SUSY DUALITIES

PHYSICS of STRONGLY INTERACTING GAUGE THEORIES:

- is there confinement?
- what is the IR spectrum of gauge invariant operators?
- is there a FIXED POINT where a CFT lives?
- .
- :

HOPE / EXPECTATION:

Can we find a set of emergent dof in terms of which we have a weakly coupled dinamics?

Dualities realise this paradigm:

- Holographic dualities
- IR dualities of SUSY gauge theories (class of universality)
  - ↳ Seiberg duality '94
    - ↳ is it enough to explore ALL SUSY IR dualities?

- NOT COVERED: geometric construction of QFTs
  - ↳ Dualities appear as consistency conditions.

4d $\mathcal{N}=1$ SQCD

$U(1)_R$ symmetry
$$[R, Q_\alpha] = -Q_\alpha \qquad R[\theta] = 1$$
$$[R, \tilde{Q}_{\dot\alpha}] = +Q_\alpha \qquad R[\bar\theta] = -1$$

$$\mathcal{L}_{SQCD} = \mathcal{L}_{sym} + \mathcal{L}_{matt}$$

- $\mathcal{L}_{sym}$ use $V = (\underbrace{\sigma^\mu}_{\text{vector}}, \underbrace{\lambda, \bar\lambda}_{\text{gauginos}}, \underbrace{D}_{\text{real scalar}})$

$$R[\sigma^\mu] = R[D] = 0 \quad , \quad R[\lambda] = 1 \quad , \quad R[\bar\lambda] = -1$$

$$\mathcal{L}_{sym} = \frac{1}{32\pi} \, \mathrm{Im}\left(\tau \int d^2\theta \, \mathrm{Tr}\, W_\alpha W^\alpha \right) =$$

$$= \mathrm{Tr}\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \bar\lambda + \frac{1}{2} D^2 \right]$$
$$+ \frac{\theta}{32\pi^2} g^2 \, \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

- $\Phi = (\varphi, \psi_\alpha, F) \quad , \quad \tilde{\Phi}$

$$R[\Phi] = R[\varphi] = r \quad , \quad R[\psi] = r-1$$

$$\mathcal{L}_{matter} = \int d^2\theta\, d^2\bar\theta\ \bar\Phi\, e^{2gV}\,\Phi + \int d^2\theta\, W(\Phi) + \int d^2\bar\theta\, \bar W(\bar\Phi)$$

$$= \overline{D_\mu\phi}\, D^\mu\phi - i\psi\,\sigma^\mu D_\mu \bar\psi + \bar F F + i\sqrt{2}\, g\, \bar\phi\lambda\psi - i\sqrt{2}\, g\, \bar\phi\bar\lambda\phi + \dots$$

EOM for aux scalars

$$\bar F_i = \frac{\partial W}{\partial \phi_i}\ , \qquad D^a = -g\, \bar\phi_i (T^a)_{ij}\phi_j$$

SCALAR POTENTIAL : $V = \bar F F + \frac{1}{2} D^2 \geq 0$

SUSY VACUA $\Longleftrightarrow$ ZEROS of scalar potential

$D^a = 0$ & $\bar F_i = \frac{\partial W}{\partial \phi_i} = 0$

$\forall a$ $\forall i$

D-Terms F-Terms

Solutions to these equations are the CLASSICAL MODULI SPACE!

$$\mathcal{M}_{cl} = \left\{ \langle \phi_i \rangle \mid \partial_{\phi_i} W = 0,\ D^a = 0\ \ \forall i, a \right\} \Big/ \text{gauge transf}$$

or

$$\mathcal{M}_{cl} = \{ \text{gauge inv operators} \} / \text{classical relations}$$

SQED

$U(1)$ theory with $Q_i \quad i = 1 \ldots N_f$

$\tilde{Q}_i \quad i = 1 \ldots N_f$

$W = 0 \rightarrow$ no F-Terms

D-Terms $\quad D = -g \sum^{N_f} (\bar{Q}_i Q_i - \bar{\tilde{Q}}_i \tilde{Q}_i)$

$$\mathcal{M}_{cl,4} = \{ 2N_f \underbrace{- \tfrac{1}{2}}_{\text{D-Term}} \underbrace{- \tfrac{1}{2}}_{\text{gauge}} \} = 2N_f - 1$$

$$\text{or} = 2N_f - 1$$

Remove one complex scalar + broken generator by the SUSY Higgs mechanism

$$\mathcal{M}_{cl} = \{ N_f \times N_f - (N_f - 1)^2 \} = 2N_f - 1$$

$M_{ij} = Q_i^a \tilde{Q}_j^a$ has rank 1

SQCD $\qquad$ N_f chirals $\qquad Q^{ai} \qquad \square$

$a = 1..N_c \qquad i = 1, \dots N_f$

$SU(N_c)$ with N_f flavors $\qquad \tilde{Q}^{ai} \qquad \bar{\square}$

- $\underline{N_f < N_c}$

D-Terms can be solved

$$Q = \begin{pmatrix} v_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & v_{N_f} & \end{pmatrix} = \tilde{Q}^T$$

$$SU(N_c) \longrightarrow SU(N_c - N_f)$$

$$\dim \mathcal{M}_{cl} = \left\{ 2 N_f N_c - \left(N_c^2 - 1 - \left((N_c - N_f)^2 - 1 \right) \right) \right\}$$

