad \text{for } i = 1, 2.$$

In the Higgs basis, $\widehat{v} = (1\,,\,0)$ and $\widehat{w} = (0\,,\,1)$, and the scalar potential is given by

$$
\begin{aligned}
\mathcal{V} &= Y_1 \mathcal{H}_1^\dagger \mathcal{H}_1 + Y_2 \mathcal{H}_2^\dagger \mathcal{H}_2 + [Y_3 e^{-i\eta} \mathcal{H}_1^\dagger \mathcal{H}_2 + \text{h.c.}] \\
&\quad + \tfrac{1}{2} Z_1 (\mathcal{H}_1^\dagger \mathcal{H}_1)^2 + \tfrac{1}{2} Z_2 (\mathcal{H}_2^\dagger \mathcal{H}_2)^2 + Z_3 (\mathcal{H}_1^\dagger \mathcal{H}_1)(\mathcal{H}_2^\dagger \mathcal{H}_2) + Z_4 (\mathcal{H}_1^\dagger \mathcal{H}_2)(\mathcal{H}_2^\dagger \mathcal{H}_1) \\
&\quad + \left\{ \tfrac{1}{2} Z_5 e^{-2i\eta} (\mathcal{H}_1^\dagger \mathcal{H}_2)^2 + \big[Z_6 e^{-i\eta} \mathcal{H}_1^\dagger \mathcal{H}_1 + Z_7 e^{-i\eta} \mathcal{H}_2^\dagger \mathcal{H}_2 \big] \mathcal{H}_1^\dagger \mathcal{H}_2 + \text{h.c.} \right\} .
\end{aligned}
$$

where Y_1, Y_2, and $Z_1, \ldots, Z_4$ are real parameters whereas Y_3, Z_5, Z_6, and Z_7 are potentially complex parameters.

The minimization of the scalar potential in the Higgs basis yields

$$
Y_1 = -\tfrac{1}{2} Z_1 v^2\,, \qquad\qquad Y_3 = -\tfrac{1}{2} Z_6 v^2\,.
$$

To understand the significance of Higgs basis parameters, we rewrite the scalar potential in the Φ-basis as follows

$$\mathcal{V} = Y_i^j(\Phi^i\Phi_j) + \tfrac{1}{2}Z_{ij}^{k\ell}(\Phi^i\Phi_k)(\Phi^j\Phi_\ell)\,,$$

where i, j, k, $\ell \in \{1,2\}$ are Higgs flavor indices and the SU(2)$_L$ indices of the scalar doublet fields have been suppressed.[5] Above, we have denoted the conjugated field by $\Phi^i \equiv (\Phi_i)^\dagger$.

It is also convenient to define:

$$V_j^i \equiv \widehat{v}^i\widehat{v}_j\,, \qquad W_j^i \equiv \widehat{w}^i\widehat{w}_j = \delta_j^i - V_j^i\,, \qquad \overline{Z}_{ij}^{k\ell} \equiv Z_{ji}^{k\ell} = Z_{ij}^{\ell k}\,.$$

The elements of Y_i^j, V_i^j, W_i^j, $Z_{ij}^{k\ell}$ and $\overline{Z}_{ij}^{k\ell}$ can be assembled into three 2×2 hermitian matrix and two 4×4 hermitian matrices.

[5] Note that $\Phi^i\Phi_j \equiv \Phi_i^-\Phi_j^+ + \Phi_i^{0\dagger}\Phi_j^0$.

$$Y = \begin{pmatrix} Y_1^{\ 1} & Y_1^{\ 2} \\ Y_2^{\ 1} & Y_2^{\ 2} \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -m_{12}^2 \\ -(m_{12}^2)^* & m_{22}^2 \end{pmatrix},$$

$$Z = \begin{pmatrix} Z_{11}^{11} & Z_{11}^{12} & Z_{11}^{21} & Z_{11}^{22} \\ Z_{12}^{11} & Z_{12}^{12} & Z_{12}^{21} & Z_{12}^{22} \\ Z_{21}^{11} & Z_{21}^{12} & Z_{21}^{21} & Z_{21}^{22} \\ Z_{22}^{11} & Z_{22}^{12} & Z_{22}^{21} & Z_{22}^{22} \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_6 & \lambda_6 & \lambda_5 \\ \lambda_6^* & \lambda_3 & \lambda_4 & \lambda_7 \\ \lambda_6^* & \lambda_4 & \lambda_3 & \lambda_7 \\ \lambda_5^* & \lambda_7^* & \lambda_7^* & \lambda_2 \end{pmatrix}.$$

and $\overline{Z} = Z(\lambda_3 \leftrightarrow \lambda_4)$, Under a change of scalar field basis, $\Phi_i \to U_i^{\ j}\Phi_j$, the matrices Y (V, W) and Z $(\overline{Z})$ transform as

$$Y \to UYU^\dagger, \qquad\qquad Z \to (U \otimes U)Z(U \otimes U)^\dagger,$$

where the Kronecker product of the 2×2 matrix A and the matrix B is given by:

$$A \otimes B = \begin{pmatrix} A_1^{\ 1}B & A_1^{\ 2}B \\ A_2^{\ 1}B & A_2^{\ 2}B \end{pmatrix}.$$

We can now identify the real coefficients of the scalar potential in the Higgs basis in terms of manifestly basis-invariant quantities:

$$Y_1 = Y_i^j \widehat{v}^i \widehat{v}_j = \mathrm{Tr}(YV)\,, \qquad\qquad Y_2 = Y_i^j \widehat{w}^i \widehat{w}_j = \mathrm{Tr}(YW)\,,$$

$$Z_1 = Z_{ij}^{k\ell} V_k{}^i V_\ell{}^j = \mathrm{Tr}\big[Z(V \otimes V)\big] = \mathrm{Tr}\big[\overline{Z}(V \otimes V)\big]\,,$$

$$Z_2 = Z_{ij}^{k\ell} W_k{}^i W_\ell{}^j = \mathrm{Tr}\big[Z(W \otimes W)\big] = \mathrm{Tr}\big[\overline{Z}(W \otimes W)\big]\,,$$

$$Z_3 = Z_{ij}^{k\ell} V_k{}^i W_\ell{}^j = \mathrm{Tr}\big[Z(V \otimes W)\big] = \mathrm{Tr}\big[Z(W \otimes V)\big]\,,$$

$$Z_4 = Z_{ij}^{k\ell} V_k{}^j W_\ell{}^i = \mathrm{Tr}\big[\overline{Z}(V \otimes W)\big] = \mathrm{Tr}\big[\overline{Z}(W \otimes V)\big]\,.$$

The complex coefficients of the scalar potential in the Higgs basis are not basis-invariant quantities. Instead, they are pseudoinvariant quantities that change by a multiplicative phase factor under a basis transformation.

Defining $X^i_j \equiv \widehat{v}^i \widehat{w}_j$, which are elements of the matrix X that transforms as $X \to (\det U)^{-1} X$. We can then identify

$$
\begin{aligned}
Y_3 &= Y_i^{\,j} \widehat{v}^i \widehat{w}_j = \mathrm{Tr}(Y X)\,,\\
Z_5 &= Z_{ij}^{k\ell} X_k{}^i X_\ell{}^j = \mathrm{Tr}\big[Z(X \otimes X)\big] = \mathrm{Tr}\big[\overline{Z}(X \otimes X)\big]\,,\\
Z_6 &= Z_{ij}^{k\ell} V_k{}^i X_\ell{}^j = \mathrm{Tr}\big[Z(X \otimes V)\big] = \mathrm{Tr}\big[Z(V \otimes X)\big]\,,\\
Z_7 &= Z_{ij}^{k\ell} X_k{}^i W_\ell{}^j = \mathrm{Tr}\big[Z(X \otimes W)\big] = \mathrm{Tr}\big[Z(W \otimes X)\big]\,.
\end{aligned}
$$

Thus, Y_3, Z_5, Z_6, and Z_7 are complex pseudoinvariant quantities that are rephased under a basis transformation $\Phi_i \to U_i{}^j \Phi_j$ as

$$
[Y_3, Z_6, Z_7] \to (\det\ U)^{-1}[Y_3, Z_6, Z_7] \quad \text{and} \quad Z_5 \to (\det\ U)^{-2} Z_5\,.
$$

The scalar mass eigenstate fields

We parameterize the invariant fields $\mathcal{H}_1$ and $\mathcal{H}_2$ as follows,

$$\mathcal{H}_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + \varphi_1^0 + iG^0\right) \end{pmatrix}, \qquad \mathcal{H}_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(\varphi_2^0 + ia^0\right) \end{pmatrix},$$

where G^+ and its hermitian conjugate G^- are the charged Goldstone bosons and G^0 is the neutral Goldstone boson.

The three remaining neutral fields mix, and the resulting neutral Higgs squared-mass matrix in the φ_1^0–φ_2^0–a^0 basis is:

$$\mathcal{M}^2 = v^2 \begin{pmatrix} Z_1 & \mathrm{Re}(Z_6 e^{-i\eta}) & -\mathrm{Im}(Z_6 e^{-i\eta}) \\ \mathrm{Re}(Z_6 e^{-i\eta}) & \frac{1}{2}[Z_{34} + \mathrm{Re}(Z_5 e^{-2i\eta})] + Y_2/v^2 & -\frac{1}{2}\mathrm{Im}(Z_5 e^{-2i\eta}) \\ -\mathrm{Im}(Z_6 e^{-i\eta}) & -\frac{1}{2}\mathrm{Im}(Z_5 e^{-2i\eta}) & \frac{1}{2}[Z_{34} - \mathrm{Re}(Z_5 e^{-2i\eta})] + Y_2/v^2 \end{pmatrix},$$

where $Z_{34} \equiv Z_3 + Z_4$.

The squared-mass matrix $\mathcal{M}^2$ is real symmetric; hence it can be diagonalized by a special real orthogonal transformation,

$$R\mathcal{M}^2 R^{\mathsf{T}} = \mathcal{M}_D^2 \equiv \mathrm{diag}\,(m_1^2\,,\,m_2^2\,,\,m_3^2)\,,$$

where m_i^2 are the eigenvalues of $\mathcal{M}^2$. We parameterize R as,

$$R = R_{12}R_{13}R_{23} = \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13} & 0 & -s_{13} \\ 0 & 1 & 0 \\ s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix}$$

$$= \begin{pmatrix} c_{13}c_{12} & -s_{12}c_{23} - c_{12}s_{13}s_{23} & -c_{12}s_{13}c_{23} + s_{12}s_{23} \\ c_{13}s_{12} & c_{12}c_{23} - s_{12}s_{13}s_{23} & -s_{12}s_{13}c_{23} - c_{12}s_{23} \\ s_{13} & c_{13}s_{23} & c_{13}c_{23} \end{pmatrix}\,,$$

where, e.g., $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Indeed, the angles θ_{12}, θ_{13} and θ_{23} are all invariant quantities since they are obtained by diagonalizing $\mathcal{M}^2$, which is manifestly basis-invariant.

The neutral physical Higgs mass eigenstates, h_1, h_2 and h_3, are given by

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = R \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \\ a^0 \end{pmatrix} = RW \begin{pmatrix} \sqrt{2}\,\mathrm{Re}\ \mathcal{H}_1^0 - v \\ \mathcal{H}_2^0 \\ \mathcal{H}_2^{0\,\dagger} \end{pmatrix},$$

which defines the unitary matrix W. The matrix RW is a function of θ_{23} and the q_{ij} given in the table below,

k	q_{k1}	q_{k2}
1	$c_{12}c_{13}$	$-s_{12} - ic_{12}s_{13}$
2	$s_{12}c_{13}$	$c_{12} - is_{12}s_{13}$
3	s_{13}	ic_{13}

The $q_{k\ell}$ are functions of θ_{12} and θ_{13}, where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. The invariant mixing angles θ_{12} and θ_{13} are defined modulo π, which are conventionally taken to lie in the region $-\frac{1}{2}\pi \leq \theta_{12}\,,\ \theta_{13} \leq \frac{1}{2}\pi$.

Explicitly,

$$RW = \begin{pmatrix} q_{11} & \frac{1}{\sqrt{2}}q_{12}^*\,e^{i\theta_{23}} & \frac{1}{\sqrt{2}}q_{12}\,e^{-i\theta_{23}} \\ q_{21} & \frac{1}{\sqrt{2}}q_{22}^*\,e^{i\theta_{23}} & \frac{1}{\sqrt{2}}q_{22}\,e^{-i\theta_{23}} \\ q_{31} & \frac{1}{\sqrt{2}}q_{32}^*\,e^{i\theta_{23}} & \frac{1}{\sqrt{2}}q_{32}\,e^{-i\theta_{23}} \end{pmatrix}.$$

In summary, we have:

$$h_k = q_{k1}\big(\sqrt{2}\,\mathrm{Re}\,\mathcal{H}_1^0 - v\big) + \frac{1}{\sqrt{2}}\big(q_{k2}^*\mathcal{H}_2^0 e^{i\theta_{23}} + \text{h.c.}\big)\,,$$

$$G^0 = \widehat{v}^i\Phi_i^0\,, \qquad G^+ = \widehat{v}^i\Phi_i^+\,, \qquad \mathcal{H}^+ = e^{i\eta}\widehat{w}^i\Phi_i^+\,.$$

It is convenient to define the positively charged Higgs field:

$$h^+ \equiv e^{i\theta_{23}}\mathcal{H}_2^+\,.$$

The $h^\pm$ squared mass is given by

$$m_\pm^2 = Y_2 + \tfrac{1}{2}Z_3 v^2\,.$$

Equivalently,

$$\mathcal{H}_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + iG + \sum_{k=1}^{3} q_{k1} h_k\right) \end{pmatrix}, \qquad e^{i\theta_{23}}\mathcal{H}_2 = \begin{pmatrix} h^+ \\ \frac{1}{\sqrt{2}}\sum_{k=1}^{3} q_{k2} h_k \end{pmatrix}.$$

Although θ_{23} is an invariant parameter, it is not physical since it can be eliminated by rephasing $\mathcal{H}_2 \to e^{-i\theta_{23}}\mathcal{H}_2$. Thus, without loss of generality, we henceforth set $\theta_{23} = 0$.

In the convention where $\theta_{23} = 0$,

$$R = \begin{pmatrix} c_{12}c_{13} & -s_{12} & -c_{12}s_{13} \\ c_{13}s_{12} & c_{12} & -s_{12}s_{13} \\ s_{13} & 0 & c_{13} \end{pmatrix} = \begin{pmatrix} q_{11} & \mathrm{Re}\ q_{12} & \mathrm{Im}\ q_{12} \\ q_{21} & \mathrm{Re}\ q_{22} & \mathrm{Im}\ q_{22} \\ q_{31} & \mathrm{Re}\ q_{32} & \mathrm{Im}\ q_{32} \end{pmatrix}.$$

Squared-mass sum rules

Using $\mathcal{M}^2 = RR^{\mathsf{T}}\mathcal{M}_D^2 R$, we obtain

$$Z_1 = \frac{1}{v^2}\sum_{k=1}^{3} m_k^2 (q_{k1})^2\,, \qquad Z_4 = \frac{1}{v^2}\left[\sum_{k=1}^{3} m_k^2 |q_{k2}|^2 - 2m_\pm^2\right]\,,$$

$$Z_5 e^{-2i\eta} = \frac{1}{v^2}\sum_{k=1}^{3} m_k^2 (q_{k2}^*)^2\,, \qquad Z_6 e^{-i\eta} = \frac{1}{v^2}\sum_{k=1}^{3} m_k^2\, q_{k1} q_{k2}^*\,,$$

In particular, $c_{13}\,\mathrm{Im}(Z_5 e^{-2i\eta}) = 2s_{13}\,\mathrm{Re}(Z_6 e^{-i\eta})$, and

$$s_{12}c_{12}c_{13} = \frac{v^2\,\mathrm{Re}(Z_6 e^{-i\eta})}{m_2^2 - m_1^2}\,,$$

$$s_{13}c_{13} = \frac{v^2\,\mathrm{Im}(Z_6 e^{-i\eta})}{m_1^2 - m_3^2 + s_{12}^2(m_2^2 - m_1^2)}$$

In terms of the Φ-basis fields,

$$\boxed{h_k = \frac{1}{\sqrt{2}}\left[\overline{\Phi}^{0i}(q_{k1}\widehat{v}_i + q_{k2}\widehat{w}_i e^{-i\eta}) + (q_{k1}\widehat{v}^{\,i} + q^*_{k2}\widehat{w}^{\,i} e^{i\eta})\overline{\Phi}^0_i\right]}$$

where the shifted neutral fields are defined by $\overline{\Phi}^0_i \equiv \Phi^0_i - v\widehat{v}_i/\sqrt{2}$ and $\overline{\Phi}^{0i} \equiv (\overline{\Phi}^0_i)^\dagger$.

