Matter Wave Interferometry

Quantum mechanics tells us that massive particles can behave as waves. These "matter waves" can be used to perform interferometry, acting as excellent quantum sensors in a variety of practical and fundamental applications. Matter wave interferometry has been carried out, for instance, with electrons, neutrons, atoms, and large molecules with masses greater than 25,000 amu (more than 2,000 atoms in each molecule). Efforts are underway to extend interferometry to even more massive objects.

Matter wave optics

In order to realize matter wave interferometers, it is necessary to create the analog of optical elements such as diffraction gratings, beam splitters, and mirrors for matter waves. There are various ways to do this, including examples such nanofabricated mechanical gratings, laser pulses, and magnetic fields.

Many, though not all, types of matter wave optics and interferometry rely on ultracold ensembles of particles.

Mechanical Transmission Gratings

In laser optics, a key example of interferometry is interference from a double slit diffraction grating. One can do the same for matter waves. A nice example for atoms is reported in Carnal and Mlynek, PRL 66, 2689 (1991).

Double slit diffraction experiments, as well as some other grating diffraction experiments, operate in the far-field Fraunhofer diffraction limit. Grating diffraction experiments have also been carried out in the near-field, Fresnel diffraction limit.

The Two-Level Atom

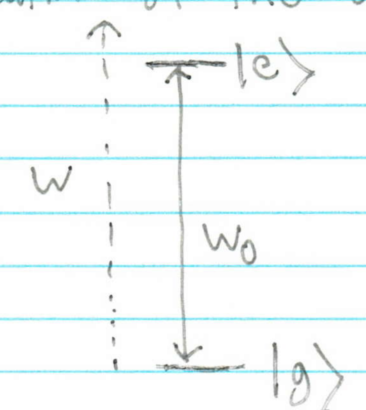
The quantum mechanical treatment of a two-level atom is a natural starting point for thinking about atom optics based on laser pulses.
Let us say that we have a ground state $|g\rangle$

and excited state $|e\rangle$ with transition frequency w_0. This system interacts with an electric

field from a laser with the form:

$$\vec{E} = \vec{E}_0 \cos(\phi - wt)$$

where ϕ corresponds to the local spatial phase at the location of the atom.



$$\delta \equiv w - w_0$$

We consider an electric dipole interaction

$$\hat{H}_{int} = -\hat{\vec{u}} \cdot \vec{E}$$

where $\hat{\vec{u}}$ is the electric dipole operator for the atom.

For example, if we consider a single electron with position $\hat{\vec{r}}$ relative to the center of the atom,

$$\hat{\vec{\mu}} = -e\hat{\vec{r}}$$

We will express the state of the atom as

$$|\psi(t)\rangle = c_1(t)|g\rangle + c_2(t)|e\rangle e^{-i\omega_0 t}$$

This is constructed to satisfy

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle \quad \text{for constant } c_1(t) \text{ and } c_2(t)$$

where $\hat{H}_0$ is the Hamiltonian for the atom in the absence of the laser field.

We have referenced the energies so that $|g\rangle$ has energy 0. So:

$$\hat{H}_0|g\rangle = 0|g\rangle = 0$$

$$\hat{H}_0|e\rangle = \hbar\omega_0|e\rangle$$

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\,\dot{c}_1(t)|g\rangle + i\hbar\,\dot{c}_2(t)|e\rangle e^{-i\omega_0 t}$$

$$+ c_2(t)\hbar\omega_0|e\rangle e^{-i\omega_0 t}$$

Since $\hat{H}_0|\psi(t)\rangle = c_2(t)\hbar\omega_0|e\rangle e^{-i\omega_0 t}$,

We indeed find that for evolution under $\hat{H}_0$,
$\dot{c}_1(t) = \dot{c}_2(t) = 0$.

We now consider evolution when the laser is pulsed on, so that the full Hamiltonian is:

$$\hat{H} = \hat{H}_0 + \hat{H}_{int}$$

$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = (\hat{H}_0 + \hat{H}_{int}) |\Psi(t)\rangle$ then yields:

$$i\hbar \dot{c}_1(t)|g\rangle + i\hbar \dot{c}_2(t)|e\rangle e^{-i\omega_0 t} =$$
$$c_1(t)\hat{H}_{int}|g\rangle + c_2(t)\hat{H}_{int}|e\rangle$$

We can find coupled differential equations by taking inner products with $\langle g|$ and $\langle e|$:

$$\dot{c}_1(t) = -ic_2(t)\Omega\cos(\phi - \omega t)e^{-i\omega_0 t}$$

$$\dot{c}_2(t) = -ic_1(t)\Omega\cos(\phi - \omega t)e^{i\omega_0 t}$$

where we define a Rabi frequency

$$\Omega \equiv \frac{1}{\hbar}\langle e|\hat{H}_{int}|g\rangle = -\frac{1}{\hbar}\langle e|\hat{\vec{\mu}}\cdot\vec{E}|g\rangle$$

For simplicity, we assume here that Ω is real.