$$= N_f^2$$

gauge inv Operators $\qquad M^i{}_j = Q^{ia} \tilde{Q}^a{}_j$

no classical relations

$$\dim \mathcal{M}_{cl} = \{ N_f \times N_f \}$$

- $\underline{N_f > N_c}$

$$\dim \mathcal{M} = \{ 2 N_c N_f - (N_c - 1)^2 \}$$

completely Higgsed gauge group

$\mathrm{Rank}(M^i{}_j) \leq N_c \quad \rightsquigarrow$ Relations

Now we have BARIONS:

$$\binom{N_f}{N_c} \quad B_{i_1 \dots i_{N_c}} = \varepsilon_{a_1 \dots a_{N_c}} Q^{a_1}_{i_1} \dots Q^{a_{N_c}}_{i_{N_c}}$$

$$\tilde{B}^{i_1 \dots i_{N_c}} = \varepsilon^{a_1 \dots a_{N_c}} \tilde{Q}^{i_1}_{a} \dots \tilde{Q}^{i_{N_c}}_{a_{N_c}}$$

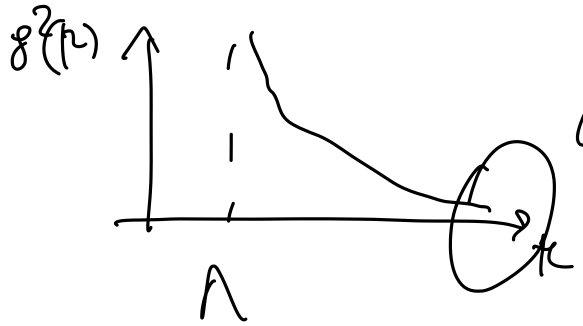
β-function

@ 1-loop $\quad \beta_g = \mu \partial_\mu g = -\frac{g^3}{16\pi^2} \underset{\substack{\uparrow \\ (3N_c - N_f)}}{b_1}$

$$\int_{\bar{g}}^{g} \frac{dg}{g^3} = -\frac{b_1}{16\pi^2} \log \frac{\mu}{\bar{\mu}}$$

$$g^2(\mu) = \frac{1}{\frac{b_1}{8\pi^2} \log\left(\frac{\mu}{\bar{\mu}}\right) + \frac{1}{\bar{g}^2(\bar{\mu})}}$$

for $N_f < 3N_c$ there's Asymptotic Freedom



$$\Lambda = \bar{\mu} \exp\left(-\frac{8\pi^2}{b_1 \bar{g}^2(\bar{\mu})}\right)$$

$$\bar{\Lambda} = \Lambda e^{\frac{i\theta}{b_1}} \quad \text{HOLO STRONG SCALE}$$

BANKS-ZAKS fixed point (Seiberg '94)

Below $N_f = 3N_c$ where β changes sign there is a fixed point! $\Rightarrow$ there is SCFT!

<u>one loop</u>

$$\beta_{NSVZ} = -\frac{g^3}{16\pi^2} \frac{\left(3N_c - N_f(1-\gamma)\right)}{1 - N_c\, g^2/8\pi^2}$$

$(\gamma = \mu\, \partial_\mu \log Z_Q)$

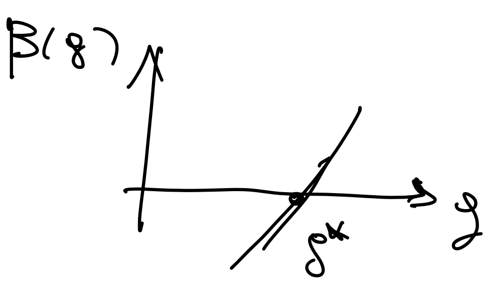
$$\gamma = -\frac{g^2}{8\pi^2}\frac{(N_c^2-1)}{N_c} + O(g^4)$$

$N_f = 3N_c - \varepsilon$ expand for small ε, g

$$\beta_{NSVZ} \simeq -\frac{g^3}{16\pi^2}\left[3N_c - \left(3(N_c^2-1) + O(\varepsilon)\right]\frac{g^2}{8\pi^2} + O(g^5)\right.$$

$$\beta_{NSVZ} = 0 \implies g = g^*$$

$$\beta'_{NSVZ}(g^*) > 0$$

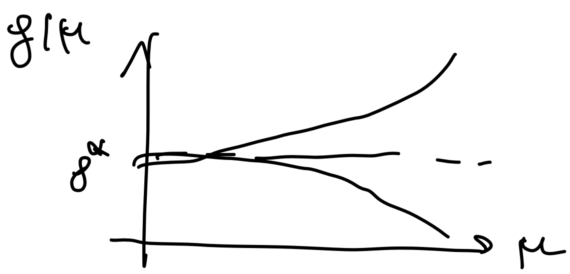


$$\beta(g) \simeq \beta(g^*) + (g-g^*)\,\beta'(g^*)$$

$$\int_{\mu_0}^{\mu} \frac{d\mu}{\mu} = \frac{1}{\beta'(g^*)}\int_{g_0}^{g}\frac{dg}{(g-g^*)}$$