We can invert the above formula to obtain:

$$\Phi_i = \begin{pmatrix} G^+\widehat{v}_i + h^+ e^{-i\eta}\widehat{w}_i \\ \dfrac{v}{\sqrt{2}}\widehat{v}_i + \dfrac{1}{\sqrt{2}}\left(iG + \displaystyle\sum_{k=1}^{3}\left(q_{k1}\widehat{v}_i + q_{k2}e^{-i\eta}\widehat{w}_i\right) h_k\right) \end{pmatrix}.$$

Plugging these results into the Higgs Lagrangian previously given yields the bosonic interactions of the Higgs mass eigenstates.

The interactions of the Higgs bosons and vector bosons are,

$$\mathscr{L}_{VVH} = \left(g m_W W_\mu^+ W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu \right) q_{k1} h_k \,,$$

$$\begin{aligned}
\mathcal{L}_{VVHH} = {} & \left[\tfrac{1}{4} g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{8c_W^2} Z_\mu Z^\mu \right] h_k h_k + \Big[\tfrac{1}{2} g^2 W_\mu^+ W^{\mu\,-} + e^2 A_\mu A^\mu \\
& \qquad + \frac{g^2}{c_W^2} \left(\tfrac{1}{2} - s_W^2 \right)^2 Z_\mu Z^\mu + \frac{2ge}{c_W} \left(\tfrac{1}{2} - s_W^2 \right) A_\mu Z^\mu \Big] h^+ h^- \\
& + \left\{ \left(\tfrac{1}{2} e g A^\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^+ \right) q_{k2} h^- h_k + \text{h.c.} \right\} ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{VHH} = {} & \frac{g}{4c_W} \epsilon_{jk\ell} q_{\ell 1} Z^\mu h_k \overleftrightarrow{\partial}_\mu h_j - \tfrac{1}{2} g \left[i q_{k2} W_\mu^+ h^- \overleftrightarrow{\partial}^\mu h_k + \text{h.c.} \right] \\
& + \left[i e A^\mu + \frac{ig}{c_W} \left(\tfrac{1}{2} - s_W^2 \right) Z^\mu \right] h^+ \overleftrightarrow{\partial}_\mu h^- \,,
\end{aligned}$$

where the sum over pairs of repeated indices $j, k = 1, 2, 3$ is implied.

The cubic and quartic Higgs self-interactions are given by,

$$
\begin{aligned}
\mathcal{L}_{3h} = &-\frac{v}{\sqrt{2}}\, h_j h_k h_\ell \Big[q_{j1}q_{k1}q_{\ell1}Z_1 + q_{j2}q^*_{k2}q_{\ell1}(Z_3+Z_4) + q_{j1}\,\mathrm{Re}(q_{k2}q_{\ell2}Z_5\, e^{-2i\theta_{23}}) \\
&\qquad +3q_{j1}q_{k1}\,\mathrm{Re}\Big(q_{\ell2}Z_6\, e^{-i\theta_{23}}\Big) + \mathrm{Re}(q^*_{j2}q_{k2}q_{\ell2}Z_7\, e^{-i\theta_{23}})\Big] \\
&\qquad -\sqrt{2}\, v\, h_k h^+ h^- \Big[q_{k1}Z_3 + \mathrm{Re}(q_{k2}\, e^{-i\theta_{23}} Z_7)\Big]\,, \\
\mathcal{L}_{4h} = &-\tfrac{1}{8} h_j h_k h_\ell h_m \Big[q_{j1}q_{k1}q_{\ell1}q_{m1}Z_1 + q_{j2}q_{k2}q^*_{\ell2}q^*_{m2}Z_2 + 2q_{j1}q_{k1}q_{\ell2}q^*_{m2}(Z_3+Z_4) \\
&\qquad +2q_{j1}q_{k1}\,\mathrm{Re}(q_{\ell2}q_{m2}Z_5\, e^{-2i\theta_{23}}) + 4q_{j1}q_{k1}q_{\ell1}\,\mathrm{Re}(q_{m2}Z_6\, e^{-i\theta_{23}}) \\
&\qquad +4q_{j1}\,\mathrm{Re}(q_{k2}q_{\ell2}q^*_{m2}Z_7\, e^{-i\theta_{23}})\Big] - \tfrac{1}{2}Z_2 h^+h^-h^+h^- \\
&\qquad -\tfrac{1}{2} h_j h_k h^+ h^- \Big[q_{j2}q^*_{k2}Z_2 + q_{j1}q_{k1}Z_3 + 2q_{j1}\,\mathrm{Re}(q_{k2}Z_7\, e^{-i\theta_{23}})\Big].
\end{aligned}
$$

It is remarkable how compact the expressions are for the Higgs boson interactions when written explicitly in terms of invariant quantities that can be directly related to observables.

The Higgs alignment limit of the 2HDM

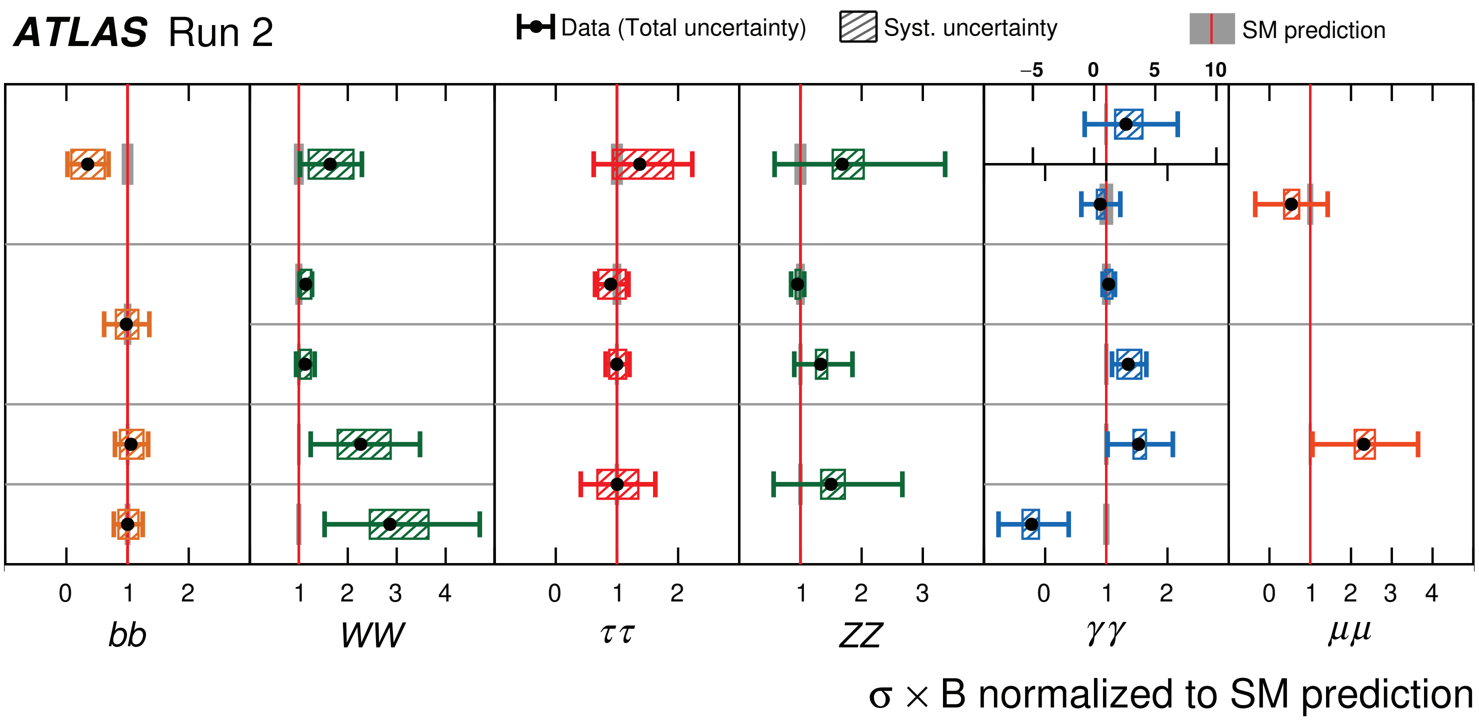
The tree-level couplings of the neutral field,

$$\varphi \equiv \sqrt{2}\,\mathrm{Re}\,\mathcal{H}_1^0 - v\,,$$

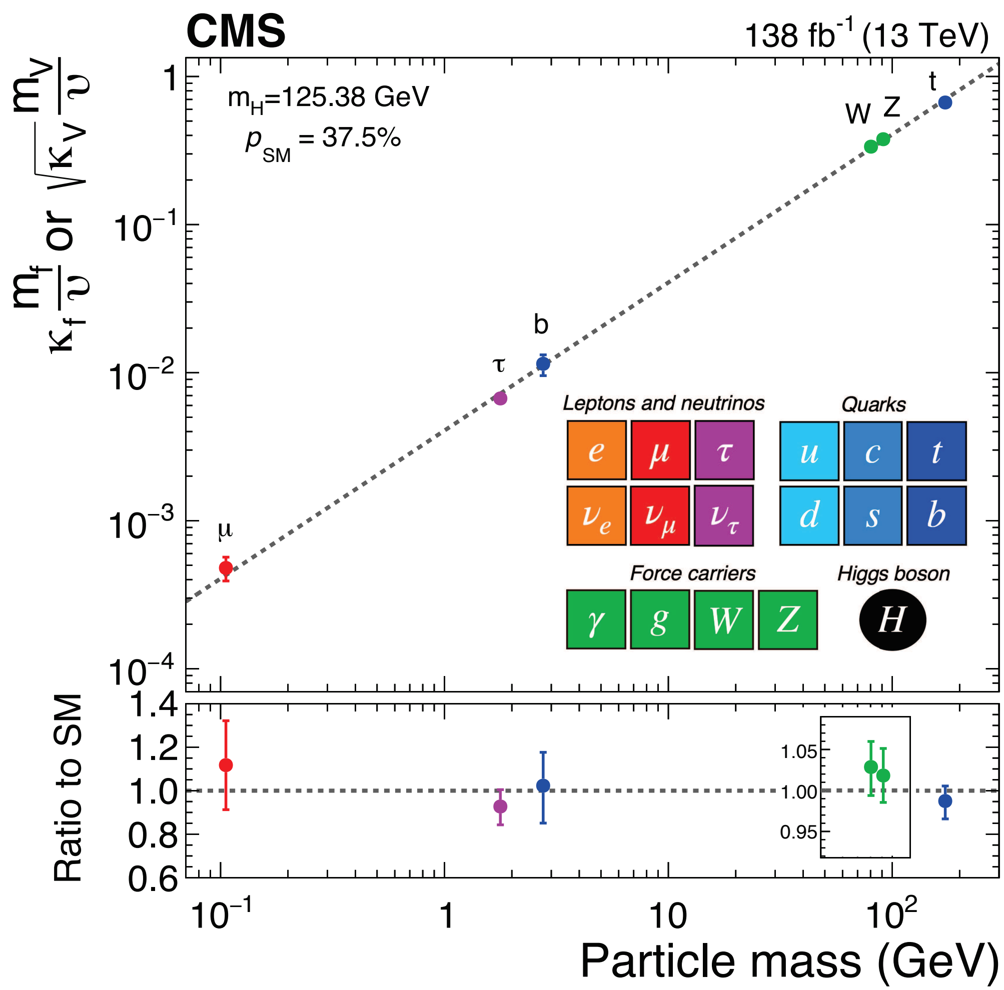
which resides in the scalar doublet $\mathcal{H}_1$ of the Higgs basis, are precisely those of the neutral Higgs field of the Standard Model (SM). However, the field φ is generally not a scalar mass eigenstate due to its mixing with the neutral scalar states that reside in $\mathcal{H}_2$.

The LHC Higgs data implies that the observed Higgs boson is SM-like. That is, the *Higgs alignment limit*, in which one of the Higgs mass eigenstates is aligned (in field space) with the Higgs vacuum expectation value (vev), is approximately realized.

The LHC data favors a SM-like Higgs boson



Ratio of observed rate to predicted SM event rate for different combinations of Higgs boson production and decay processes, as observed by the ATLAS Collaboration (based on 139 fb^{-1} of data). The horizontal bar on each point denotes the 68% confidence interval. The narrow grey bands indicate the theory uncertainties in the SM cross section times the branching fraction predictions. The p-value for compatibility of the measurement and the SM prediction is 72%. Taken from The ATLAS Collaboration, "A detailed map of Higgs boson interactions by the ATLAS experiment ten years after the discovery," Nature 607, no. 7917, 52-59 (2022) [arXiv:2207.00092 [hep-ex]].



The measured coupling modifiers of the Higgs boson to fermions and heavy gauge bosons, observed by the CMS Collaboration, as functions of fermion or gauge boson mass, where v is the vacuum expectation value of the Higgs field. For gauge bosons, the square root of the coupling modifier is plotted, to keep a linear proportionality to the mass, as predicted in the SM. The p-value with respect to the SM prediction is 37.5%. Taken from The CMS Collaboration, "A portrait of the Higgs boson by the CMS experiment ten years after the discovery," Nature 607, no. 7917, 60-68 (2022) [arXiv:2207.00043 [hep-ex]].

In the alignment limit where h_1 is identified as SM-like ($m_1 \simeq 125$ GeV),

$$\frac{g_{h_1VV}}{g_{h_{\rm SM}VV}} = q_{11} = c_{12}c_{13} \simeq 1\,, \qquad \text{where } V = W \text{ or } Z\,,$$

it then follows that s_{12}, $s_{13} \ll 1$. Thus,

$$s_{12} \equiv \sin\theta_{12} \simeq \frac{{\rm Re}(Z_6 e^{-i\eta})v^2}{m_2^2 - m_1^2} \ll 1\,,$$

$$s_{13} \equiv \sin\theta_{13} \simeq -\frac{{\rm Im}(Z_6 e^{-i\eta})v^2}{m_3^2 - m_1^2} \ll 1\,.$$

$${\rm Im}(Z_5 e^{-2i\eta}) \simeq \frac{(m_2^2 - m_1^2)s_{12}s_{13}}{v^2} \simeq -\frac{{\rm Im}(Z_6^2 e^{-2i\eta})v^2}{m_3^2 - m_1^2} \ll 1\,.$$

We also obtain the following approximate mass relations,

$$m_1^2 \simeq v^2\big[Z_1 - s_{12}\,{\rm Re}(Z_6 e^{-i\eta}) + s_{13}\,{\rm Im}(Z_6 e^{-i\eta})\big]\,,$$

$$m_2^2 - m_3^2 \simeq v^2\big[{\rm Re}(Z_5 e^{-2i\eta}) + s_{12}\,{\rm Re}(Z_6 e^{-i\eta}) + s_{13}\,{\rm Im}(Z_6 e^{-i\eta})\big]\,,$$

$$m_2^2 - m_\pm^2 \simeq \tfrac{1}{2}v^2\big[Z_4 + {\rm Re}(Z_5 e^{-2i\eta}) + 2s_{12}\,{\rm Re}(Z_6 e^{-i\eta})\big]\,.$$

Conditions for approximate Higgs alignment

1. The *decoupling limit* is achieved if $m_2,\ m_3 \gg v \simeq 246$ GeV (under the assumption that Z_6 is at most an $\mathcal{O}(1)$ parameter). That is, $Y_2 \gg v^2$.

2. Approximate Higgs alignment without decoupling is achieved if $|Z_6| \ll 1$, while all Higgs squared masses are of $\mathcal{O}(v^2)$.[6]

Remark: Although the tree-level couplings of $\varphi \equiv \sqrt{2}\,\mathrm{Re}\,\mathcal{H}_1^0 - v$ coincide with those of the SM Higgs boson, the one-loop couplings can differ due to the exchange of the other Higgs states (if not too heavy). For example, the $h^\pm$ loop contributes to the decays of the SM-like Higgs boson to $\gamma\gamma$ and γZ.

[6] More precisely, we require that $|Z_6| \ll \Delta m_{j1}^2/v^2$, where $\Delta m_{j1}^2 \equiv m_j^2 - m_1^2$ for $j = 2, 3$.