Since for $|i\rangle = |g\rangle, |e\rangle$, $\langle i|\hat{x}|i\rangle = \langle i|\hat{y}|i\rangle = \langle i|\hat{z}|i\rangle = 0$, $\langle i|\hat{u}\cdot\vec{E}|i\rangle = 0$. Expanding the cosine:

$$\dot{c}_1(t) = -i c_2(t)\frac{\Omega}{2}\left(e^{-i\phi}e^{i\delta t} + e^{i\phi}e^{-i(\omega+\omega_0)t}\right)$$

$$\dot{c}_2(t) = -i c_1(t)\frac{\Omega}{2}\left(e^{i\phi}e^{-i\delta t} + e^{-i\phi}e^{i(\omega+\omega_0)t}\right)$$

We now will make what is known as the

<u>rotating wave approximation.</u>

We will drop the rapidly oscillating terms at frequency $\omega+\omega_0$. This assumes that

$\omega+\omega_0 >> \Omega, \delta$, which is typically a valid

approximation for atom optics. Our differential equations now reduce to:

$$\dot{c}_1(t) = -i\frac{\Omega}{2}e^{-i\phi}e^{i\delta t}c_2(t)$$

$$\dot{c}_2(t) = -i\frac{\Omega}{2}e^{i\phi}e^{-i\delta t}c_1(t)$$

For the initial conditions $c_1(0)=1$, $c_2(0)=0$, the solution is:

$$c_1(t) = \left[\cos\left(\frac{\Omega' t}{2}\right) - i\frac{\delta}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right)\right] e^{i\delta t/2}$$

$$c_2(t) = -i\frac{\Omega}{\Omega'}\sin\left(\frac{\Omega' t}{2}\right) e^{-i\delta t/2} e^{i\phi}$$

for $\Omega' \equiv \sqrt{\Omega^2+\delta^2}$.

Population oscillates between $|g\rangle$ and $|e\rangle$ at a rate corresponding to Ω'. This is called Rabi flopping. For larger $|\delta|$, the oscillations occur more quickly but have a lower amplitude.

We can consider the specific case of resonant driving so that $\delta=0$. Then:

$$c_1(t) = c_1(0)\cos\left(\frac{\Omega t}{2}\right) - ie^{-i\phi}c_2(0)\sin\left(\frac{\Omega t}{2}\right)$$

$$c_2(t) = -ie^{i\phi}c_1(0)\sin\left(\frac{\Omega t}{2}\right) + c_2(0)\cos\left(\frac{\Omega t}{2}\right)$$

There are two especially interesting values of t. First, if $\Omega t = \frac{\pi}{2}$, then:

$$|g\rangle \to \frac{1}{\sqrt{2}}\left(|g\rangle - ie^{i\phi}|e\rangle\right) \qquad \Omega t = \frac{\pi}{2}$$

$$|e\rangle \to \frac{1}{\sqrt{2}}\left(-ie^{-i\phi}|g\rangle + |e\rangle\right) \qquad \frac{\pi}{2} \text{ pulse}$$

It is important to note here that when a photon is absorbed to transfer the atom from $|g\rangle$ to $|e\rangle$, the atom also absorbs the photons momentum $\hbar\vec{k}$, where $\vec{k}$ is the wave number of the laser. So if the atom has initial momentum $\vec{p}_0$, we can think of state $|g\rangle$ having momentum $\vec{p}_0$ and state $|e\rangle$ having momentum $\hbar\vec{k}$. A $\frac{\pi}{2}$ pulse thus serves as a beam splitter. Note that when a photon is absorbed to go from $|g\rangle$ to $|e\rangle$, the local laser phase $e^{i\phi}$ is imprinted onto the atom's wavefunction.

When a photon is emitted to go from $|e\rangle$ to $|g\rangle$, the conjugate phase factor $e^{-i\phi}$ is imprinted.