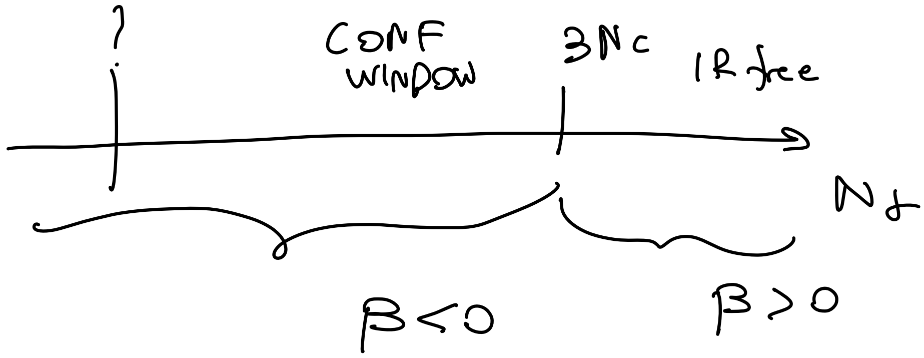
$$\downarrow$$

$$\boxed{\frac{g-g^*}{(g_0-g^*)} = \left(\frac{\mu}{\mu_0}\right)^{\beta'(g^*)}}$$



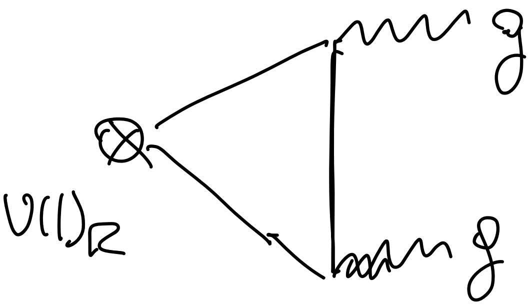
SQCD

$SU(N_c)$



SQCD $\quad SU(N_c) \quad N_f\, Q,\ N_f\, \tilde{Q} \quad W=0$

	$SU(N_f)$	$SU(N_f)$	$U(1)_A$	$U(1)_B$	$U(1)_R$
Q	$\square$	1	1	1	$\frac{N_f-N_c}{N_f}$
$\tilde{Q}$	1	$\square$	1	-1	$\frac{N_f-N_c}{N_f}$



$$A_{Rgg} = \sum_i q_i\, T(R_i)$$

$$= 2N_f\,(R[Q]-1)\, T(\square)^{=1/2}$$

$$+ \underbrace{R[\lambda]}_{=1}\, T(\underbrace{adj}_{N_c}) = 0$$

$$R[Q] = R[\tilde{Q}] = \frac{N_f - N_c}{N_f}$$

In SCFT there is a relation between R and D

R: R-symmetry generator

D: Dilation

For 4d $\mathcal{N}=1$ SCFT

for a chiral op $\Delta = \frac{3}{2} R$

anti chiral op $\Delta = -\frac{3}{2} R$

UNITARITY BOUND IN CFT

$$\Delta \geq \frac{(d-2)}{2} = 1$$

in $d=4$

Δ: of every op.