Higgs-fermion Yukawa interactions

The Higgs–fermion Yukawa couplings (in the Φ-basis):

$$-\mathscr{L}_Y = (\widehat{\boldsymbol{y}}_{\boldsymbol{u}\boldsymbol{i}})_m{}^n \left[\Phi^{0i}\,\widehat{\bar{u}}{}^m_L \widehat{u}_{nR} - (\Phi^-)^i\,\widehat{\bar{d}}{}^m_L \widehat{u}_{nR}\right] + (\widehat{\boldsymbol{y}}^i_{\boldsymbol{d}})_m{}^n \left[\Phi^+_i \widehat{\bar{u}}{}^m_L \widehat{d}_{nR} + \Phi^0_i\,\widehat{\bar{d}}{}^m_L \widehat{d}_{nR}\right]$$
$$+ (\widehat{\boldsymbol{y}}^i_{\boldsymbol{e}})_m{}^n \left[\Phi^+_i \widehat{\bar{\nu}}{}^m_L \widehat{e}_{nR} + \Phi^0_i\,\widehat{\bar{e}}{}^m_L \widehat{e}_{nR}\right] + \text{h.c.}\,,$$

where $f_R \equiv \frac{1}{2}(1+\gamma_5)f$ and $f_L \equiv \frac{1}{2}(1-\gamma_5)f$ [with four-component fermion fields $f = u, d, \nu, e$]. The hatted fields correspond to the fermion interaction-eigenstates, and m, n are fermion flavor labels. We have also defined

$$\widehat{\boldsymbol{y}}^i_{\boldsymbol{f}} \equiv (\widehat{\boldsymbol{y}}_{\boldsymbol{f}\,i})^\dagger\,, \quad \text{for } f = u, d, e.$$

We can construct invariant matrix Yukawa couplings $\widehat{\boldsymbol{\kappa}}^{\boldsymbol{F}}$ and $\widehat{\boldsymbol{\rho}}^{\boldsymbol{F}}$ (where $F = U$, D, E) as follows:

$$\widehat{\boldsymbol{\kappa}}^{\boldsymbol{F}} \equiv \widehat{v}^j \, \widehat{\boldsymbol{y}}_{\boldsymbol{f}\,j} \,, \qquad\qquad \widehat{\boldsymbol{\rho}}^{\boldsymbol{F}} \equiv e^{i\eta} \widehat{w}^j \, \widehat{\boldsymbol{y}}_{\boldsymbol{f}\,j} \,.$$

we end up with

$$\begin{aligned}
-\mathscr{L}_Y = & \Big\{ (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{U}})_m{}^n \left[\mathcal{H}_1^{0\dagger} \, \widehat{\overline{u}}{}_L^m \widehat{u}_{nR} - \mathcal{H}_1^- \widehat{\overline{d}}{}_L^m \widehat{u}_{nR} \right] + (\widehat{\boldsymbol{\rho}}^{\boldsymbol{U}})_m{}^n \left[\mathcal{H}_2^{0\dagger} \, \widehat{\overline{u}}{}_L^m \widehat{u}_{nR} - \mathcal{H}_2^- \widehat{\overline{d}}{}_L^m \widehat{u}_{nR} \right] + \text{h.c.} \Big\} \\
& + \Big\{ (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{D}})^\dagger{}_m{}^n \left[\mathcal{H}_1^+ \widehat{\overline{u}}{}_L^m \widehat{d}_{nR} + \mathcal{H}_1^0 \, \widehat{\overline{d}}{}_L^m \widehat{d}_{nR} \right] + (\widehat{\boldsymbol{\rho}}^{\boldsymbol{D}})^\dagger{}_m{}^n \left[\mathcal{H}_2^+ \widehat{\overline{u}}{}_L^m \widehat{d}_{nR} + \mathcal{H}_2^0 \, \widehat{\overline{d}}{}_L^m \widehat{d}_{nR} \right] + \text{h.c.} \Big\} \\
& + \Big\{ (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{E}})^\dagger{}_m{}^n \left[\mathcal{H}_1^+ \widehat{\overline{\nu}}{}_L^m \widehat{e}_{nR} + \mathcal{H}_1^0 \, \widehat{\overline{e}}{}_L^m \widehat{e}_{nR} \right] + (\widehat{\boldsymbol{\rho}}^{\boldsymbol{E}})^\dagger{}_m{}^n \left[\mathcal{H}_2^+ \widehat{\overline{\nu}}{}_L^m \widehat{e}_{nR} + \mathcal{H}_2^0 \, \widehat{\overline{e}}{}_L^m \widehat{e}_{nR} \right] + \text{h.c.} \Big\} .
\end{aligned}$$

The fermion mass matrices can be identified by setting the scalar fields to their vevs.

$$(\widehat{\boldsymbol{M}}_{\boldsymbol{U}})_m{}^n = \frac{v}{\sqrt{2}} (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{U}})_m{}^n \,, \qquad (\widehat{\boldsymbol{M}}_{\boldsymbol{F}})_m{}^n = \frac{v}{\sqrt{2}} (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{F}})^\dagger{}_m{}^n \,, \quad \text{for } F = D,\, E.$$

Diagonalizations of the fermion mass matrices are accomplished via the singular value decomposition of linear algebra. Introducing the unitary matrices L_f and R_f $(f = u, d, e)$, where[7]

$$\widehat{f}_{mL} = (L_f)_m{}^n f_{nL}\,, \qquad \widehat{f}_{mR} = (R_u)_m{}^n f_{nR}\,,$$

the diagonalization equations are:

$$L_u^\dagger \widehat{\boldsymbol{M}}_{\boldsymbol{U}} R_u \equiv \boldsymbol{M}_{\boldsymbol{U}} = \mathrm{diag}(m_u\,,\, m_c\,,\, m_t)\,,$$

$$L_d^\dagger \widehat{\boldsymbol{M}}_{\boldsymbol{D}} R_d \equiv \boldsymbol{M}_{\boldsymbol{D}} = \mathrm{diag}(m_d\,,\, m_s\,,\, m_b)\,,$$

$$L_e^\dagger \widehat{\boldsymbol{M}}_{\boldsymbol{E}} R_e \equiv \boldsymbol{M}_{\boldsymbol{E}} = \mathrm{diag}(m_e\,,\, m_\mu\,,\, m_\tau)\,,$$

where the diagonalized masses are real and nonnegative. Since no right-handed neutrino field has been introduced so far, the neutrinos are exactly massless.

[7] Since the neutrinos are massless (prior to introducing the neutrino mass generation mechanism), one is free to define $\widehat{\nu}_{mL} = (L_e)_m{}^n \nu_{nL}$.

To write out the corresponding Higgs–fermion Yukawa interactions, it is convenient to define

$$\boldsymbol{\kappa}^{\boldsymbol{U}} \equiv L_u^\dagger \,\widehat{\boldsymbol{\kappa}}^{\boldsymbol{U}} R_u = \frac{\sqrt{2}}{v} \boldsymbol{M}_{\boldsymbol{U}} \,,$$

$$\boldsymbol{\kappa}^{\boldsymbol{D}} \equiv L_d^\dagger \,\widehat{\boldsymbol{\kappa}}^{\boldsymbol{D}\dagger} R_d = \frac{\sqrt{2}}{v} \boldsymbol{M}_{\boldsymbol{D}} \,,$$

$$\boldsymbol{\kappa}^{\boldsymbol{E}} \equiv L_e^\dagger \,\widehat{\boldsymbol{\kappa}}^{\boldsymbol{E}\dagger} R_e = \frac{\sqrt{2}}{v} \boldsymbol{M}_{\boldsymbol{E}} \,,$$

which are diagonal with positive entries by construction, and

$$\boldsymbol{\rho}^{\boldsymbol{U}} \equiv L_u^\dagger \,\widehat{\boldsymbol{\rho}}^{\boldsymbol{U}} R_u \,,$$

$$\boldsymbol{\rho}^{\boldsymbol{D}\dagger} \equiv L_d^\dagger \,\widehat{\boldsymbol{\rho}}^{\boldsymbol{D}\dagger} R_d \,,$$

$$\boldsymbol{\rho}^{\boldsymbol{E}\dagger} \equiv L_e^\dagger \,\widehat{\boldsymbol{\rho}}^{\boldsymbol{E}\dagger} R_e \,,$$

which are arbitrary complex coupling matrices that are independent of the fermion masses.

That is,

$$\boldsymbol{\kappa}^{\boldsymbol{F}} = \frac{\sqrt{2}\boldsymbol{M_F}}{v} = \widehat{v}^i \boldsymbol{y_f}_i\,, \qquad\qquad \boldsymbol{\rho}^{\boldsymbol{F}} = e^{i\eta}\widehat{w}^i \boldsymbol{y_f}_i\,,$$

or equivalently,

$$\boldsymbol{y_f}_i = \frac{\sqrt{2}}{v}\boldsymbol{M_F}\widehat{v}_i + e^{-i\eta}\boldsymbol{\rho}^{\boldsymbol{F}}\,\widehat{w}_i\,.$$

The Yukawa Lagrangian in the Φ-basis in terms of fermion mass eigenstates is therefore:

$$\begin{aligned}
-\mathscr{L}_Y = (\boldsymbol{y_{ui}})_p{}^n & \left[\Phi^{0i}\,\delta^p_m \bar{u}^m_L u_{nR} - (\Phi^-)^i\,(\boldsymbol{K}^\dagger)_m{}^p \bar{d}^m_L u_{nR}\right] \\
& +(\boldsymbol{y}^i_{\boldsymbol{d}})_p{}^n\left[\Phi^+_i \boldsymbol{K}_m{}^p \bar{u}^m_L d_{nR} + \Phi^0_i\,\delta^p_m \bar{d}^m_L d_{nR}\right] \\
& +(\boldsymbol{y}^i_{\boldsymbol{e}})_m{}^n\left[\Phi^+_i \bar{\nu}^m_L e_{nR} + \Phi^0_i\,\bar{e}^m_L e_{nR}\right] + \text{h.c.}\,,
\end{aligned}$$

where $\boldsymbol{K} \equiv L^\dagger_u L_d$ is the CKM mixing matrix.

Exercise: Rewrite $\mathscr{L}_Y$ above in terms of the Higgs basis fields $\mathcal{H}_1$ and $\mathcal{H}_2$.

In terms of the quark mass-eigenstate fields and the scalar mass eigenstate fields, the Yukawa Lagrangian is given by:

$$
\begin{aligned}
-\mathscr{L}_Y = {} & \overline{U}\left\{\frac{\boldsymbol{M_U}}{v}q_{k1} + \frac{1}{\sqrt{2}}\left[q^*_{k2}\,\boldsymbol{\rho}^{\boldsymbol{U}}P_R + q_{k2}\,\boldsymbol{\rho}^{\boldsymbol{U}\dagger}P_L\right]\right\}Uh_k - \frac{i}{v}\overline{U}\boldsymbol{M_U}\gamma_5 UG^0 \\
& +\overline{D}\left\{\frac{\boldsymbol{M_D}}{v}q_{k1} + \frac{1}{\sqrt{2}}\left[q_{k2}\,\boldsymbol{\rho}^{\boldsymbol{D}\dagger}P_R + q^*_{k2}\,\boldsymbol{\rho}^{\boldsymbol{D}}P_L\right]\right\}Dh_k + \frac{i}{v}\overline{D}\boldsymbol{M_D}\gamma_5 DG^0 \\
& +\overline{E}\left\{\frac{\boldsymbol{M_E}}{v}q_{k1} + \frac{1}{\sqrt{2}}\left[q_{k2}\,\boldsymbol{\rho}^{\boldsymbol{E}\dagger}P_R + q^*_{k2}\,\boldsymbol{\rho}^{\boldsymbol{E}}P_L\right]\right\}Eh_k + \frac{i}{v}\overline{E}\boldsymbol{M_E}\gamma_5 EG^0 \\
& +\left\{\overline{U}\left[\boldsymbol{K}\boldsymbol{\rho}^{\boldsymbol{D}\dagger}P_R - \boldsymbol{\rho}^{\boldsymbol{U}\dagger}\boldsymbol{K}P_L\right]Dh^+ + \overline{N}\boldsymbol{\rho}^{\boldsymbol{E}\dagger}P_R Eh^+ + \text{h.c.}\right\} \\
& +\left\{\frac{\sqrt{2}}{v}\overline{U}\left[\boldsymbol{K}\boldsymbol{M_D}P_R - \boldsymbol{M_U}\boldsymbol{K}P_L\right]DG^+ + \frac{\sqrt{2}}{v}\overline{N}\boldsymbol{M_E}P_R EG^+ + \text{h.c.}\right\},
\end{aligned}
$$

where there is an implicit sum over $k \in \{1,2,3\}$, $P_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)$, and the mass-eigenstate fields of the down-type quarks, the up-type quarks, the charged leptons and the neutrinos are $D = (d,s,b)^{\mathsf{T}}$, $U \equiv (u,c,t)^{\mathsf{T}}$, $E = (e,\mu,\tau)^{\mathsf{T}}$, and $N = (\nu_e,\nu_\mu,\nu_\tau)^{\mathsf{T}}$, respectively.

In general, the matrices $\boldsymbol{\rho}^{\boldsymbol{F}}$ are complex and flavor-nondiagonal, resulting in flavor-changing neutral current (FCNC) processes and new sources of CP violation (beyond the CKM matrix $\boldsymbol{K}$) mediated at tree level by the exchange of the h_k.

REMARK: In the exact Higgs alignment limit where h_1 is the SM-like Higgs boson, $s_{12} = s_{13} = 0$, or equivalently

$$q_{11} = q_{22} = -iq_{32} = 1 \quad \text{and} \quad q_{21} = q_{31} = q_{12} = 0\,.$$

One easily checks that h_1 possesses the Yukawa couplings of the SM Higgs boson:

$$-\mathscr{L}_Y = \frac{1}{v} \sum_{F=U,D,E} \overline{F}\, \boldsymbol{M}_{\boldsymbol{F}} F\, h_1\,.$$

Nevertheless, tree-level FCNCs and CP violation mediated by h_2 and h_3 are still present.

Eliminating the tree-level Higgs-mediated FCNCs

A phenomenologically acceptable model must provide an explanation for the approximate flavor diagonality and reality of the $\boldsymbol{\rho^F}$ matrices.

A *natural* way[8] to achieve this result is to impose a symmetry on the dimension-four terms of the Higgs Lagrangian.[9] This symmetry is manifestly realized in a particular scalar field basis that henceforth defines the Φ-basis.

Example: Impose a $\mathbb{Z}_2$ discrete symmetry, $\Phi_1 \to \Phi_1$ and $\Phi_2 \to -\Phi_2$ on the dimension-four terms of the Higgs Lagrangian in the Φ-basis, which sets $\lambda_6 = \lambda_7 = 0$ and sets two of the four Higgs-quark Yukawa coupling matrices to zero. Two possible $\mathbb{Z}_2$ charge assignments for the quark fields are shown in the table below.

	Φ_1	Φ_2	U_R	D_R	$U_L,\ D_L$	Yukawa couplings
Type I	$+$	$-$	$-$	$-$	$+$	$\boldsymbol{y_u^1} = \boldsymbol{y_d^1} = 0$
Type II	$+$	$-$	$-$	$+$	$+$	$\boldsymbol{y_u^1} = \boldsymbol{y_d^2} = 0$

[8]Natural means without fine-tuning the parameters of $\mathscr{L}_Y$.

[9]We allow for soft symmetry-breaking dimension-two terms in $\mathscr{L}_Y$, which will generate FCNCs at loop order that are consistent with experimental constraints.

The corresponding basis-independent conditions are,

$$\text{Type I:}\quad \epsilon^{ij}\,\boldsymbol{y_d}_i\,\boldsymbol{y_u}_j = 0\,, \qquad \Longrightarrow \qquad \boldsymbol{\kappa^D}\boldsymbol{\rho^U} - \boldsymbol{\rho^D}\boldsymbol{\kappa^U} = 0\,,$$

$$\text{Type II:}\quad \delta_i^j\,\boldsymbol{y_d}^i\boldsymbol{y_u}_j = 0\,, \qquad \Longrightarrow \qquad \boldsymbol{\kappa^D}\boldsymbol{\kappa^U}^\dagger + \boldsymbol{\rho^D}\boldsymbol{\rho^U}^\dagger = 0\,,$$

In the Φ-basis, we define $\tan\beta \equiv |v_2/v_1|$ and $\xi \equiv \arg(v_2/v_1)$,

$$\widehat{v} = (\cos\beta\,,\, e^{i\xi}\sin\beta)\,, \qquad \widehat{w} = (-e^{-i\xi}\sin\beta\,,\,\cos\beta)\,.$$

Using $\boldsymbol{y_f}_i = \sqrt{2}(\boldsymbol{M_F}/v)\widehat{v}_i + e^{-i\eta}\boldsymbol{\rho^F}\widehat{w}_i$, it follows that $\boldsymbol{\rho^U}$ and $\boldsymbol{\rho^D}$ are diagonal matrices given by[10]

$$\text{Type I:}\quad \boldsymbol{\rho^U} = \frac{e^{i(\xi+\eta)}\sqrt{2}\boldsymbol{M_U}\cot\beta}{v}\,, \qquad \boldsymbol{\rho^D} = \frac{e^{i(\xi+\eta)}\sqrt{2}\boldsymbol{M_D}\cot\beta}{v}\,,$$

$$\text{Type II:}\quad \boldsymbol{\rho^U} = \frac{e^{i(\xi+\eta)}\sqrt{2}\boldsymbol{M_U}\cot\beta}{v}\,, \qquad \boldsymbol{\rho^D} = -\,\frac{e^{i(\xi+\eta)}\sqrt{2}\boldsymbol{M_D}\tan\beta}{v}\,.$$

[10] To obtain $\boldsymbol{\rho^E}$, replace $\boldsymbol{D}$ with $\boldsymbol{E}$ in the formulae above.