If we instead perform a π pulse with $\Omega t = \pi$:

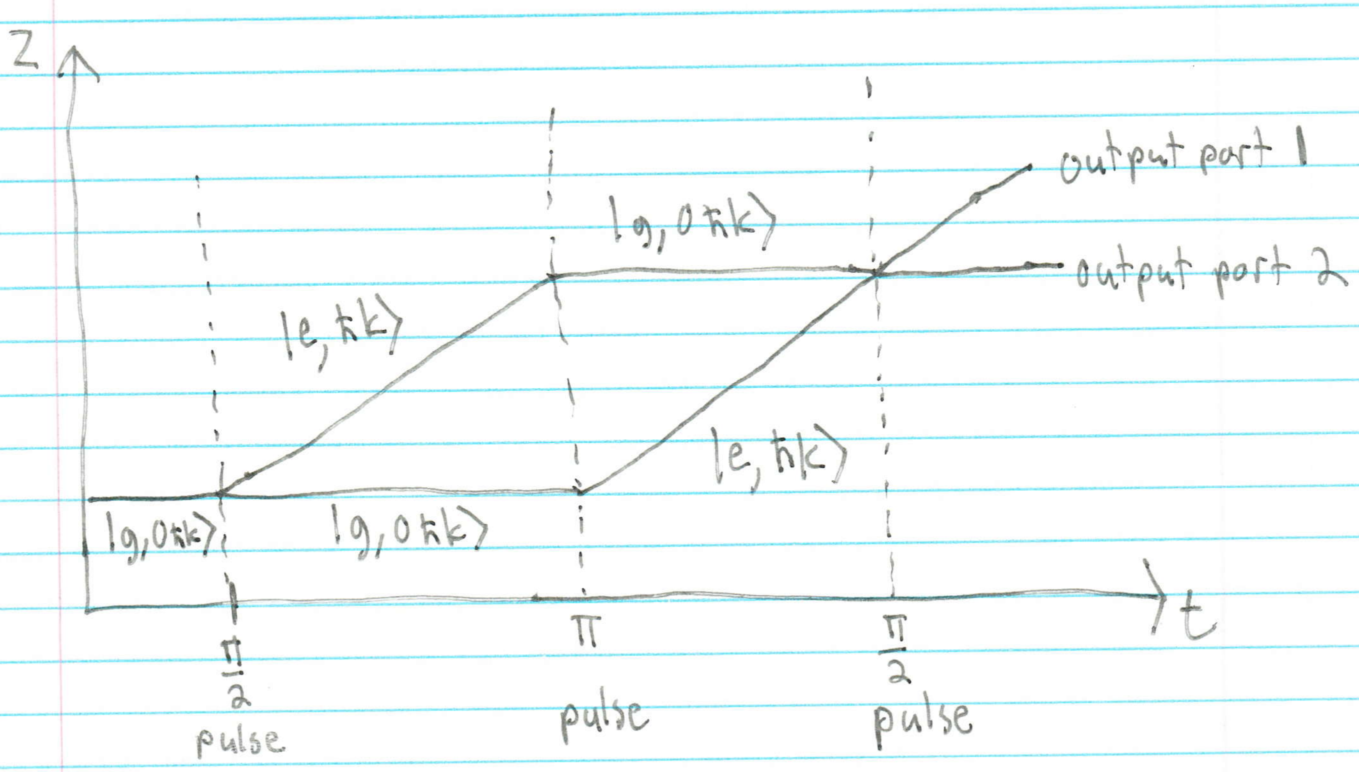
$$|g\rangle \rightarrow -ie^{i\phi}|e\rangle$$

$\Omega t = \pi$

π pulse

$$|e\rangle \rightarrow -ie^{-i\phi}|g\rangle$$

This corresponds to a mirror operation, with population "reflected" from one state to the other. One can use a sequence of beam splitters and mirrors to make an interferometer:



Atom optics based on a pure two-level system are perhaps the simplest form of atom optics and will be used in experiments like AION and MAGIS. For this form of atom optics, it is desirable to use a transition with a long-lived excited state $|e\rangle$ to limit signal degradation from the excited state undergoing decay to spontaneous emission. One can use, for instance, the 698 nm clock transition in strontium. Not all atoms have such transitions readily available, and the fact that such transitions are necessarily weak puts demanding requirements on the laser system being used. Therefore, oftentimes multiphoton transitions that create an effective two-level system are used in atom interferometry. The ideal type of atom optics to use depends on the specific application.

AC Stark Shifts

In a two-level system, the presence of a laser field will lead to energy shifts known as AC Stark shifts. Not only can these lead to systematic errors to be accounted for, but they can themselves serve as useful tools for atom optics. It is useful to make a transformation to coefficients:

$$\tilde{c}_1(t) = c_1(t), \quad \tilde{c}_2(t) = c_2(t)\, e^{i\delta t}$$

We can substitute these into our differential equations:

$$\dot{c}_1(t) = \dot{\tilde{c}}_1(t) = -i\frac{\Omega}{2}\tilde{c}_2(t)\, e^{-i\phi}$$

$$\dot{c}_2(t) = \dot{\tilde{c}}_2(t)\, e^{-i\delta t} - i\delta\, \tilde{c}_2(t)\, e^{-i\delta t} = -i\frac{\Omega}{2} e^{i\phi} e^{-i\delta t}\, \tilde{c}_1(t)$$

$$\Rightarrow \quad \dot{\tilde{c}}_2(t) = -i\frac{\Omega}{2} e^{i\phi}\, \tilde{c}_1(t) + i\delta\, \tilde{c}_2(t)$$