$\frac{(d-2)}{2}$: free scalar

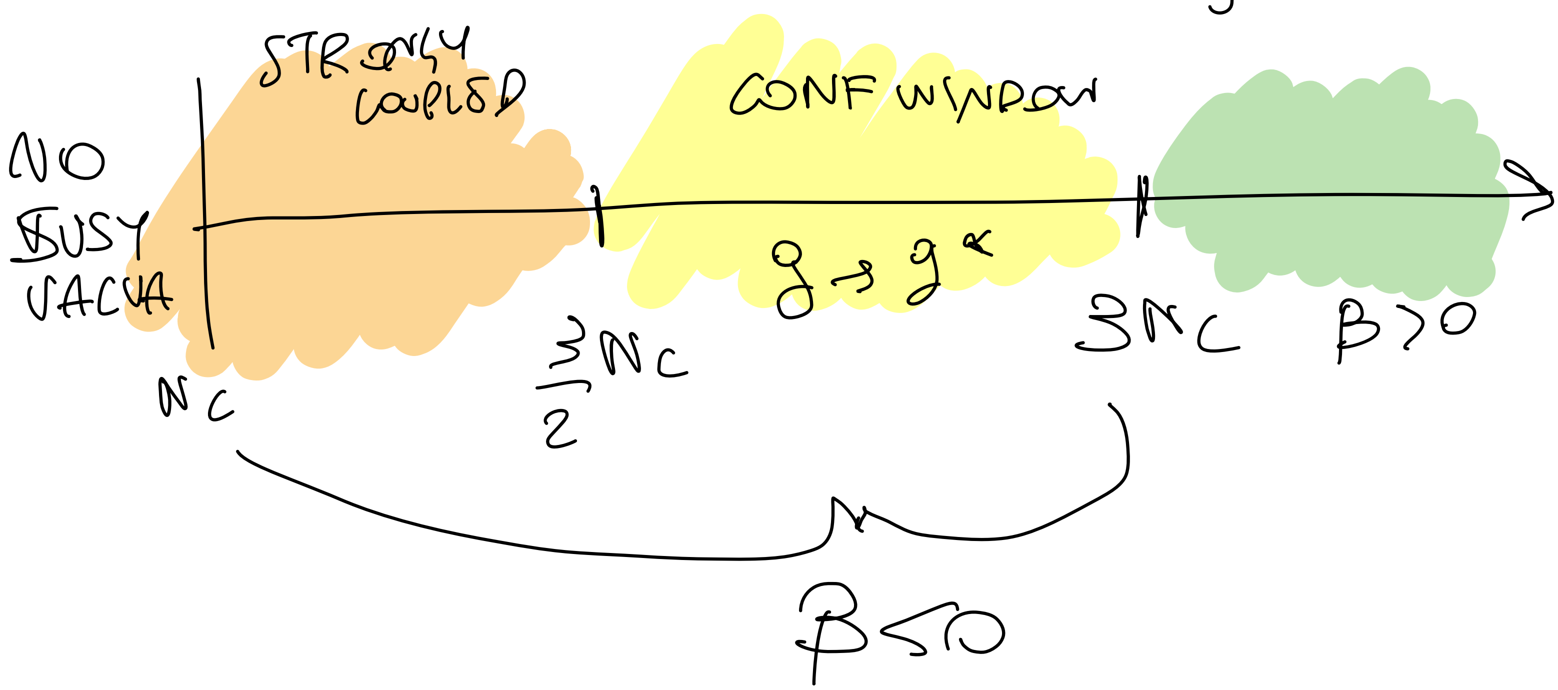
In SQCD mesons

$$M^i_j = Q^{ia}\tilde{Q}_{aj},$$

$$\Delta(M^i_j) = \frac{3}{2} R[M^i_j] = \frac{3}{2}\, 2R[Q] = 3\frac{(N_f-N_c)}{N_f} \geq 1$$

$$\Longrightarrow \quad N_f \geq \frac{3}{2}N_c$$

below mesons become free



Seiberg duality for $SU(N_c)$ SQCD with $N_f > N_c + 1$:

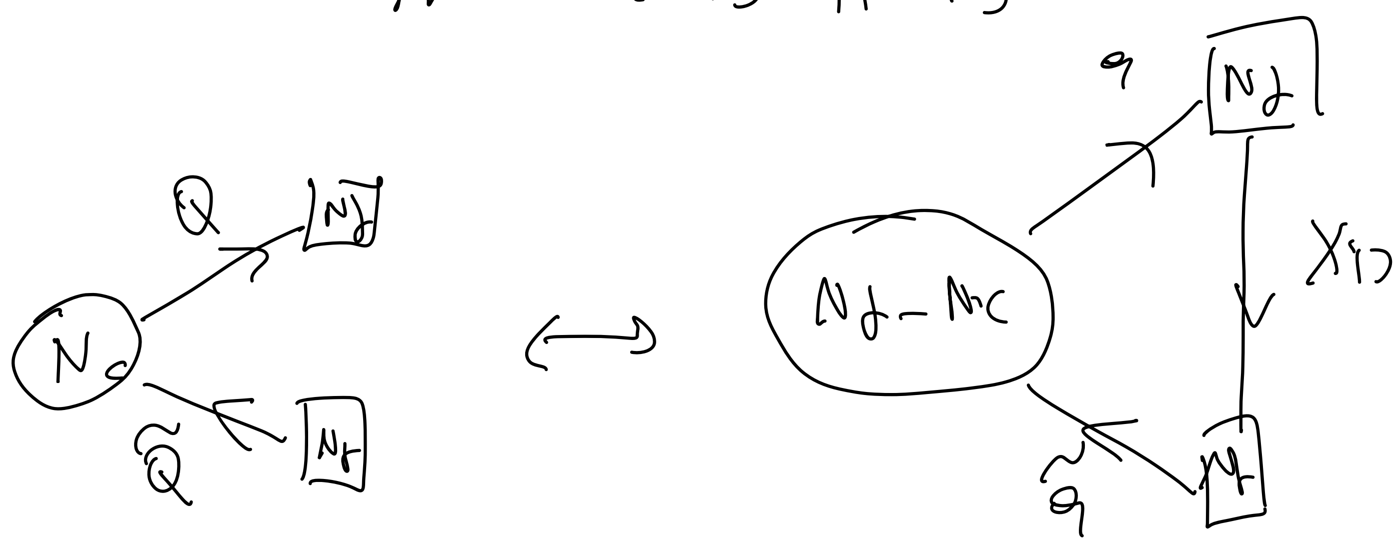
eSQCD $\quad SU(N_c)$, $N_f\, Q$, $N_f\, \tilde{Q}$, $W = 0$

is IR DUAL

mSQCD $\quad SU(N_f - N_c)$, $N_f\, q$, $N_f\, \tilde{q}$

$+ N_f^2$ singlets X_{ij}

$$W = X_{ij}\, q_i\, \tilde{q}_j$$



eSQCD $\qquad\qquad$ mSQCD

		$SU(N_f)$	$SU(N_f)$	$U(1)_R$	$U(1)_B$
E	Q	$\square$	1	$r = \frac{N_f - N_c}{N_f}$	1
	$\tilde{Q}$	1	$\bar{\square}$	$r = \frac{N_f - N_c}{N_f}$	-1
M	q	$\square$	1	$r' = \frac{N_c}{N_f}$	$N_c/(N_f - N_c$
	$\tilde{q}$	1	$\bar{\square}$	$r' = \frac{N_c}{N_f}$	$-N_c/(N_f - N_c$
	X_{ij}	$\square$	$\bar{\square}$	$\frac{2(N_f - N_c)}{N_f}$	0