REMARK: The Φ-basis defined above is not quite unique. One always has the option to interchange the roles of Φ_1 and Φ_2 by defining a Φ' basis via $\Phi' = U\Phi$, where

$$U = \begin{pmatrix} 0 & e^{-i\xi} \\ e^{i\zeta} & 0 \end{pmatrix} .$$

The softly broken $\mathbb{Z}_2$ symmetry is also manifestly realized in the Φ'-basis, where the previously tabulated $\mathbb{Z}_2$ charges of Φ_1 and Φ_2 are interchanged. In particular, in light of

$$\begin{pmatrix} \sin\beta \\ e^{i\zeta}\cos\beta \end{pmatrix} = U \begin{pmatrix} \cos\beta \\ e^{i\xi}\sin\beta \end{pmatrix} ,$$

we conclude that $\beta' = \frac{1}{2}\pi - \beta$ and $\xi' = \zeta$. Moreover, due to the pseudoinvariant nature of $e^{i\eta}$, we see that $e^{i\eta'} = (\det\ U)^{-1}e^{i\eta}$. Using $\det U = -e^{i(\zeta-\xi)}$, it follows that $e^{i(\xi'+\eta')} = -e^{i(\xi+\eta)}$.

Thus, with respect to the parameters of the Φ'-basis, the results obtained previously are modified by interchanging $\tan\beta \leftrightarrow \cot\beta$ and multiplying the resulting expressions by -1.

Conditions for a CP-conserving scalar potential and vacuum

Consider what happens if we transform between two Higgs bases. To transform to another Higgs basis, we can employ $\Phi_i \to U_i{}^j \Phi_j$, where $U = \mathrm{diag}(1, e^{i\chi})$, in which case $\eta \to \eta - \chi$. Hence,

$$[Y_3, Z_6, Z_7] \to e^{-i\chi}[Y_3, Z_6, Z_7] \quad \text{and} \quad Z_5 \to e^{-2i\chi} Z_5 \,,$$

whereas Y_1, Y_2 and $Z_{1,2,3,4}$ are invariant.

The 2HDM scalar potential and vacuum are CP-invariant if one can find a choice of χ such that all the coefficients of the scalar potential in the Higgs basis are real after imposing the scalar potential minimum conditions. This conditions is satisfied if and only if $\mathrm{Im}(Z_5^* Z_6^2) = \mathrm{Im}(Z_5^* Z_7^2) = \mathrm{Im}(Z_6^* Z_7) = 0$.

The conditions for a CP-invariant scalar potential and vacuum are $\mathrm{Im}(Z_5^* Z_6^2) = \mathrm{Im}(Z_5^* Z_7^2) = \mathrm{Im}(Z_6^* Z_7) = 0$, implying the existence of a *real Higgs basis* (where all Higgs basis scalar potential parameters are real). These conditions are satisfied if

1. $\mathrm{Im}(Z_5 e^{-2i\eta}) = \mathrm{Im}(Z_6 e^{-i\eta}) = \mathrm{Im}(Z_7 e^{-i\eta}) = 0\,,$

   or

2. $\mathrm{Im}(Z_5 e^{-2i\eta}) = \mathrm{Re}(Z_6 e^{-i\eta}) = \mathrm{Re}(Z_7 e^{-i\eta}) = 0\,.$

In both cases the neutral scalar squared-mass matrix assumes a block diagonal form consisting of a 2×2 mass matrix that yields the squared-masses of two neutral CP-even Higgs bosons and a 1×1 mass matrix corresponding to the squared mass of a neutral CP-odd Higgs boson (identified as h_3 or h_2, respectively).

The CP-conserving 2HDM

Without loss of generality, we work in a real Higgs basis and any associated Φ-basis in which all scalar potential parameters and the corresponding scalar vevs are real (with $\tan\beta \equiv v_2/v_1$ either positive or negative). In particular, $\eta = 0$ mod π.[11] Under a real orthogonal basis transformation, $\Phi_i \to \mathcal{R}_i{}^j \Phi_j$,

$$[Y_3, Z_6, Z_7, \varepsilon, \tan\beta] \to \det\mathcal{R}\,[Y_3, Z_6, Z_7, \varepsilon, \tan\beta]\,,$$

where $\varepsilon \equiv e^{i\eta} = \pm 1$ and $\det\mathcal{R} = \pm 1$. It is convenient to choose

$$\varepsilon \equiv e^{i\eta} = \begin{cases} \operatorname{sgn} Z_6\,, & \text{if } Z_6 \neq 0, \\ \operatorname{sgn} Z_7\,, & \text{if } Z_6 = 0 \text{ and } Z_7 \neq 0. \end{cases}$$

[11] The case of $Z_6 = Z_7 = 0$ must be treated separately since in this case $\eta = 0 \bmod \frac{1}{2}\pi$.

The neutral Higgs squared-mass matrix in a real Higgs basis is:

$$\mathcal{M}^2 = \begin{pmatrix} Z_1 v^2 & \varepsilon Z_6 v^2 & 0 \\ \varepsilon Z_6 v^2 & Y_2 + \frac{1}{2}(Z_3 + Z_4 + Z_5)v^2 & 0 \\ 0 & 0 & Y_2 + \frac{1}{2}(Z_3 + Z_4 - Z_5)v^2 \end{pmatrix}.$$

Diagonalizing the neutral scalar squared-mass matrix, only one nontrivial mixing angle θ_{12} is required, since $\theta_{13} = \theta_{23} = 0$. The scalar mass eigenstates are identified as two neutral CP-even scalars h_1 and h_2 and a CP-odd scalar h_3

$$h_1 = \left(\sqrt{2}\,\mathrm{Re}\,\mathcal{H}_1^0 - v\right)\cos\theta_{12} - \sqrt{2}\,\mathrm{Re}\,\mathcal{H}_2^0 \sin\theta_{12}\,,$$

$$h_2 = \left(\sqrt{2}\,\mathrm{Re}\,\mathcal{H}_1^0 - v\right)\sin\theta_{12} + \sqrt{2}\,\mathrm{Re}\,\mathcal{H}_2^0 \cos\theta_{12}\,,$$

$$h_3 = \sqrt{2}\,\mathrm{Im}\,\mathcal{H}_2^0\,,$$

with corresponding masses $m_i \equiv m_{h_i}$.

The squared masses of two neutral CP-even scalars, h_1 and h_2 and the CP-odd scalar h_3 are:

$$m_{1,2}^2 = \tfrac{1}{2}\Big\{Y_2 + \big(Z_1 + \tfrac{1}{2}Z_{345}\big)v^2 \pm \sqrt{\big[Y_2 - \big(Z_1 - \tfrac{1}{2}Z_{345}\big)v^2\big]^2 + 4Z_6^2 v^4}\,\Big\}\,,$$
$$m_3^2 = Y_2 + \tfrac{1}{2}(Z_3 + Z_4 - Z_5)v^2 = m_\pm^2 + \tfrac{1}{2}(Z_4 - Z_5)v^2\,,$$

where $Z_{345} \equiv Z_3 + Z_4 + Z_5$, with no mass ordering of h_1, h_2, h_3 implied. The mixing angle θ_{12} (where $|\theta_{12}| \leq \frac{1}{2}\pi$) is obtained from

$$\sin^2\theta_{12} = \frac{Z_1 v^2 - m_1^2}{m_2^2 - m_1^2}\,,$$
$$\sin\theta_{12}\cos\theta_{12} = \frac{\varepsilon Z_6 v^2}{m_2^2 - m_1^2}.$$

Conventional notation for the CP-conserving 2HDM

If h_1 (identified as the SM-like Higgs boson) is the lighter of the two CP-even scalars, then the standard CP-conserving 2HDM conventions define

$$\begin{aligned} h \equiv h_1 &= -\big(\sqrt{2}\,\mathrm{Re}\,\Phi_1^0 - vc_\beta\big)\sin\alpha + \big(\sqrt{2}\,\mathrm{Re}\,\Phi_2^0 - vs_\beta\big)\cos\alpha\,, \\ H \equiv -\varepsilon h_2 &= \quad \big(\sqrt{2}\,\mathrm{Re}\,\Phi_1^0 - vc_\beta\big)\cos\alpha + \big(\sqrt{2}\,\mathrm{Re}\,\Phi_2^0 - vs_\beta\big)\sin\alpha\,, \\ A \equiv \varepsilon h_3 &= -\sqrt{2}\big[\mathrm{Im}\,\Phi_1^0 s_\beta - \mathrm{Im}\,\Phi_2^0 c_\beta\big]\,, \\ H^\pm \equiv \varepsilon h^\pm &= -\Phi_1^\pm s_\beta + \Phi_2^\pm c_\beta\,. \end{aligned}$$

where h and H are CP-even (with $m_h < m_H$), A is CP-odd, and

$$\beta - \alpha = \varepsilon\,\theta_{12} + \tfrac{1}{2}\pi\,.$$

Define the quantities: $s_{\beta-\alpha} \equiv \sin(\beta-\alpha)$ and $c_{\beta-\alpha} \equiv \cos(\beta-\alpha)$. By convention, $|\theta_{12}| \leq \frac{1}{2}\pi$ which implies that $0 \leq s_{\beta-\alpha} \leq \pi$.

k	q_{k1}	q_{k2}
1	$s_{\beta-\alpha}$	$\varepsilon\, c_{\beta-\alpha}$
2	$-\varepsilon\, c_{\beta-\alpha}$	$s_{\beta-\alpha}$
3	0	i

$q_{k\ell}$ for the CP-conserving 2HDM when $h_1 = h$ is identified with the SM-like Higgs boson.

Hence, the squared-mass sum rules previously derived imply that

$$Z_1 v^2 = m_h^2 s_{\beta-\alpha}^2 + m_H^2 c_{\beta-\alpha}^2\,,$$

$$s_{\beta-\alpha} c_{\beta-\alpha} = -\frac{Z_6 v^2}{m_H^2 - m_h^2}\,,$$

which yields an explicit expression for $c_{\beta-\alpha}$,

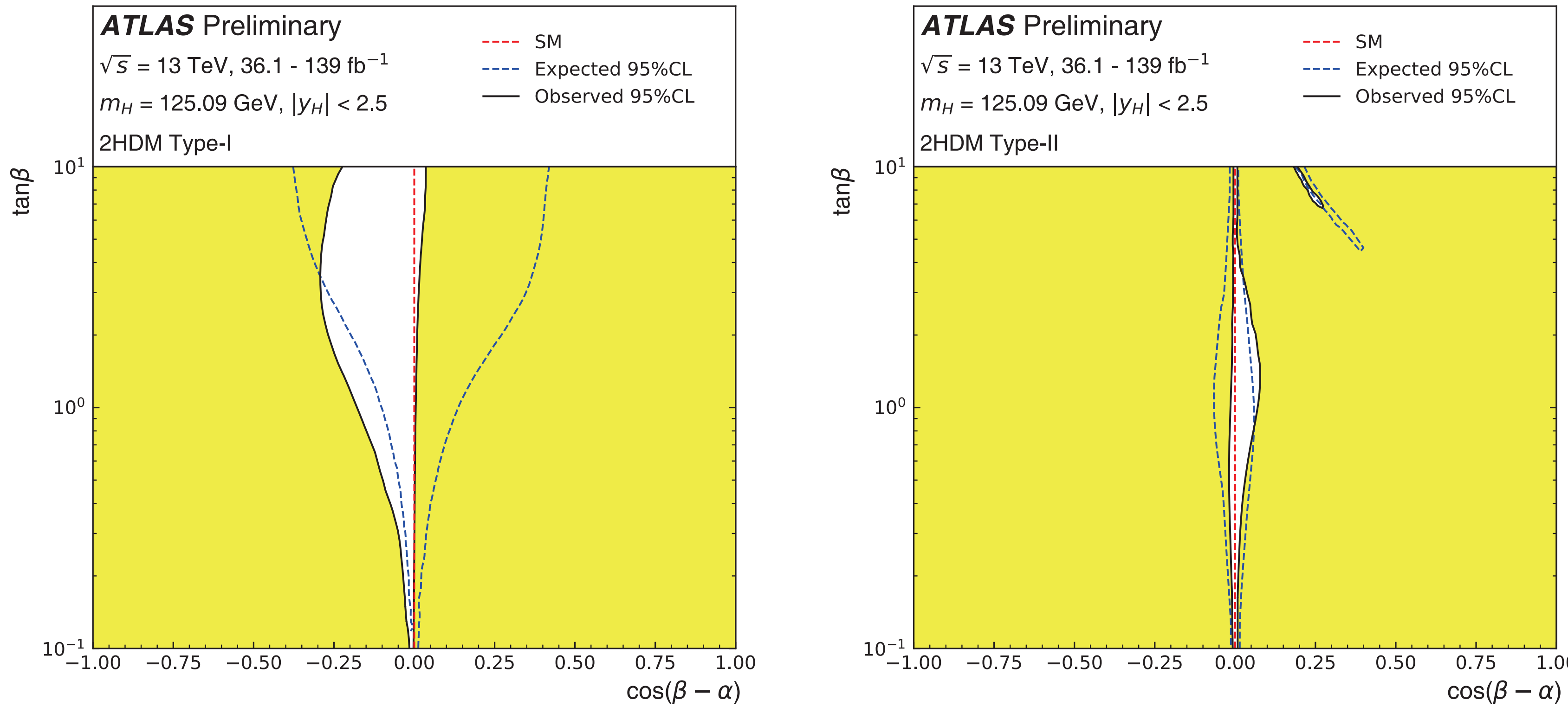
$$\varepsilon c_{\beta-\alpha} = \frac{-|Z_6| v^2}{\sqrt{(m_H^2 - m_h^2)(m_H^2 - Z_1 v^2)}} \leq 0\,.$$

The Higgs alignment limit of the CP-conserving 2HDM

Approximate Higgs alignment corresponds to $|c_{\beta-\alpha}| \ll 1$, which is achieved if $m_H \gg v$ (decoupling limit) or if $|Z_6| \ll 1$ [Higgs alignment without decoupling if $m_H \sim \mathcal{O}(v)$].

$$\begin{aligned}
|c_{\beta-\alpha}| &\simeq \frac{|Z_6|v^2}{m_H^2 - m_h^2} \ll 1\,, \\
m_h^2 &\simeq v^2(Z_1 + Z_6 c_{\beta-\alpha})\,, \\
m_H^2 - m_A^2 &\simeq v^2(Z_5 - Z_6 c_{\beta-\alpha})\,, \\
m_H^2 - m_{H^\pm}^2 &\simeq \tfrac{1}{2}v^2(Z_4 + Z_5 - 2Z_6 c_{\beta-\alpha})\,.
\end{aligned}$$

LHC constraints on Higgs alignment in the 2HDM



Regions excluded by fits to the measured rates of the productions and decay of the Higgs boson (assumed to be h of the 2HDM). Contours at 95% CL. The observed bestfit values for $\cos(\beta-\alpha)$ are -0.006 for the Type-I 2HDM and 0.002 for the Type-II 2HDM. Taken from ATLAS Collaboration, ATLAS-CONF-2021-053 (2 November 2021).

