QTFP Winter School 2023 - QSimFP Tutorial

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1. The equations of motion for a shallow fluid flow are,

$$\partial_t \phi + \frac{1}{2}\mathbf{v}^2 + gh = 0, \qquad \partial_t h + \nabla \cdot (h\mathbf{v}) = 0.$$

   where $\mathbf{v} = \nabla\phi$ is a d-dimensional velocity field, $g \simeq 9.81\text{ms}^{-2}$ is the gravitational accleration and h is the height of the fluid. By perturbing $\phi \to \phi + \delta\phi$ and $h \to h + \delta h$, show that velocity potential fluctuations obey,

$$(\partial_t + \nabla \cdot \mathbf{v})(\partial_t + \mathbf{v} \cdot \nabla)\delta\phi - g\nabla \cdot (h\nabla\delta\phi) = 0. \tag{1}$$

2. Using (1), find a dispersion relation for $\delta\phi$ in a homogeneous fluid, i.e. when $\mathbf{v}$ and h are constant. Give an expression for the wave speed c in a frame co-moving with the fluid.

3. Show that (1) can be written in the form,

$$\partial_\mu(f^{\mu\nu}\partial_\nu\delta\phi) = 0, \qquad \partial_\mu = \begin{pmatrix} \partial_t \\ \partial_i \end{pmatrix}, \tag{2}$$

   where $i = 1...d$, and find the components of $f^{\mu\nu}$. Hint: it will help to write,

$$f^{\mu\nu} = \begin{pmatrix} f^{tt} & f^{tj} \\ f^{it} & f^{ij} \end{pmatrix}.$$

4. Write the form of the metric tensor $g^{\mu\nu}$ such that $f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$, where g is the determinant of the metric. Find a condition for which the identification above is valid.

   You may find the following facts helpful. A matrix M of the form,

$$M = C\begin{pmatrix} a^2 - b^i b^i & -b^i \\ -b^j & \delta^{ij} \end{pmatrix},$$

   has inverse and determinant,

$$M^{-1} = \frac{1}{a^2 C}\begin{pmatrix} -1 & -b^j \\ -b^i & a^2\delta^{ij} - b^i b^j \end{pmatrix}, \qquad \det(M) = -a^2 C^{d+1},$$

   where C is a conformal factor, a is a scalar and b is a $1 \times d$ vector.