$R[W] = 2$ since $R[d^2\theta] = -2$

$\downarrow$

$R[X_{ij} q_i \tilde{q}_j] = 2 = R[q] + R[\tilde{q}] + R[X_{ij}]$

$\hookrightarrow R[X_{ij}] = 2 - 2R[q]$

$= 2 - \frac{2N_c}{N_f}$

OPERATOR MAP

E: $M_{ij} = Q_i^e \tilde{Q}_j^e$ + BARIONS
mesons B, $\tilde{B}$

M: $m_{ij} = q_i^e \tilde{q}_j^e$

$W = q_i \tilde{q}_j X_{ij}$

e.o.m for X_{ij} $\quad \dfrac{\partial W}{\partial X_{ij}} = q_i^e \tilde{q}_j^e = 0$

mesons are killed by F-Terms
they are <u>not</u> in the spectrum

So we have X_{ij}, $b^{\cdots}$, $\tilde{b}^{\cdots}$

Ele $\qquad\qquad\qquad\qquad$ Mag

$$M_{ij} \xleftrightarrow[\square \times \bar{\square}]{SU(N_f)\times SU(N_f)} X_{ij}$$

$$R[M_{ij}] = 2R[Q] = 2\frac{(N_f - N_c)}{N_f} = R[X_{ij}]$$

$$B_{i_1 \ldots i_{N_c}} \longleftrightarrow \varepsilon_{i_1 \ldots i_{N_c} j_1 \ldots j_{N_f - N_c}}\, b^{j_1 \ldots j_{N_f - N_c}}$$

$$\binom{N_f}{N_c} \longleftrightarrow \binom{N_f}{N_f - N_c}$$

$$R[B^{\cdot\cdot}] = N_c R[Q] = N_c \frac{(N_f - N_c)}{N_f}$$

$$\parallel$$

$$R[b^{\cdot\cdot}] = (N_f - N_c) R[q] = (N_f - N_c)\frac{N_c}{N_f}$$

m SQCD

Consider SQCD with $SU(N_f - N_c)$ with X_{ij} and $W = 0$

$$\beta^{1\text{-loop}} = -\tilde{b}_1 \, g^3$$

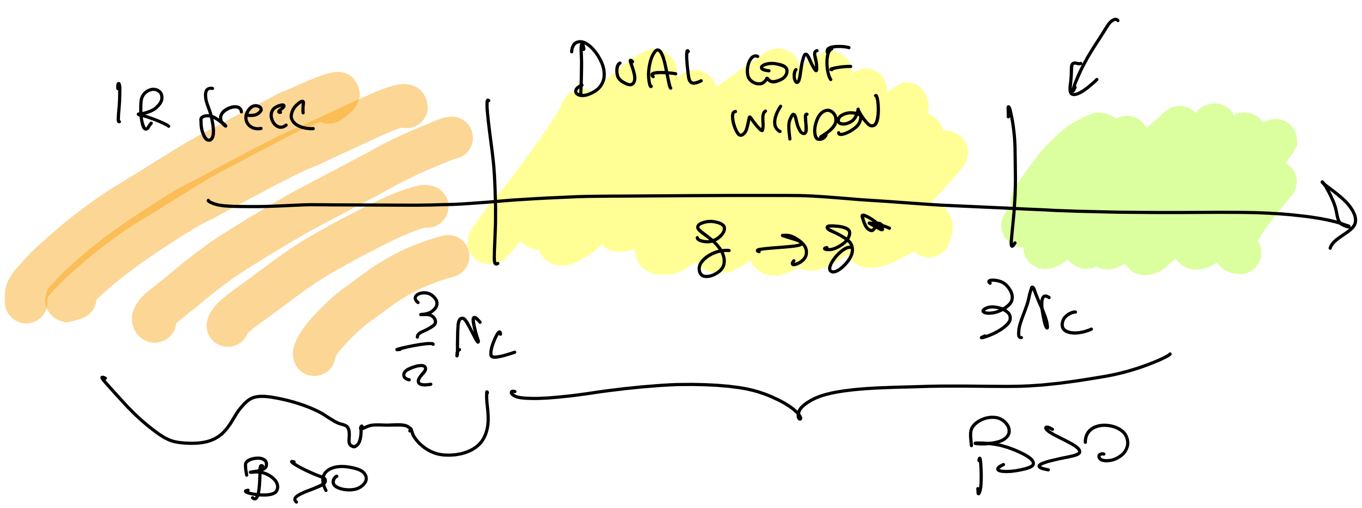
$$\tilde{b}_1 = 3(N_f - N_c) - N_f = 2N_f - 3N_c$$

so $\beta > 0$ for $N_f \leq \frac{3}{2} N_c$

$$R[M_{ij}] = R[q_i \tilde{q}_j] = \frac{2N_c}{N_f}$$

$$\Delta[q_i \tilde{q}_j] = \frac{3}{2} R[q\tilde{q}] = \frac{3N_c}{N_f} \leq 1$$