Higgs couplings of the CP-conserving 2HDM

Using the previous expressions obtained for the general 2HDM, one can derive the Higgs couplings in the CP-conserving 2HDM. Here are a few examples:

$$\mathscr{L}_{VVH} = \left(g m_W W_\mu^+ W^{\mu\,-} + \frac{g}{2c_W} m_Z Z_\mu Z^\mu\right)\left[s_{\beta-\alpha} h + c_{\beta-\alpha} H\right],$$

$$\begin{aligned}\mathscr{L}_{VHH} = {} & \frac{g}{2c_W} A Z^\mu \overleftrightarrow{\partial}_\mu \left(c_{\beta-\alpha} h - s_{\beta-\alpha} H\right) + \frac{g}{2c_W} Z^\mu G \overleftrightarrow{\partial}_\mu \left(s_{\beta-\alpha} h + c_{\beta-\alpha} H\right) \\ & - \tfrac{1}{2} g \Big\{ i W_\mu^+ \Big[G^- \overleftrightarrow{\partial}^\mu \left(s_{\beta-\alpha} h + c_{\beta-\alpha} H\right) \\ & \qquad + H^- \overleftrightarrow{\partial}^\mu \left(c_{\beta-\alpha} h - s_{\beta-\alpha} H + iA\right)\Big] + \text{h.c.}\Big\},\end{aligned}$$

$$\begin{aligned}\mathscr{L}_{VVHH} = {} & \left[\tfrac{1}{4} g^2 W_\mu^+ W^{\mu\,-} + \frac{g^2}{8c_W^2} Z_\mu Z^\mu\right](hh + HH + AA) \\ & + \Big\{\left(\tfrac{1}{2} e g A^\mu W_\mu^+ - \frac{g^2 s_W^2}{2c_W} Z^\mu W_\mu^+\right)\Big[G^-(s_{\beta-\alpha} h + c_{\beta-\alpha} H) \\ & \qquad + H^-(c_{\beta-\alpha} h - s_{\beta-\alpha} H + iA)\Big] + \text{h.c.}\Big\}.\end{aligned}$$

Yukawa couplings of the CP-conserving 2HDM

It is straightforward to derive the Yukawa couplings of the physical Higgs bosons in the CP-conserving 2HDM.

$$
\begin{aligned}
\mathscr{L}_Y = & -\frac{1}{v}(\overline{U}\boldsymbol{M_U}U + \overline{D}\boldsymbol{M_D}D + \overline{E}\boldsymbol{M_E}E)(hs_{\beta-\alpha} + Hc_{\beta-\alpha}) \\
& -\frac{1}{\sqrt{2}}\varepsilon\big[\overline{U}\big(\boldsymbol{\rho^U}P_R + \boldsymbol{\rho^{U\dagger}}P_L\big)U + \overline{D}\big(\boldsymbol{\rho^D}P_L + \boldsymbol{\rho^{D\dagger}}P_R\big)D \\
& \qquad\qquad + \overline{E}\big(\boldsymbol{\rho^E}P_L + \boldsymbol{\rho^{E\dagger}}P_R\big)E\big](hc_{\beta-\alpha} - Hs_{\beta-\alpha}) \\
& -\frac{i}{\sqrt{2}}\varepsilon\big[\overline{U}(\boldsymbol{\rho^{U\dagger}}P_L - \boldsymbol{\rho^U}P_R)U + (\overline{D}\boldsymbol{\rho^{D\dagger}}P_R - \boldsymbol{\rho^D}P_L)D \\
& \qquad\qquad + \overline{E}(\boldsymbol{\rho^{E\dagger}}P_R - \boldsymbol{\rho^E}P_L)E\big]A \\
& -\varepsilon\Big\{\overline{U}\Big[\boldsymbol{K}\boldsymbol{\rho^{D\dagger}}P_R - \boldsymbol{\rho^{U\dagger}}\boldsymbol{K}P_L\Big]DH^+ + \overline{N}\boldsymbol{\rho^{E\dagger}}P_R E H^+ + \text{h.c.}\Big\}.
\end{aligned}
$$

If we impose the discrete $\mathbb{Z}_2$ symmetry to eliminate tree-level Higgs-mediated FCNCs, one obtains the following relations[12]

$$\text{Type I:} \quad \boldsymbol{\rho}^{\boldsymbol{U}} = \frac{\sqrt{2}\boldsymbol{M}_{\boldsymbol{U}}\,\varepsilon\cot\beta}{v}, \qquad \boldsymbol{\rho}^{\boldsymbol{D}} = \frac{\sqrt{2}\boldsymbol{M}_{\boldsymbol{D}}\,\varepsilon\cot\beta}{v},$$

$$\text{Type II:} \quad \boldsymbol{\rho}^{\boldsymbol{U}} = \frac{\sqrt{2}\boldsymbol{M}_{\boldsymbol{U}}\,\varepsilon\cot\beta}{v}, \qquad \boldsymbol{\rho}^{\boldsymbol{D}} = -\frac{\sqrt{2}\boldsymbol{M}_{\boldsymbol{D}}\,\varepsilon\tan\beta}{v}.$$

thereby promoting $\varepsilon\tan\beta$ to a physical parameter.

Plugging corresponding $\boldsymbol{\rho}^{\boldsymbol{U}}$ and $\boldsymbol{\rho}^{\boldsymbol{D}}$ into our previous formulae, one can derive the Type-I and Type-II Yukawa couplings of the CP-conserving 2HDM. For example,

[12] To obtain $\boldsymbol{\rho}^{\boldsymbol{E}}$, replace $\boldsymbol{D}$ with $\boldsymbol{E}$ in the formulae above.

$$
\begin{aligned}
\mathscr{L}_{II} = -\frac{h}{v}\Big\{ &\big(s_{\beta-\alpha} + c_{\beta-\alpha}\cot\beta\big)\big(\overline{U}\boldsymbol{M_U}U + \text{h.c.}\big) \\
&+ \big(s_{\beta-\alpha} - c_{\beta-\alpha}\tan\beta\big)\big(\overline{D}\boldsymbol{M_D}D + \overline{E}\boldsymbol{M_E}E + \text{h.c.}\big)\Big\} \\
-\frac{H}{v}\Big\{ &\big(c_{\beta-\alpha} - s_{\beta-\alpha}\cot\beta\big)\big(\overline{U}\boldsymbol{M_U}U + \text{h.c.}\big) \\
&+ \big(c_{\beta-\alpha} + s_{\beta-\alpha}\tan\beta\big)\big(\overline{D}\boldsymbol{M_D}D + \overline{E}\boldsymbol{M_E}E + \text{h.c.}\big)\Big\} \\
+ i\frac{A}{v}\Big\{ &\cot\beta\,\overline{U}\boldsymbol{M_U}\gamma_5 U + \tan\beta\big(\overline{D}\boldsymbol{M_D}\gamma_5 D + \overline{E}\boldsymbol{M_E}\gamma_5 E\big) + \text{h.c.}\Big\} \\
+ \frac{\sqrt{2}}{v}\Big\{ &H^+\Big[\overline{U}\big(\boldsymbol{M_U}\boldsymbol{K}P_L\cot\beta + \boldsymbol{K}\boldsymbol{M_D}P_R\tan\beta\big)D + \overline{N}\boldsymbol{M_E}P_R\tan\beta\, E\Big] + \text{h.c.}\Big\}.
\end{aligned}
$$

REMARK: Note that $\mathscr{L}_{II}$ is invariant under $\Phi_i \to \mathcal{R}_i{}^j\Phi_j$ with $\det\mathcal{R} = \pm 1$. In the 2HDM literature, it is conventional to restrict $\det\mathcal{R} = +1$ by taking the Higgs vevs, or equivalently $\tan\beta$, nonnegative (i.e., $0 \leq \beta \leq \frac{1}{2}\pi$), in which case ε is fixed by the sign of $c_{\beta-\alpha}$ [recall that $\varepsilon c_{\beta-\alpha} \leq 0$].

The MSSM Higgs Sector

Tree-level MSSM Higgs sector

The tree-level Higgs sector of the MSSM is a CP-conserving Type-II 2HDM, with a scalar potential with quartic terms constrained by supersymmetry. It is convenient to define

$$H_{di} \equiv \epsilon_{ij}\Phi_1^j = \left((\Phi_1^0)^\dagger\,,\,-\Phi_1^-\right)\,, \qquad\qquad H_{ui} = \Phi_{2i} = \left(\Phi_2^+\,,\,\Phi_2^0\right)\,,$$

where i and j are SU(2) indices and $\Phi_1^j \equiv (\Phi_{1j})^\dagger$. Then the MSSM scalar Higgs potential is given by

$$\begin{aligned}\mathcal{V} &= M_d^2 H_d^\dagger H_d + M_u^2 H_u^\dagger H_u + (M_{ud}^2 \epsilon^{ij} H_{ui} H_{dj} + \text{h.c.}) \\ &\quad + \tfrac{1}{8}(g^2 + g'^{\,2})(H_u^\dagger H_u - H_d^\dagger H_d)^2 + \tfrac{1}{2} g^2 |H_d^\dagger H_u|^2\,,\end{aligned}$$

where $M_d^2 \equiv |\mu|^2 + m_{H_d}^2$, $M_u^2 \equiv |\mu|^2 + m_{H_u}^2$, and $M_{ud}^2 \equiv b$ [cf. Stephen Martin's lectures].

In particular,

$$\epsilon^{ij} H_{ui} H_{dj} = H_u^+ H_d^- - H_u^0 H_d^0 = -\Phi_1^\dagger \Phi_2 \,.$$

The quartic Higgs couplings are related to the electroweak gauge couplings g and g':

$$\lambda_1 = \lambda_2 = -\lambda_3 - \lambda_4 = \tfrac{1}{4}(g^2 + g'^{\,2})\,, \quad \lambda_4 = -\tfrac{1}{2} g^2\,, \quad \lambda_5 = \lambda_6 = \lambda_7 = 0\,.$$

The Φ-basis, where the above relations satisfied, corresponds to the scalar field basis in which the supersymmetry of the dimension-four terms of the scalar potential is manifestly realized. The supersymmetry is softly broken by the scalar squared-mass parameters, $m^2_{H_d}$, $m^2_{H_u}$, and b.

REMARK: Note that M^2_{ud}, the only potentially complex parameter that appears in the scalar potential, can be chosen real by an appropriate rephasing of the Higgs doublet fields, which defines a real scalar field basis.

In the real scalar field basis, the minimum of the Higgs scalar potential is

$$\langle H_d^0 \rangle = \frac{v_d}{\sqrt{2}} = \frac{v\cos\beta}{\sqrt{2}}\,, \qquad\qquad \langle H_u^0 \rangle = \frac{v_u}{\sqrt{2}} = \frac{v\sin\beta}{\sqrt{2}}\,,$$

where v_d and v_u are real, with $v \equiv (v_d^2 + v_u^2)^{1/2} \simeq 246$ GeV. Consequently, the tree-level MSSM Higgs scalar potential and vacuum are CP-conserving. Moreover, one can redefine $H_d \to -H_d$ or $H_u \to -H_u$ (if necessary) such that v_d and v_u are nonnegative. In this case, the parameter $\tan\beta \equiv v_u/v_d$ is nonnegative and $0 \leq \beta \leq \frac{1}{2}\pi$. One can now transform to a real Higgs basis where

$$Y_1 = -\tfrac{1}{2}Z_1 v^2\,, \qquad Y_2 = m_A^2 + \tfrac{1}{8}(g^2 + g'^{\,2})v^2\cos^2 2\beta\,,$$

$$Y_3 = -\tfrac{1}{2}Z_6 v^2\,, \qquad Z_1 = Z_2 = \tfrac{1}{4}(g^2 + g'^{\,2})\cos^2 2\beta\,,$$

$$Z_3 = Z_5 + \tfrac{1}{4}(g^2 - g'^{\,2})\,, \qquad Z_4 = Z_5 - \tfrac{1}{2}g^2\,,$$

$$Z_5 = \tfrac{1}{4}(g^2 + g'^{\,2})\sin^2 2\beta\,, \qquad Z_7 = -Z_6 = \tfrac{1}{4}(g^2 + g'^{\,2})\sin 2\beta\cos 2\beta\,.$$

The properties of the tree-level MSSM Higgs sector can now be derived using the results previously obtained in this lecture. For example, the following tree-level mass bounds are satisfied:

$$m_h^2 \leq \min\{m_Z^2 \cos^2 2\beta\,,\, m_A^2 + m_Z^2 \sin^2 2\beta\}\,,$$

$$m_H^2 \geq \max\{m_Z^2 \cos^2 2\beta\,,\, m_A^2 + m_Z^2 \sin^2 2\beta\}\,.$$

In particular, $m_h \leq m_Z$, in conflict with the observed Higgs boson mass of 125 GeV. We will see shortly that the radiative corrections to above inequalities are significant in the MSSM, and parameter regimes exist in which the upper bound on the mass m_h can be raised to a value above 125 GeV, thereby restoring the consistency with the observed Higgs boson data.

The tree-level properties of the MSSM Higgs sector can be rederived directly in the scalar field basis where supersymmetry is manifestly realized. One immediately identifies the charged Higgs bosons and the CP-odd neutral scalar,

$$H^\pm = H_d^\pm \sin\beta + H_u^\pm \cos\beta\,,$$
$$A^0 = \sqrt{2}\left(\mathrm{Im}\, H_d^0 \sin\beta + \mathrm{Im}\, H_u^0 \cos\beta\right)\,.$$

Likewise, the two CP-even neutral scalars h and H,

$$h^0 = -(\sqrt{2}\,\mathrm{Re}\, H_d^0 - v_d)\sin\alpha + (\sqrt{2}\,\mathrm{Re}\, H_u^0 - v_u)\cos\alpha\,,$$
$$H^0 = (\sqrt{2}\,\mathrm{Re}\, H_d^0 - v_d)\cos\alpha + (\sqrt{2}\,\mathrm{Re}\, H_u^0 - v_u)\sin\alpha\,.$$

are obtained by diagonalizing the CP-even scalar squared-mass matrix with respect to the basis $\{\sqrt{2}\,\mathrm{Re}\, H_d^0 - v_d\,,\ \sqrt{2}\,\mathrm{Re}\, H_u^0 - v_u\}$

$$\mathcal{M}^2 = \begin{pmatrix} m_A^2 \sin^2\beta + m_Z^2 \cos^2\beta & -(m_A^2 + m_Z^2)\sin\beta\cos\beta \\ -(m_A^2 + m_Z^2)\sin\beta\cos\beta & m_A^2 \cos^2\beta + m_Z^2 \sin^2\beta \end{pmatrix}\,.$$

All scalar masses and couplings can be expressed in terms of two parameters, usually chosen to be m_A and $\tan\beta$. The masses of the neutral CP-odd and charged Higgs bosons are given by

$$m_A^2 = \frac{2M_{ud}^2}{\sin 2\beta} = M_d^2 + M_u^2\,,$$

after using the scalar potential minimum conditions, and

$$m_{H^\pm}^2 = m_A^2 + m_W^2\,.$$

The squared masses of the CP-even Higgs bosons h^0 and H^0 are eigenvalues of $\mathcal{M}^2$. The trace and determinant of $\mathcal{M}^2$ yield

$$m_h^2 + m_H^2 = m_A^2 + m_Z^2\,, \qquad\qquad m_h^2 m_H^2 = m_A^2 m_Z^2 \cos^2 2\beta\,,$$

where the CP-even Higgs squared masses are given by:

$$m_{H,h}^2 = \tfrac{1}{2}\left(m_A^2 + m_Z^2 \pm \sqrt{(m_A^2 + m_Z^2)^2 - 4m_Z^2 m_A^2 \cos^2 2\beta}\,\right)\,.$$

It is standard practice to choose the mixing angle α to lie in the range $|\alpha| \leq \frac{1}{2}\pi$. However, because the off-diagonal element of $\mathcal{M}^2$ is negative, it follows that $-\frac{1}{2}\pi \leq \alpha \leq 0$. Hence, $0 \leq \beta - \alpha \leq \pi$. The following formulae are easily derived:

$$\cos\alpha = \sqrt{\frac{m_A^2 \sin^2\beta + m_Z^2 \cos^2\beta - m_h^2}{m_H^2 - m_h^2}}\,,$$

$$\sin\alpha = -\sqrt{\frac{m_H^2 - m_A^2 \sin^2\beta - m_Z^2 \cos^2\beta}{m_H^2 - m_h^2}}\,.$$

$$\cos(\beta - \alpha) = \frac{m_Z^2 \sin 2\beta \cos 2\beta}{\sqrt{(m_H^2 - m_h^2)(m_H^2 - m_Z^2 \cos^2 2\beta)}}\,,$$

$$\sin(\beta - \alpha) = \sqrt{\frac{m_H^2 - m_Z^2 \cos^2 2\beta}{m_H^2 - m_h^2}}\,.$$

The Higgs alignment limit is realized in the decoupling limit when $m_H \gg m_h$, which yields $|\cos(\beta - \alpha)| \ll 1$.

Yukawa couplings of the MSSM Higgs sector

The MSSM Higgs sector employs Type-II Higgs–fermion Yukawa couplings as a consequence of supersymmetry rather than a $\mathbb{Z}_2$ symmetry. Nevertheless, the dimension-four terms of the tree-level MSSM Higgs Lagrangian respect the $\mathbb{Z}_2$ symmetry defined by the Type-II $\mathbb{Z}_2$ charges previously given.[13] Hence, the tree-level MSSM Higgs–fermion Yukawa couplings are given by $\mathscr{L}_{II}$ of the CP-conserving 2HDM.

The tree-level Higgs couplings to charginos and neutralinos can also be derived following the recipe given in Stephen Martin's lectures.

[13] In the MSSM, this $\mathbb{Z}_2$ symmetry is softly broken due to the nonzero parameter M^2_{ud} in the scalar potential.