Dual meson become free for $N_f \geq 3N_c$



In the dual conf window

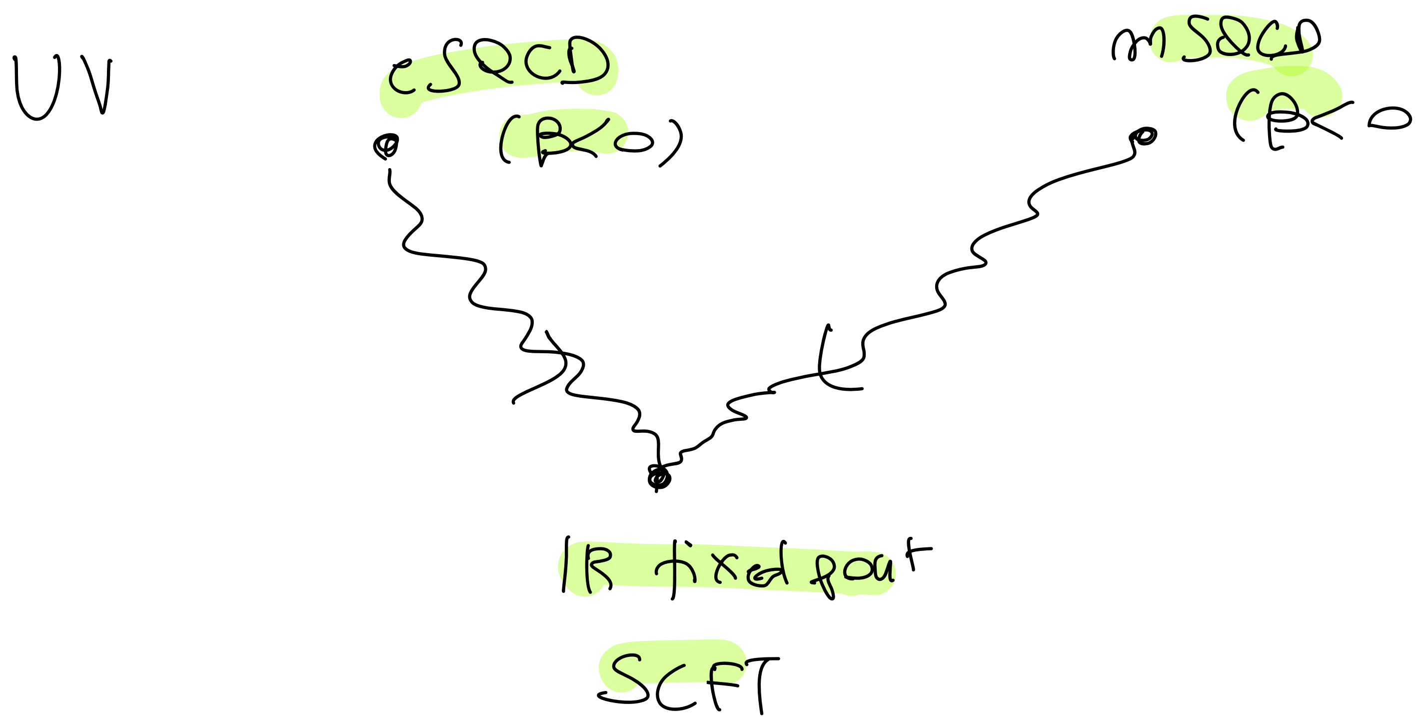
$\Delta[q\tilde{q}] < 2$, $\Delta[X_{ij}] = 1$

they are free

$\Rightarrow \Delta[X_{ij}\, q_i \tilde{q}_j] < 3$ it is relevant

this deformation drives the mSQCD to the same fixed point of the eSQCD

In the conf window $\frac{3}{2}N_c < N_f < 3N_c$



- $N_c + 1 < N_f \leq \frac{3}{2} N_c$

e SQCD is very hard! it is strongly coupled ($\beta < 0$)

but m SQCD is IR free ($\beta > 0$)

⇝ Paradigm realised

we have a description in terms of weakly coupled fields

- $N_f = N_c + 1$

We have <u>S-confinement</u>

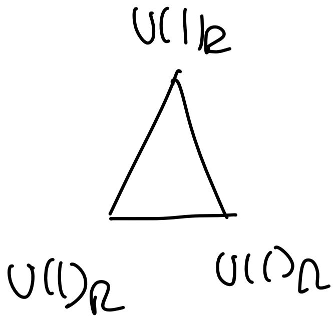
(confinement without symmetry breaking)

the theory is described by

$$W_{eff} \sim (\det M - B_i M_{ij} \hat{B}_j)$$

NO GAUGE DYNAMICS!