The One-Loop Corrected MSSM Higgs Masses

We begin by expanding the neutral components of the scalar Higgs fields are expanded around their vevs:

$$H^0_{d,u} \equiv \frac{h_{d,u} + i a_{d,u} + v_{d,u}}{\sqrt{2}}\,,$$

and plugging this result into the MSSM scalar Higgs potential,

$$\mathcal{V} = \mathcal{V}_0 + t_d h_d + t_u h_u + \tfrac{1}{2}(\mathcal{M}^2_e)_{ij} h_i h_j + \tfrac{1}{2}(\mathcal{M}^2_o)_{ij} a_i a_j + \cdots\,,$$

where repeated indices $i, j = d, u$ are summed over, and cubic or quartic terms in the scalar fields are not explicitly shown.

Explicitly, the linear (tadpole) terms in the scalar potential are given by

$$t_d \equiv \left.\frac{\partial\mathcal{V}}{\partial h_d}\right|_{h=a=0} = v_d\left(M_d^2 + \tfrac{1}{8}G^2(v_d^2 - v_u^2) - b\frac{v_u}{v_d}\right),$$

$$t_u \equiv \left.\frac{\partial\mathcal{V}}{\partial h_u}\right|_{h=a=0} = v_u\left(M_u^2 + \tfrac{1}{8}G^2(v_u^2 - v_d^2) - b\frac{v_d}{v_u}\right),$$

where $G^2 \equiv g^2 + g'^{\,2}$.

Likewise, the quadratic terms in the scalar fields yield 2×2 CP-even and CP-odd scalar squared-mass matrices [in the (h_d, h_u) basis]:

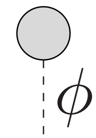
$$\mathcal{M}_e^2 \equiv \left.\frac{\partial^2 V}{\partial h_i \partial h_j}\right|_{h=a=0} = \begin{pmatrix} M_d^2 + \frac{1}{8}G^2(3v_d^2 - v_u^2) & -\frac{1}{4}G^2 v_u v_d - b \\ -\frac{1}{4}G^2 v_u v_d - b & M_u^2 + \frac{1}{8}G^2(3v_u^2 - v_d^2) \end{pmatrix},$$

$$\mathcal{M}_o^2 \equiv \frac{\partial^2 V}{\partial a_i \partial a_j}\bigg|_{h=a=0} = \begin{pmatrix} M_d^2 + \frac{1}{8}G^2(v_d^2 - v_u^2) & b \\ b & M_u^2 + \frac{1}{8}G^2(v_u^2 - v_d^2) \end{pmatrix}.$$

All parameters appearing in the above formulae should be interpreted as bare (unrenormalized) parameters. We ensure that $v_{u,d}$ are stationary points of the full one-loop effective potential by enforcing the tadpole cancellation condition:

$$-i(t_{d,u} + T_{d,u}) = 0\,,$$

where $-iT_{d,u}$ consist of the sum of all Feynman diagrams contributing to the one-point 1PI Green functions of h_d and h_u, respectively.



$$-iT_\phi$$

The sum of all one-loop tadpole graphs at zero external momentum contributing to the one-point 1PI Green function is denoted by $-iT_\phi$.

REMARK: For simplicity, we take the gaugino mass parameters, the μ parameter, and the A-terms to be real, thus neglecting potential CP-violating effects that could arise from CP-violating parameters in the sparticle sector. Under this assumption, there is no mixing at one loop between CP-even and CP-odd Higgs scalar eigenstates, and we can treat the analysis of the CP-even and CP-odd scalar squared-mass matrices separately.

Using the tadpole cancellation condition, the CP-odd scalar squared-mass matrix simplifies to

$$\mathcal{M}_o^2 = \begin{pmatrix} b\dfrac{v_u}{v_d} - \dfrac{T_d}{v_d} & b \\ b & b\dfrac{v_d}{v_u} - \dfrac{T_u}{v_u} \end{pmatrix}.$$

Diagonalizing $\mathcal{M}_o^2$ and expanding to leading order in $T_{u,d}$, the bare masses for the CP-odd scalar A and the Goldstone boson G are found:

$$m_A^2 = \frac{v^2}{v_u v_d} b - \frac{v_u^2}{v^2}\frac{T_d}{v_d} - \frac{v_d^2}{v^2}\frac{T_u}{v_u}\,, \qquad m_G^2 = -\frac{1}{v^2}\left(T_d v_d + T_u v_u\right)\,.$$

Solving for b, M_d^2 and M_u^2 and making use of the tadpole cancellation condition,

$$\begin{aligned}
b &= \left(\frac{v_u v_d}{v^2}\right) m_A^2 + \left(\frac{v_u}{v}\right)^4 \frac{T_d}{v_u} + \left(\frac{v_d}{v}\right)^4 \frac{T_u}{v_d}\,,\\
M_d^2 &= \left(\frac{v_u}{v}\right)^2 m_A^2 + \left[\left(\frac{v_u}{v}\right)^4 - 1\right]\frac{T_d}{v_d} + \left(\frac{v_d v_u}{v^2}\right)^2 \frac{T_u}{v_u} + \tfrac{1}{8}G^2(v_u^2 - v_d^2)\,,\\
M_u^2 &= \left(\frac{v_d}{v}\right)^2 m_A^2 + \left(\frac{v_u v_d}{v^2}\right)^2 \frac{T_d}{v_d} + \left[\left(\frac{v_d}{v}\right)^4 - 1\right]\frac{T_u}{v_u} - \tfrac{1}{8}G^2(v_u^2 - v_d^2)\,.
\end{aligned}$$

Inserting these results into $\mathcal{M}_e^2$, we obtain

$$\mathcal{M}_e^2 = \begin{pmatrix} \mathcal{M}_{dd}^2 & \mathcal{M}_{du}^2 \\ \mathcal{M}_{du}^2 & \mathcal{M}_{uu}^2 \end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{M}_{dd}^2 &= m_A^2 s_\beta^2 + m_Z^2 c_\beta^2 + \frac{T_d}{v_d}(s_\beta^4 - 1) + \frac{T_u}{v_u} s_\beta^2 c_\beta^2 \,, \\
\mathcal{M}_{uu}^2 &= m_A^2 c_\beta^2 + m_Z^2 s_\beta^2 + \frac{T_d}{v_d} s_\beta^2 c_\beta^2 + \frac{T_u}{v_u}(c_\beta^4 - 1) \,, \\
\mathcal{M}_{du}^2 &= -(m_A^2 + m_Z^2) s_\beta c_\beta - \frac{T_u}{v_u} c_\beta^3 s_\beta - \frac{T_d}{v_d} s_\beta^3 c_\beta \,,
\end{aligned}$$

with $m_Z^2 \equiv \frac{1}{4} G^2 v^2$.

The eigenvalues of $\mathcal{M}_e^2$ are the bare squared masses, m_H^2 and m_h^2, where

$$m_{H,h}^2 = \frac{1}{2}\left(\mathcal{M}_{dd}^2 + \mathcal{M}_{uu}^2 \pm \sqrt{(\mathcal{M}_{dd}^2 - \mathcal{M}_{uu}^2)^2 + 4\left[\mathcal{M}_{du}^2\right]^2}\right).$$

It is noteworthy that the tree-level sum rule,

$$\mathrm{Tr}\mathcal{M}_e^2 = m_Z^2 + \mathrm{Tr}\mathcal{M}_o^2\,,$$

still holds when $v_{u,d}$ are stationary points of the full one-loop effective potential. In particular, one can check that

$$m_h^2 + m_H^2 = m_Z^2 + m_A^2 + m_G^2\,,$$

where $m_h^2 + m_H^2 = \mathcal{M}_{dd}^2 + \mathcal{M}_{uu}^2$ and $m_G^2 = -\left(T_d v_d + T_u v_u\right)/v^2$.

We can extend the above analysis to include the charged Higgs boson and Goldstone boson fields. Starting from the MSSM Higgs scalar potential, one can identify the terms that are quadratic in the charged scalar fields by replacing $H^0_{d,u}$ with their vacuum expectation values, $\langle H^0_{d,u}\rangle = v_{d,u}/\sqrt{2}$:

$$V \supset (\mathcal{M}^2_{\pm})_{ij} H_i^+ H_j^- \,,$$

where repeated indices $i, j = d, u$ are summed over and

$$\mathcal{M}^2_{\pm} = \begin{pmatrix} M_d^2 + \frac{1}{4}g^2 v_u^2 + \frac{1}{8}G^2(v_d^2 - v_u^2) & b + \frac{1}{4}g^2 v_u v_d \\ b + \frac{1}{4}g^2 v_u v_d & M_u^2 + \frac{1}{4}g^2 v_d^2 + \frac{1}{8}G^2(v_u^2 - v_d^2) \end{pmatrix} .$$

We can eliminate M_d^2 and M_u^2 via the tadpole cancellation equation.

We then end up with

$$\mathcal{M}_{\pm}^2 = \begin{pmatrix} \left(b + \frac{1}{4}g^2 v_u v_d\right)\dfrac{v_u}{v_d} - \dfrac{T_d}{v_d} & b + \frac{1}{4}g^2 v_u v_d \\ b + \frac{1}{4}g^2 v_u v_d & \left(b + \frac{1}{4}g^2 v_u v_d\right)\dfrac{v_d}{v_u} - \dfrac{T_u}{v_u} \end{pmatrix}.$$

Comparing with our previous expressions for m_A^2 and m_G^2, it immediately follows that

$$m_{H^\pm}^2 = m_A^2 + m_W^2\,, \qquad\qquad m_{G^\pm}^2 = m_G^2\,,$$

after using $m_W^2 = \frac{1}{4}g^2 v^2$.

It is convenient to replace the bare masses (denoted by a lower case m) by physical masses (denoted by an upper case M) in the one-loop approximation:

$$m_\phi^2 = M_\phi^2 - \mathrm{Re}\,\Sigma_{\phi\phi}(M_\phi^2)\,, \quad \text{for } \phi = h, H, A, H^\pm\,,$$
$$m_V^2 = M_V^2 - \mathrm{Re}\,A_{VV}(M_V^2)\,, \quad \text{for } V = W^\pm, Z\,,$$

where $-i\Sigma_{\phi\phi}$ is the sum of all one-particle irreducible, connected Feynman diagams contributing to the self-energy of the scalar field ϕ, and the external legs are amputated, and A_{VV} is the coefficient of $g_{\mu\nu}$ that appears in the self-energy of the vector boson V.

Although the physical Higgs masses are gauge invariant quantities, it is convenient to work in the Landau gauge where the gauge parameter $\xi = 0$ and the Goldstone boson pole masses are zero. Thus, evaluating the equation for m_ϕ^2 with $\phi = G$ and $G^\pm$, respectively, with $M_G = M_{G^\pm} = 0$, it follows that[14]

$$m_G^2 = M_G^2 - \Sigma_{GG}(0) = -\Sigma_{GG}(0)\,,$$
$$m_{G^\pm}^2 = M_{G^\pm}^2 - \Sigma_{G^+G^-}(0) = -\Sigma_{G^+G^-}(0)\,,$$

which implies that

$$\Sigma_{GG}(0) = \Sigma_{G^+G^-}(0) = \frac{T_d c_\beta + T_u s_\beta}{v}\,.$$

[14]Note that the absorptive parts of $\Sigma_{GG}(0)$ and $\Sigma_{G^+G^-}(0)$ are zero. Thus in the CP-conserving limit, $\Sigma_{GG}(0)$ and $\Sigma_{G^+G^-}(0)$ are both real quantities.

Working to one-loop accuracy, we end up with:

$$\begin{aligned} M_{H^\pm}^2 = M_W^2 + M_A^2 + \mathrm{Re}\,\Sigma_{H^+H^-}(M_W^2 + M_A^2) \\ - \mathrm{Re}\,A_{WW}(M_W^2) - \mathrm{Re}\,\Sigma_{AA}(M_A^2)\,, \end{aligned}$$

since $\Sigma_{H^+H^-}(M_W^2 + M_A^2)$ differs from $\Sigma_{H^+H^-}(M_{H^\pm}^2)$ by terms of two-loop order in perturbation theory. To complete the computation, one must explicitly evaluate the contributions of the MSSM particle spectrum to the three one-loop self-energy functions that appear in the equation above.

In contrast to the one-loop computation of $m_{H^\pm}$, the tree-level expressions for the squared masses of the CP-even neutral Higgs bosons depend on $\tan\beta$. Consequently, the counterterms associated with the parameters v_u and v_d are now relevant.

The renormalized VEVs are given in terms of the scalar wave function renormalization constants, at one-loop accuracy, by

$$v_{d,r} = Z_{H_d}^{-1/2} v_d = v_d\big(1 - \tfrac{1}{2}\delta Z_{H_d}\big)\,,$$
$$v_{u,r} = Z_{H_u}^{-1/2} v_u = v_u\big(1 - \tfrac{1}{2}\delta Z_{H_u}\big)\,,$$

and the counterterms for the vevs are defined by

$$\delta v_d \equiv v_{d,r} - v_d = -\tfrac{1}{2} v_d \delta Z_{H_d}\,, \qquad \delta v_u \equiv v_{u,r} - v_u = -\tfrac{1}{2} v_u \delta Z_{H_u}\,.$$

The neutral Higgs masses depend on the bare parameter $\tan\beta$, which can be replaced by a renormalized parameter and a counterterm,

$$\tan\beta \to \tan\beta - \delta\tan\beta\ ,$$

where

$$\frac{\delta\tan\beta}{\tan\beta} = \frac{v_d}{v_u}\,\delta\left(\frac{v_u}{v_d}\right) = \frac{\delta v_u}{v_u} - \frac{\delta v_d}{v_d} = \tfrac{1}{2}\big(\delta Z_{H_d} - \delta Z_{H_u}\big)\ .$$

Likewise, we can express the shifts of the parameters s_β and c_β in terms of $\delta\tan\beta$:

$$s_\beta \to s_\beta - \delta s_\beta = s_\beta - c_\beta^3\,\delta\tan\beta\,,$$

$$c_\beta \to c_\beta - \delta c_\beta = c_\beta + c_\beta^2 s_\beta\,\delta\tan\beta\,.$$

Using the above results,

$$\mathcal{M}^2_{dd} = M_A^2 s_\beta^2 + M_Z^2 c_\beta^2 + \delta\mathcal{M}^2_{dd}\,,$$
$$\mathcal{M}^2_{uu} = M_A^2 c_\beta^2 + M_Z^2 s_\beta^2 + \delta\mathcal{M}^2_{uu}\,,$$
$$\mathcal{M}^2_{du} = -(M_A^2 + M_Z^2) s_\beta c_\beta + \delta\mathcal{M}^2_{du}\,,$$

where β is the one-loop renormalized parameter and

$$\begin{aligned}
\delta\mathcal{M}^2_{dd} &= -\mathrm{Re}\,\Sigma_{AA}(M_A^2) s_\beta^2 - \mathrm{Re}\,A_{ZZ}(M_Z^2) c_\beta^2 + \frac{T_d}{v_d}(s_\beta^4 - 1) + \frac{T_u}{v_u} s_\beta^2 c_\beta^2 \\
&\quad - 2 s_\beta c_\beta^3 (M_A^2 - M_Z^2)\delta\tan\beta\,, \\
\delta\mathcal{M}^2_{uu} &= -\mathrm{Re}\,\Sigma_{AA}(M_A^2) c_\beta^2 - \mathrm{Re}\,A_{ZZ}(M_Z^2) s_\beta^2 + \frac{T_d}{v_d} s_\beta^2 c_\beta^2 + \frac{T_u}{v_u}(c_\beta^4 - 1) \\
&\quad + 2 s_\beta c_\beta^3 (M_A^2 - M_Z^2)\delta\tan\beta\,, \\
\delta\mathcal{M}^2_{du} &= \left[\mathrm{Re}\,\Sigma_{AA}(M_A^2) + \mathrm{Re}\,A_{ZZ}(M_Z^2)\right] s_\beta c_\beta - \frac{T_d}{v_d} s_\beta^3 c_\beta - \frac{T_u}{v_u} c_\beta^3 s_\beta \\
&\quad + (M_A^2 + M_Z^2) c_\beta^2 c_{2\beta}\delta\tan\beta\,.
\end{aligned}$$

Using

$$m_H^2 = M_H^2 - \mathrm{Re}\,\Sigma_{HH}(M_H^2)\,,$$

$$m_h^2 = M_h^2 - \mathrm{Re}\,\Sigma_{hh}(M_h^2)\,,$$

one can perturbatively expand the expressions for m_H^2 and m_h^2 at one-loop accuracy and rewrite the bare squared-mass parameters in terms of physical (renormalized) parameters. In particular,

$$M_H^2 - \mathrm{Re}\,\Sigma_{HH}(\widehat{M}_H^2) = \widehat{M}_H^2 + \tfrac{1}{2}\left(\delta\mathcal{M}_{dd}^2 + \delta\mathcal{M}_{uu}^2\right) + \frac{(M_Z^2 - M_A^2)c_{2\beta}(\delta\mathcal{M}_{dd}^2 - \delta\mathcal{M}_{uu}^2) - 2(M_Z^2 + M_A^2)s_{2\beta}\delta\mathcal{M}_{du}^2}{2(\widehat{M}_H^2 - \widehat{M}_h^2)}\,,$$

$$M_h^2 - \mathrm{Re}\,\Sigma_{hh}(\widehat{M}_h^2) = \widehat{M}_h^2 + \tfrac{1}{2}\left(\delta\mathcal{M}_{dd}^2 + \delta\mathcal{M}_{uu}^2\right) - \frac{(M_Z^2 - M_A^2)c_{2\beta}(\delta\mathcal{M}_{dd}^2 - \delta\mathcal{M}_{uu}^2) - 2(M_Z^2 + M_A^2)s_{2\beta}\delta\mathcal{M}_{du}^2}{2(\widehat{M}_H^2 - \widehat{M}_h^2)}\,,$$

where $\widehat{M}_{H,h}^2 \equiv \dfrac{1}{2}\left(M_Z^2 + M_A^2 \pm \sqrt{(M_A^2 - M_Z^2)^2 + 4M_A^2 M_Z^2 s_{2\beta}^2}\right)$.

Note that $\widehat{M}^2_{H,h}$ are the eigenvalues of the tree-level CP-even Higgs boson squared-mass matrix with the bare parameters m_A, m_Z, and β replaced by the corresponding physical (renormalized) masses M_A and M_Z and the one-loop renormalized parameter β. One can also employ this squared-mass matrix to define the mixing angle α, which can be expressed in terms of M_A^2, M_Z^2, and the renormalized parameter β as follows:

$$\cos 2\alpha = \frac{(M_Z^2 - M_A^2)c_{2\beta}}{\widehat{M}_H^2 - \widehat{M}_h^2}\,, \qquad \sin 2\alpha = \frac{-(M_Z^2 + M_A^2)s_{2\beta}}{\widehat{M}_H^2 - \widehat{M}_h^2}\,.$$

Using the above expressions, one can derive the following useful identity:

$$M_A^2 \sin[2(\beta-\alpha)] = -M_Z^2 \sin[2(\beta+\alpha)]\,.$$

It then follows that

$$M_H^2 = \widehat{M}_H^2 + \mathrm{Re}\,\Sigma_{HH}(\widehat{M}_H^2) + \delta\mathcal{M}^2_{dd}\cos^2\alpha + \delta\mathcal{M}^2_{uu}\sin^2\alpha + \delta\mathcal{M}^2_{du}\sin 2\alpha\,,$$
$$M_h^2 = \widehat{M}_h^2 + \mathrm{Re}\,\Sigma_{hh}(\widehat{M}_h^2) + \delta\mathcal{M}^2_{dd}\sin^2\alpha + \delta\mathcal{M}^2_{uu}\cos^2\alpha - \delta\mathcal{M}^2_{du}\sin 2\alpha\,.$$

Plugging in the expressions previously obtained for $\delta\mathcal{M}^2_{dd}$, $\delta\mathcal{M}^2_{uu}$, and $\delta\mathcal{M}^2_{du}$, into the above equations, we obtain

$$\begin{aligned}
M_H^2 &= \widehat{M}_H^2 + \mathrm{Re}\,\Sigma_{HH}(\widehat{M}_H^2) - \cos^2(\beta+\alpha)\,\mathrm{Re}\,A_{ZZ}(M_Z^2) - s^2_{\beta-\alpha}\,\mathrm{Re}\,\Sigma_{AA}(M_A^2) \\
&\quad + \frac{T_d}{v_d}\big[s^2_\beta s^2_{\beta-\alpha} - \cos^2\alpha\big] + \frac{T_u}{v_u}\big[c^2_\beta s^2_{\beta-\alpha} - \sin^2\alpha\big] + 2m_Z^2 c_\beta^2 \sin[2(\beta+\alpha)]\delta\tan\beta\,, \\
M_h^2 &= \widehat{M}_h^2 + \mathrm{Re}\,\Sigma_{hh}(\widehat{M}_h^2) - \sin^2(\beta+\alpha)\,\mathrm{Re}\,A_{ZZ}(M_Z^2) - c^2_{\beta-\alpha}\,\mathrm{Re}\,\Sigma_{AA}(M_A^2) \\
&\quad + \frac{T_d}{v_d}\big[s^2_\beta c^2_{\beta-\alpha} - \sin^2\alpha\big] + \frac{T_u}{v_u}\big[c^2_\beta c^2_{\beta-\alpha} - \cos^2\alpha\big] - 2m_Z^2 c_\beta^2 \sin[2(\beta+\alpha)]\delta\tan\beta\,.
\end{aligned}$$

It is convenient to evaluate the one-loop tadpole functions with respect to the neutral CP-even Higgs boson mass basis:[15]

$$T_H \equiv T_u\sin\alpha + T_d\cos\alpha\,, \qquad\qquad T_h \equiv T_u\cos\alpha - T_d\sin\alpha\,.$$

[15] Since T_u and T_d are one-loop quantities, it is consistent to define T_h and T_H at one-loop accuracy by employing the mixing angle α whose definition is based on tree-level relations.

One can then rewrite the expressions for M_H^2 and m_h^2 in a more useful form,

$$M_H^2 = \widehat{M}_H^2 + \mathrm{Re}\,\Sigma_{HH}(\widehat{M}_H^2) - \cos^2(\beta+\alpha)\,\mathrm{Re}\,A_{ZZ}(M_Z^2) - s_{\beta-\alpha}^2\,\mathrm{Re}\,\Sigma_{AA}(M_A^2)$$
$$+ c_{\beta-\alpha}^2\Sigma_{GG}(0) - 2c_{\beta-\alpha}\frac{T_H}{v} + 2m_Z^2 c_\beta^2 \sin[2(\beta+\alpha)]\delta\tan\beta\,,$$
$$M_h^2 = \widehat{M}_h^2 + \mathrm{Re}\,\Sigma_{hh}(\widehat{M}_h^2) - \sin^2(\beta+\alpha)\,\mathrm{Re}\,A_{ZZ}(M_Z^2) - c_{\beta-\alpha}^2\,\mathrm{Re}\,\Sigma_{AA}(M_A^2)$$
$$+ s_{\beta-\alpha}^2\Sigma_{GG}(0) - 2s_{\beta-\alpha}\frac{T_h}{v} - 2m_Z^2 c_\beta^2 \sin[2(\beta+\alpha)]\delta\tan\beta\,,$$

where

$$\Sigma_{GG}(0) = \frac{1}{v}\left[T_H c_{\beta-\alpha} + T_h s_{\beta-\alpha}\right].$$

One also obtains the one-loop correction to the tree-level squared-mass sum rule of the MSSM Higgs sector,

$$M_h^2 + M_H^2 = M_A^2 + M_Z^2 + \mathrm{Re}\,\Sigma_{hh}(\widehat{M}_h^2) + \mathrm{Re}\,\Sigma_{HH}(\widehat{M}_H^2) - \mathrm{Re}\,\Sigma_{AA}(M_A^2)$$
$$- \mathrm{Re}\,A_{ZZ}(M_Z^2) - \Sigma_{GG}(0)\,.$$

A notable prediction of the MSSM is that the tree-level mass of the lightest CP-even Higgs boson is bounded from above, and its maximal value is achieved in the case of $\beta = \frac{1}{2}\pi$ and $M_A > M_Z$. In this limit, $v_d = 0$ and $v_u = v$, in which case $t_d = T_d = 0$ and there is no mixing of h_u and h_d (i.e., $\alpha = 0$). It then follows that $\widehat{M}_h = M_Z$ and $\widehat{M}_H = M_A$, and the expressions for M_h^2 and M_H^2 simplify to

$$M_h^2 = M_Z^2 + \mathrm{Re}\,\Sigma_{hh}(M_Z^2) - \mathrm{Re}\,A_{ZZ}(M_Z^2) - \frac{T_h}{v}\,,$$
$$M_H^2 = M_A^2 + \mathrm{Re}\,\Sigma_{HH}(M_A^2) - \mathrm{Re}\,\Sigma_{AA}(M_A^2)\,,$$

independently of the value of $\delta\tan\beta$.

The MSSM Higgs Mass in the Decoupling Limit

In the Higgs decoupling limit where $M_A \gg M_Z$, it follows that $c_{\beta-\alpha} = 0$ and $s_{\beta-\alpha} = 1$. In this limit at one-loop accuracy[16]

$$
\begin{aligned}
M_h^2 &= c_{2\beta}^2\big[M_Z^2 - \mathrm{Re}\, A_{ZZ}(M_Z^2)\big] + \mathrm{Re}\,\Sigma_{hh}(M_Z^2 c_{2\beta}^2) - \frac{T_h}{v} + 4M_Z^2 c_\beta^2 s_{2\beta} c_{2\beta}\delta\tan\beta\,, \\
M_H^2 &= M_A^2 + s_{2\beta}^2\big[M_Z^2 - \mathrm{Re}\, A_{ZZ}(M_Z^2)\big] + \mathrm{Re}\,\Sigma_{HH}(M_A^2) - \mathrm{Re}\,\Sigma_{AA}(M_A^2) \\
&\qquad\qquad - 4M_Z^2 c_\beta^2 s_{2\beta} c_{2\beta}\delta\tan\beta\,.
\end{aligned}
$$

It is instructive to look at the leading contributions to the one-loop radiatively corrected mass of the SM-like Higgs boson of the MSSM. Numerically, the leading effect is due to the loop contributions of the top quarks and the supersymmetric top-quark partners. Because of the dependence on the couplings of the top quark and top squarks that depend on the Higgs–top-quark Yukawa coupling y_t, it is sufficient to evaluate the leading m_t^4 behavior of the self-energy functions that appear in the formulae above,

[16] At one-loop accuracy, one may replace $m_Z^2\,\delta\tan\beta$ with $M_Z^2\,\delta\tan\beta$, since $\delta\tan\beta$ is a one-loop quantity.

One can check that there are no terms that behave like m_t^4 in neglect the term in $A_{ZZ}(M_Z^2)$ and in the expression for $\delta \tan\beta$. Hence, we are left with extracting the leading m_t^4 behavior of

$$M_h^2 = M_Z^2 c_{2\beta}^2 + \mathrm{Re}\,\Sigma_{hh}(M_Z^2 c_{2\beta}^2) - \frac{T_h}{v}\,,$$

due to loops of top quarks and their supersymmetric scalar partners. At one-loop order in the limit of $M_Z \ll M_t \ll M_A, M_S$, where M_S is the geometric mean of the two top-squark squared masses, $M_S^2 \equiv m_{\widetilde{t}_1} m_{\widetilde{t}_2}$,

$$\boxed{M_h^2 \simeq M_Z^2 c_{2\beta}^2 + \frac{3g^2 m_t^4}{8\pi^2 m_W^2}\left[\ln\left(\frac{M_S^2}{m_t^2}\right) + \frac{X_t^2}{M_S^2}\left(1 - \frac{X_t^2}{12M_S^2}\right)\right]},$$

where $m_t X_t \equiv v(a_t s_\beta - \mu y_t c_\beta)/\sqrt{2}$ is the off-diagonal entry of the top-squark squared-mass matrix, and a_t and μ have been assumed to be real (for simplicity).

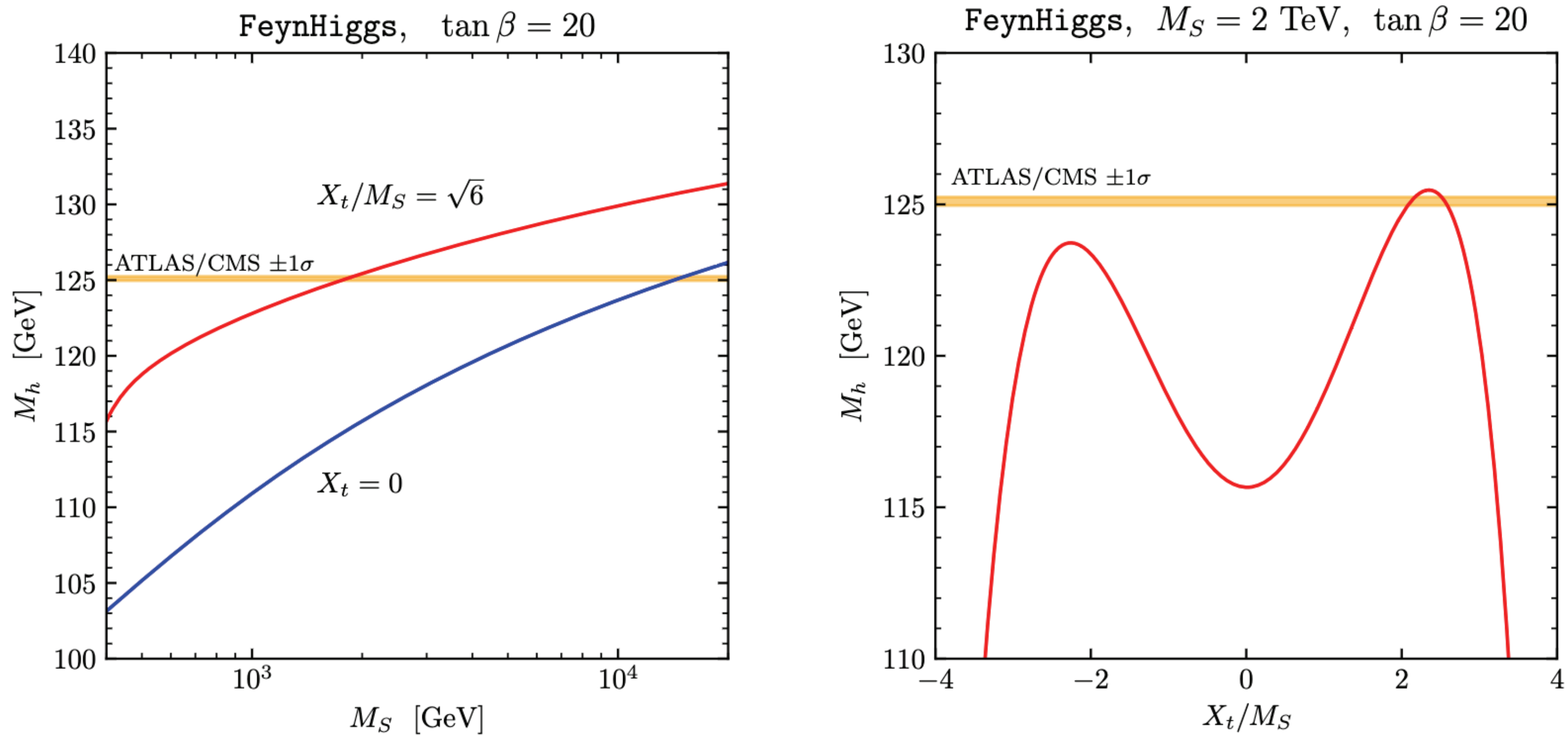


Figure 1: *The lighter CP-even Higgs mass in the MSSM as a function of a common SUSY mass parameter M_S and of the stop mixing parameter X_t (normalized to M_S). Both parameters are defined in the $\overline{DR}$ scheme at the scale $Q = M_S$.*

Taken from: from P. Slavich, S. Heinemeyer, et al., "Higgs-mass predictions in the MSSM and beyond," Eur. Phys. J. C **81**, 450 (2021) [arXiv:2012.15629 [hep-ph]]. This review article summarizes the efforts of the "Precision SUSY Higgs Mass Calculation Initiative" and represents the state of the art of the radiatively corrected MSSM Higgs sector.

The observed Higgs mass of 125 GeV suggests that if the MSSM is realized in Nature, then the effective scale of SUSY breaking (M_S) is likely to be on the heavy side (i.e., closer to 10 TeV) rather than of $\mathcal{O}(1\ \mathrm{TeV})$ as initially proposed for a solution to the hierarchy problem.