ANOMALY MATCH



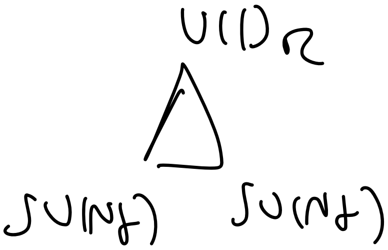
$$A_{RRR} = \sum_{\text{ferms}} q_R^3$$

$$eSQCD = \overbrace{2 N_f N_c}^{Q,\tilde{Q}} (R[Q]-1)^3 + (N_c^2-1)\underbrace{R[\lambda]}_{=1} =$$

$$= -\frac{2N_f^4}{N_c^2} + N_c^2 - 1$$

same!

$$mSQCD = \overbrace{2N_f(N_f-N_c)}^{q,\tilde{q}} (R[q]-1)^3 + ((N_f-N_c)^2-1)\underbrace{R[\lambda]}_{1} + N_f^2 (R[X_{ij}]-1)^3$$



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CONSISTENCY checks

- In eSQCD integrate-out 1 flavor

$$\delta W_e = m\, Q^e_{N_f}\, \tilde{Q}^e_{N_f}$$

eSQCD $\longrightarrow$ eSQCD

$SU(N_c)$ $\quad$ $SU(N_c)$

N_f $\quad$ $N_f - 1$

- In mSQCD

$$\delta W_m = m\, X_{N_f N_f}$$

$$W + \delta W_m = q^a_i\, \tilde{q}^a_j\, X_{ij} + m\, X_{N_f N_f}$$

e.o.m $X_{N_f N_f} \Rightarrow q^a_{N_f}\, \tilde{q}^a_{N_f} + m = 0$

$$\langle q^a_{N_f}\, \tilde{q}^a_{N_f} \rangle \neq 0$$

$\Downarrow$ Higgs

$SU(N_f - N_c)$ $\longrightarrow$ $SU(N_f - N_c - 1)$

N_f $\quad$ $N_f - 1$

$$
\begin{tikzcd}
SU(N_c),\ N_f \arrow[r, leftrightarrow] \arrow[d, "\text{mass def}"'] & \begin{array}{c} SU(N_f - N_c),\ N_f \\ W = Xq\tilde{q} \end{array} \arrow[d, "\text{Higgs}"] \\
SU(N_c),\ N_f - 1 \arrow[r, leftrightarrow] & \begin{array}{c} SU(N_f - N_c - 1),\ N_f - 1 \\ W = \cancel{X}q\tilde{q} \end{array}
\end{tikzcd}
$$

The SUPER-CONF INDEX

Kinney–Maldacena–Minwalla–Raju '05

Romelsberger '05, '07

Is a refined Witten index

$$I = \mathrm{Tr}\, (-1)^F e^{-\beta \delta} \mu_i^{m_i}$$

$$\delta = \frac{1}{2} \{Q, Q^\dagger\}$$

m_i charges commuting with Q

Q is one of the super-charges

states with $\delta > 0$ come in bosonic/fermic pairs $\Rightarrow$ only $\delta = 0$ states contribute

$\rightsquigarrow$ Dolan–Osborn 2008

the index can be written as a <u>matrix integral</u>

it is written in terms of elliptic - gamma functions

$$\Gamma(z;p,q) = \prod_{k,n=0}^{\infty} \frac{(1-p^{k+1} q^{n+1} z^{-1})}{(1-p^{k} q^{n} z)}$$

- A chiral mult of R-charge r in the fund of SU(Nc) contributes as:

$$\prod_{a=1}^{Nc} \Gamma((pq)^{r/2} z_a ; pq)$$

$\uparrow$ Cartan of SU(Nc)

- A vect mult in the adjoint of SU(Nc)

$$\int \prod_{r=1}^{N-1} \frac{dz_i}{z_i} \prod_{i<j} \frac{1}{\Gamma(z_i/z_j ; p,q)}$$

Example

$$I_{N_c,N_f}(\vec{x},\vec{y},b,r) = \int \prod_{a=1}^{N_c-1} \frac{dz_a}{z_a}$$

$$\prod_{a<b}^{N_c} \frac{1}{\Gamma(z_a/z_b ; p,q)} \times$$

$$\times \prod_{a=1}^{Nc} \prod_{j=1}^{Nf} \Gamma((pq)^{r/2} z_a^{-1} x_j, b)$$

$$\Gamma((pq)^{r/2} z_a y_j^{-1} b)$$

$$SU(N_f)_{\vec{x}} \times SU(N_f)_{\vec{y}} \times U(1)_b$$

$$r = \frac{N_f - N_c}{N_f}$$

Work mathematicians Spiridonov, Rains, ...

Seiberg - duality - identity:

$$I_{N_f, N_c}(\vec{x}, \vec{y}, b, r) = I_{N_f, N_f - N_c}(\vec{x}, \vec{y}, b, r)$$

$$\underbrace{\prod_{i,j=1}^{N_f} \Gamma\left((pq)^{\frac{N_f - N_c}{N_f}} x_i y_j^{-1}\right)}_{I_{x,y}}$$

- After Seiberg '94 dual of $SU(N_c)$ SQCD many! IR dualities have been found

example Intriligator-Pouliot '95

$$USp(2N_c) \overset{Q}{—} \boxed{2N_f} \quad \leftrightarrow \quad USp(2N_f-2N_c-4) \overset{q}{—} \boxed{2N_f} \circlearrowright X_{ij}$$

$$W=0 \qquad\qquad W = X^{ab} q_i q_j$$

FOR $N_f = 2N_c+4$

$$USp(2N_c) — \boxed{2N_c+4} \quad \leftrightarrow \quad \boxed{2N_c+4} \circlearrowright X_{ij}$$

s-confining

⋮

$\Downarrow$

3d $\mathcal{N}=2$ DUALITIES (4 supercharges)

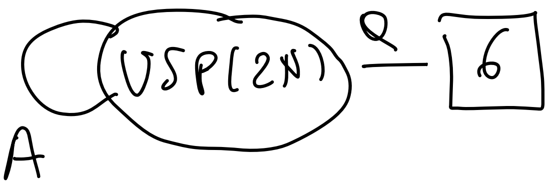
obtain MANY! dualities including for Chern-Simons theories!

4d $\mathcal{N}=1$ theories with TENSOR MATTER

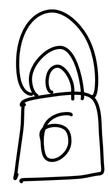
Csaki–Schmaltz–Skiba '98

classified ALL s-confining $\mathcal{N}=1$ theories

Example



$\longleftrightarrow$



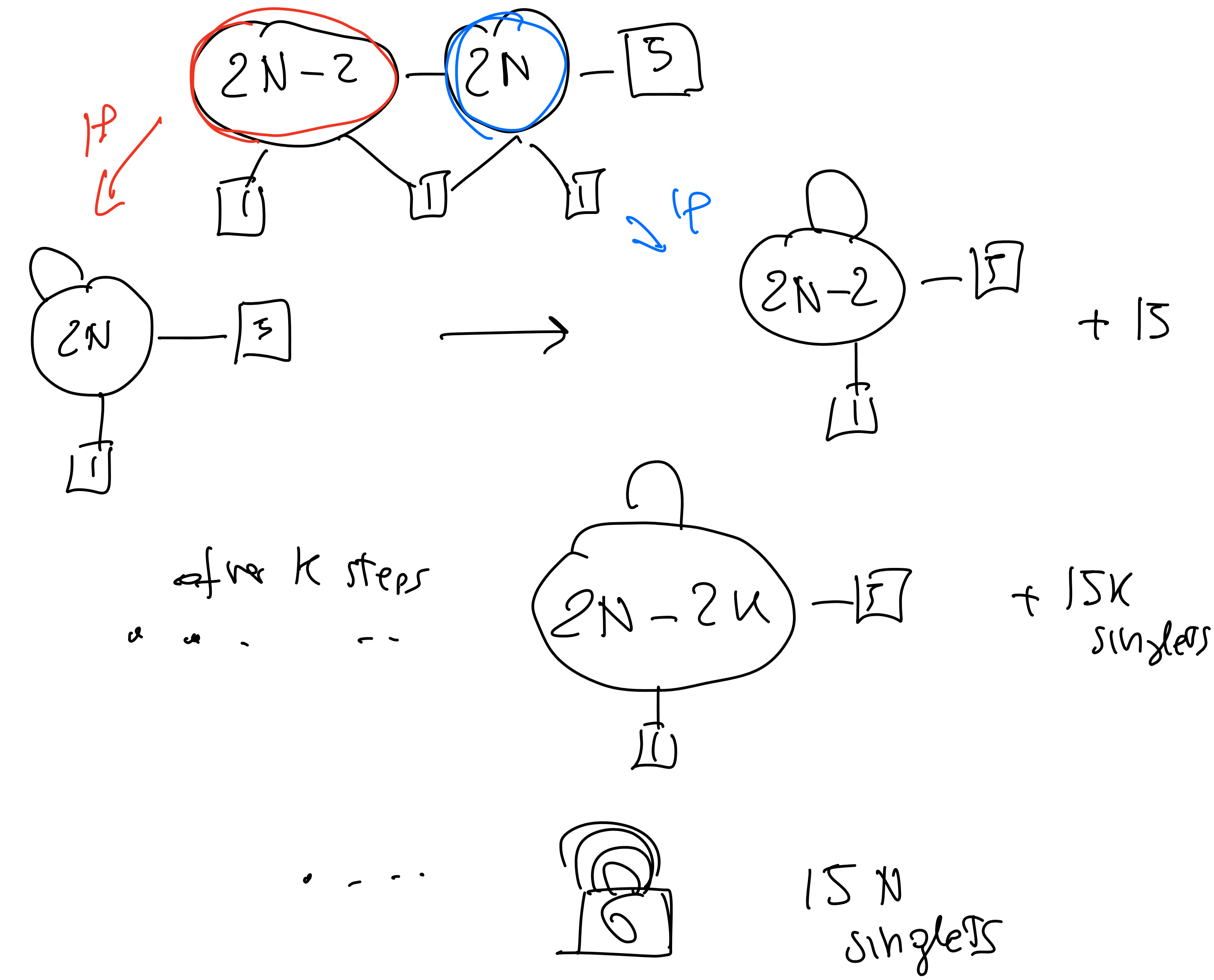
$15N$ chirals in the antisymmetric of $SU(6)$

↑ anti of USp(2N)

$SU(6) \times U(1)$

$$M_k = \mathrm{Tr}_N\left(Q A^k Q\right) \longleftrightarrow M^{(k)}_{ab} \quad 1 \le a < b \le 6$$

$$k = 1 \dots N$$



the sequential deconfinement proves
the s-confining dualities
in terms of "Seiberg" duality

In 3d there are MIRROR DUALITIES can also be derived assuming only Seiberg duality the MIRROR dualization algorithm [21].