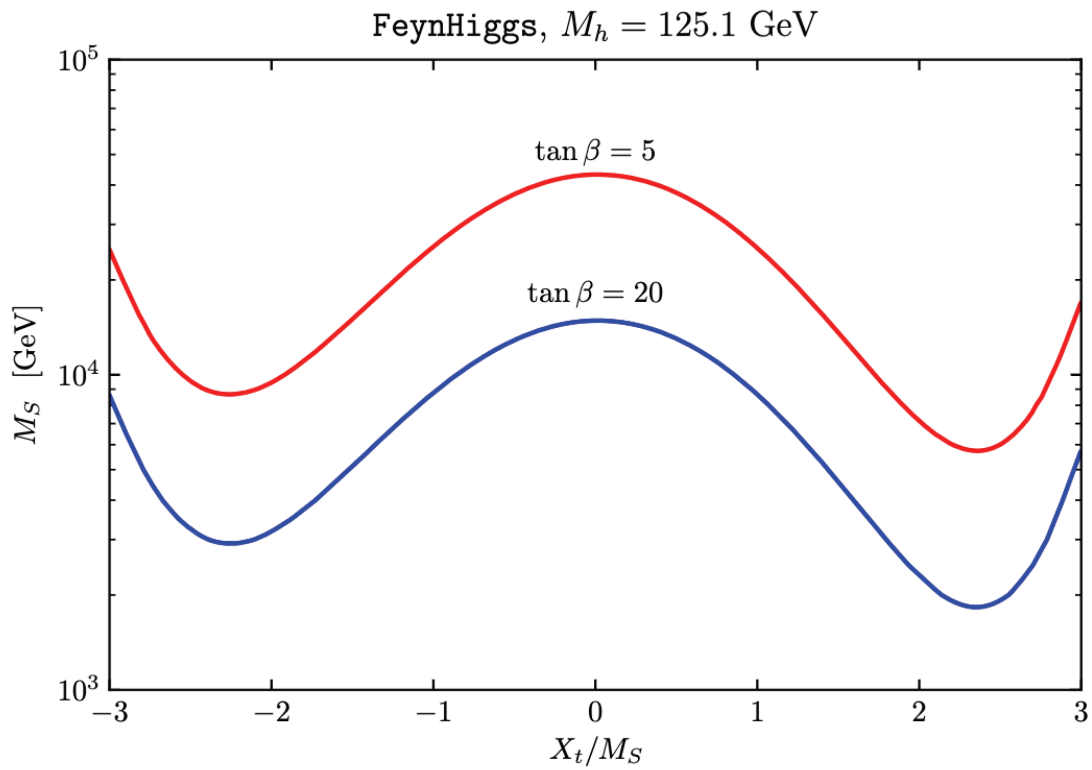


Figure 2: *Values of the SUSY mass parameter M_S and of the stop mixing parameter X_t (normalized to M_S) that lead to the prediction $M_h = 125.1$ GeV, in a simplified MSSM scenario with degenerate SUSY masses, for $\tan\beta = 20$ (blue) or $\tan\beta = 5$ (red).*

The MSSM Higgs Mass via the Renormalization Group

The leading logarithmic behavior of the radiatively corrected Higgs mass can be understood quite easily using the renormalization group equations (RGEs) of the SM. In the decoupling limit, there exists a scale M_S below which the effective field theory of the MSSM coincides with that of the SM. At the scale M_S, we can employ the MSSM relation $M_h^2 = M_Z^2 c_{2\beta}^2$. Equivalently, $\lambda(M_S) = \frac{1}{8}(g^2 + g'^{\,2})c_{2\beta}^2$, which serves as a boundary condition of the RGE for λ,

$$\frac{d\lambda}{dt} = \beta_\lambda \,, \qquad \text{where } t \equiv \ln\mu.$$

In first approximation, we can take the right-hand side of above equation to be independent of t, in which case

$$\lambda(m_t) = \lambda(M_S) - \tfrac{1}{2}\beta_\lambda \ln\left(\frac{M_S^2}{m_t^2}\right) \,.$$

The one-loop beta function for λ in the Standard Model (SM) is given by

$$16\pi^2\beta_\lambda = 24\lambda^2 + \tfrac{3}{8}\left[2g^4 + \left(g^2 + g'^2\right)^2\right] - 2\sum_i N_{c_i} y_i^4$$
$$-\lambda\left(9g^2 + 3g'^2 - 4\sum_i N_{c_i} y_i^2\right),$$

with $y_i = g m_{f_i}/(\sqrt{2} m_W)$ and $N_{ci} = 3$ [$N_{ci} = 1$] for quarks [charged leptons]. To obtain the leading logarithmic behavior of the radiatively corrected Higgs mass, it suffices to retain the term in β_λ that is proportional to y_t^4:

$$\beta_\lambda = -\frac{3y_t^4}{8\pi^2} = -\frac{3g^4 m_t^4}{32\pi^2 m_W^4}\,.$$

Finally, we can identify

$$M_h^2 = 2\lambda(m_t^2)v^2 = M_Z^2 c_{2\beta}^2 + \frac{3g^2 m_t^4}{8\pi^2 m_W^2}\ln\left(\frac{M_S^2}{m_t^2}\right)\,,$$

in agreement with the leading logarithmic behavior of the radiatively corrected Higgs mass.

The MSSM Wrong-Higgs Couplings

The tree-level MSSM Lagrangian consists of SUSY-conserving mass and interaction terms, supplemented by soft SUSY-breaking operators. In particular, all tree-level dimension-four gauge invariant interactions must respect supersymmetry.

When supersymmetry is broken, in principle all SUSY-breaking operators consistent with gauge invariance can be generated in the effective low-energy theory below the scale of SUSY breaking. The MSSM Higgs sector provides an especially illuminating example of this phenomenon.

In particular, if the masses of all the Higgs bosons lie below the SUSY-breaking scale M_S, then the low-energy effective theory below M_S, is the most general 2HDM.[17]

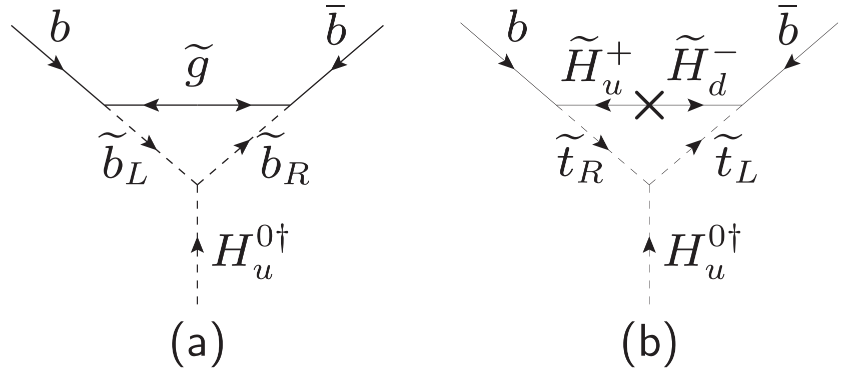
For simplicity, we will focus on the Higgs couplings to the third generation of quarks (neglecting the generation indices and the couplings to leptons). Using the MSSM Higgs field notation and the two-component spinor formalism, the 2HDM Yukawa Lagrangian (prior to imposing any symmetry constraints) is given by:

$$\begin{aligned}\mathscr{L}_Y = &- y_t(H_u^0 t\bar{t} - H_u^+ b\bar{t}) - w_t(H_d^{0\dagger} t\bar{t} + H_d^+ b\bar{t}) \\ &- y_b(H_d^0 b\bar{b} - H_d^- t\bar{b}) - w_b(H_u^{0\dagger} b\bar{b} + H_u^- t\bar{b}) + \text{h.c.}\end{aligned}$$

[17] Due to CP-violating effects generated by non-removable phases that may exist above M_S in the MSSM, the corresponding 2HDM scalar potential and Yukawa couplings may be CP-violating.

Imposing supersymmetry on the Yukawa Lagrangian implies that we must eliminate the nonholomorphic couplings by setting $w_t = w_b = 0$, which yields the Type-II Yukawa interactions.

Under the assumption that all SUSY particle masses (characterized by a mass scale M_S) are significantly heavier than the heaviest scalar of the Higgs sector, one can formally integrate out all the SUSY particles below the scale M_S. The resulting low-energy effective theory is the non-supersymmetric 2HDM. In this effective theory, the so-called wrong-Higgs Yukawa couplings, w_t and w_b, are nonzero.



One-loop MSSM contributions to the wrong-Higgs Yukawa couplings to $b\bar{b}$. In diagram (b), the $\times$ serves as a reminder that the exchanged charged higgsino is a Dirac fermion that is comprised of a pair of two-component fermions, $\widetilde{H}_u^+$ and $\widetilde{H}_d^-$.

The Feynman rule for the $H_u^{0\dagger} b\bar{b}$ vertex is $-iw_b$. The dominant contributions to this quantity are generated at one-loop order due to the two Feynman diagrams exhibited in the figure above.[18] We shall simplify the analysis by ignoring squark mixing, although a more complete calculation must take this into account since we will be assuming that μ, a_b, and a_t are nonzero. Finally, we shall ignore CP-violating effects by taking μ, a_b, and a_t and M_3 to be real parameters. In what follows, we shall first assume that μ and M_3 are positive real parameters (a condition we shall later relax).

[18]We shall neglect subdominant corrections to w_b/y_b that are proportional to y_b, g^2, and $g'^{\,2}$.

We employ Feynman rules obtained from the following interaction Lagrangians. First, the gluino–squark–quark Lagrangian is given by

$$\mathscr{L}_{\rm int} = -\sqrt{2} g_s (\boldsymbol{T^a})_j{}^k \sum_q \left[\widetilde{g}_a q_k \, \widetilde{q}_L^{\dagger j} + \widetilde{g}_a^\dagger q^{\dagger j} \, \widetilde{q}_{Lk} - \widetilde{g}_a \bar{q}^j \, \widetilde{q}_{Rk} - \widetilde{g}_a^\dagger \bar{q}_k^\dagger \, \widetilde{q}_R^{\dagger j} \right],$$

where the squark fields are taken to be in the same basis as the quarks.

Second, the couplings of Higgs bosons to squarks are given by

$$\begin{aligned} \mathscr{L}_{H\widetilde{q}\widetilde{q}} = {} & \mu \big[y_t (\widetilde{t}_L^\dagger \widetilde{t}_R H_d^0 + \widetilde{b}_L^\dagger \widetilde{t}_R H_d^-) + y_b (\widetilde{b}_L^\dagger \widetilde{b}_R H_u^0 + \widetilde{t}_L^\dagger \widetilde{b}_R H_u^+) \big] \\ & - a_t \widetilde{t}_R^\dagger (\widetilde{t}_L H_u^0 - \widetilde{b}_L H_u^+) - a_b \widetilde{b}_R^\dagger (\widetilde{b}_L H_d^0 - \widetilde{t}_L H_d^-) + \text{h.c.} \end{aligned}$$

Third, the higgsino couplings to $q\widetilde{q}$ are given by:

$$\begin{aligned} \mathscr{L}_{\widetilde{H} q \widetilde{q}} = & - y_t \big[\widetilde{H}_u^0 (t \widetilde{t}_R^\dagger + \bar{t} \widetilde{t}_L) - \widetilde{H}_u^+ (b \widetilde{t}_R^\dagger + \bar{t} \widetilde{b}_L) \big] \\ & - y_b \big[\widetilde{H}_d^0 (b \widetilde{b}_R^\dagger + \bar{b} \widetilde{b}_L) - \widetilde{H}_d^- (t \widetilde{b}_R^\dagger + \bar{b} \widetilde{t}_L) \big] + \text{h.c.} \end{aligned}$$

Finally, in the approximation where the gauge couplings are neglected, the chargino masses and the gluino mass are obtained from

$$\mathscr{L}_{\rm mass} = -\tfrac{1}{2}M_3\widetilde{g}\widetilde{g} - M_2\widetilde{W}^+\widetilde{W}^- - \mu\widetilde{H}_u^+\widetilde{H}_d^- + \text{h.c.},$$

where the mixing of gauginos and higgsinos (proportional to g) is neglected. The gluino of mass $M_{\widetilde{g}} = M_3$ is a Majorana fermion, and the charged Dirac fermion of mass $M_{\widetilde{H}^\pm} = \mu$ comprises the pair of two-component higgsino fields, $\widetilde{H}_u^+$ and $\widetilde{H}_d^-$.

Under the assumption that $M_S \gg m_{H^\pm}$, one can compute the leading contribution to the wrong-Higgs coupling diagrams by setting all external four-momenta equal to zero. Performing the integration over the loop momentum then yields the Passarino–Veltman function $C_0(0,0,0;m_a^2,m_b^2,m_c^2)$, where the arguments of C_0 are the squared masses of the particles appearing in the loop.

The Passarino–Veltman function C_0

We work in $d = 4 - 2\epsilon$ dimensions and employ dimensional regularization.

$$C_0(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -16\pi^2 i\mu^{2\epsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{D_C},$$

where $p = -(p_1 + p_2)$ and

$$D_C \equiv (q^2 - m_a^2 + i\varepsilon)[(q + p_1)^2 - m_b^2 + i\varepsilon][(q + p_1 + p_2)^2 - m_c^2 + i\varepsilon],$$

The following integral expression for C_0 can be derived:

$$C_0(p_1^2, p_2^2, p^2; m_a^2, m_b^2, m_c^2) = -\int_0^1 dx \int_0^x \frac{dy}{D - i\varepsilon},$$

after dropping terms of $\mathcal{O}(\epsilon)$, where

$$D \equiv p^2 x^2 + p_2^2 y^2 + (p_1^2 - p_2^2 - p^2)xy + (m_c^2 - m_a^2 - p^2)x$$
$$+ (m_b^2 - m_c^2 + p^2 - p_1^2)y + m_a^2 .$$

Thus, we obtain

$$
\begin{aligned}
-iw_b\delta_{jk} = {} & (i\mu y_b)2g_s^2\big(\boldsymbol{T^a T^a}\big)_{jk}\, i^3 M_3\,\frac{i}{16\pi^2} C_0(0,0,0;\,,M_3^2, m^2_{\widetilde{b}_L}, m^2_{\widetilde{b}_R}) \\
& + (-ia_t)(-iy_t)(-iy_b)\delta_{jk}\, i^3\mu\,\frac{i}{16\pi^2} C_0(0,0,0;\,,\mu^2, m^2_{\widetilde{t}_L}, m^2_{\widetilde{t}_R})\,,
\end{aligned}
$$

where j, k are color indices and the factor of i^3 derives from the numerators of the three propagators in the loop.

The above result is usually expressed in terms of the function

$$
\begin{aligned}
\mathcal{I}(m_a, m_b, m_c) & \equiv -C_0(0,0,0; m_a^2, m_b^2, m_c^2) \\
& = \frac{m_a^2 m_b^2 \ln(m_a^2/m_b^2) + m_b^2 m_c^2 \ln(m_b^2/m_c^2) + m_c^2 m_a^2 \ln(m_c^2/m_a^2)}{(m_a^2 - m_b^2)(m_b^2 - m_c^2)(m_a^2 - m_c^2)}\,,
\end{aligned}
$$

where $\mathcal{I}(m,m,m) = 1/(2m^2)$.

Hence, our final result for the wrong-Higgs coupling is

$$w_b = y_b \left[\frac{C_F \alpha_s \mu M_3}{2\pi} \mathcal{I}(M_{\widetilde{g}}, m_{\widetilde{b}_L}, m_{\widetilde{b}_R}) + \frac{\mu a_t y_t}{16\pi^2} \mathcal{I}(M_{\widetilde{H}^\pm}, m_{\widetilde{t}_L}, m_{\widetilde{t}_R}) \right] ,$$

where $(\boldsymbol{T^a T^a})_{jk} = C_F \delta_{jk}$, with $C_F = 4/3$, is the Casimir operator in the fundamental representation of $\mathrm{SU}(3)_C$. The above result was derived under the assumption that M_3 and μ are positive. However, it can be shown that this result remains valid if M_3 and μ are real quantities of either sign.

A remarkable feature of the above result is that, in the limit of $M_S \gg m_{H^\pm}$, expression for w_b given above does not decouple if μ, M_3, $a_t \sim \mathcal{O}(M_S)$. That is, apart from the one-loop suppression factor, the contribution of w_b to the Yukawa interactions of the effective low-energy 2HDM theory can yield significant deviations from the Type-II Yukawa interactions of the tree-level MSSM Higgs sector.

For example, setting $\langle H_u^0\rangle = v_u/\sqrt{2}$ and $\langle H_d^0\rangle = v_d/\sqrt{2}$ yields

$$m_b = \frac{y_b v}{\sqrt{2}}\cos\beta\left(1+\frac{w_b\tan\beta}{y_b}\right) \equiv \frac{y_b v}{\sqrt{2}}\cos\beta(1+\Delta_b)\,,$$

which defines the quantity Δ_b. The dominant contributions to Δ_b are $\tan\beta$-enhanced, with $\Delta_b \simeq (w_b/y_b)\tan\beta$. Thus, the tree-level relation between the b-quark mass and the b-quark Yukawa coupling receives a significant radiative correction if $\tan\beta$ is large. This can significantly modify the tree-level predictions for the couplings of $b\bar{b}$ to the Higgs bosons of the MSSM.

Exercise: Derive the following expression for the $hb\bar{b}$ coupling:

$$g_{hb\bar{b}} = -\frac{m_b\sin\alpha}{v\cos\beta}\left[1-\left(\frac{\Delta_b}{1+\Delta_b}\right)(1+\cot\alpha\cot\beta)\right]\,.$$

Show that $g_{hb\bar{b}}$ reduces to its SM value when $m_A \gg m_Z$. Obtain the corresponding expressions for $g_{Hb\bar{b}}$, $g_{Ab\bar{b}}$, and $g_{H^+b\bar{t}}$.