This corresponds to a Hamiltonian matrix

$$H = \hbar \begin{pmatrix} 0 & \frac{\Omega}{2} e^{-i\phi} \\ \frac{\Omega}{2} e^{i\phi} & -\delta \end{pmatrix}$$

so that $i\hbar \begin{pmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{pmatrix} = H \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$

The energy eigenvalues for the coupled system are:

$$E_{g,e} = \frac{\hbar}{2}\left(-\delta \pm \sqrt{\Omega^2+\delta^2}\right) = \frac{\hbar}{2}\left(-\delta \pm \delta\sqrt{1+\Omega^2/\delta^2}\right.$$

For the shifted ground state, for $|\delta| >> |\Omega|$,

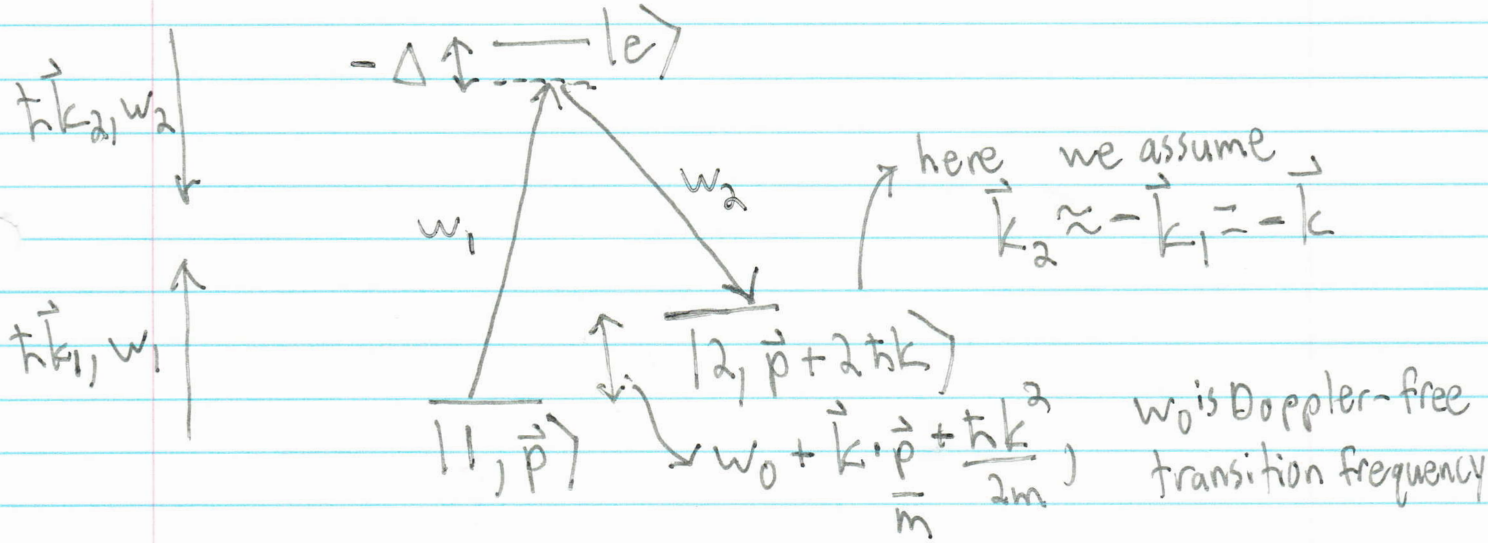
$$E_g \approx \frac{\hbar\Omega^2}{4\delta}$$

Ω^2 is proportional to the laser intensity, so this leads to an intensity-dependent energy shift.

Matter Wave Optics Using Velocity Selective Raman Transitions

Moler et al., PRA 45, 342 (1992)

Velocity-selective Raman transitions can be used to create an effective two-level system between two different internal states with different momentum:



If we neglect AC Stark shifts, the effective Hamiltonian for a pulse starting at time t_0 is:

$$\hat{H} = \hbar\frac{\Omega_{eff}}{2}\left[e^{i(\phi+\delta(t-t_0))}|2\rangle\langle 1| + e^{-i(\phi+\delta(t-t_0))}|1\rangle\langle 2|\right]$$

where $\delta \equiv \omega_0 + \vec{k}\cdot\frac{\vec{p}}{m} + \frac{\hbar k^2}{2m} - (\omega_1 - \omega_2)$

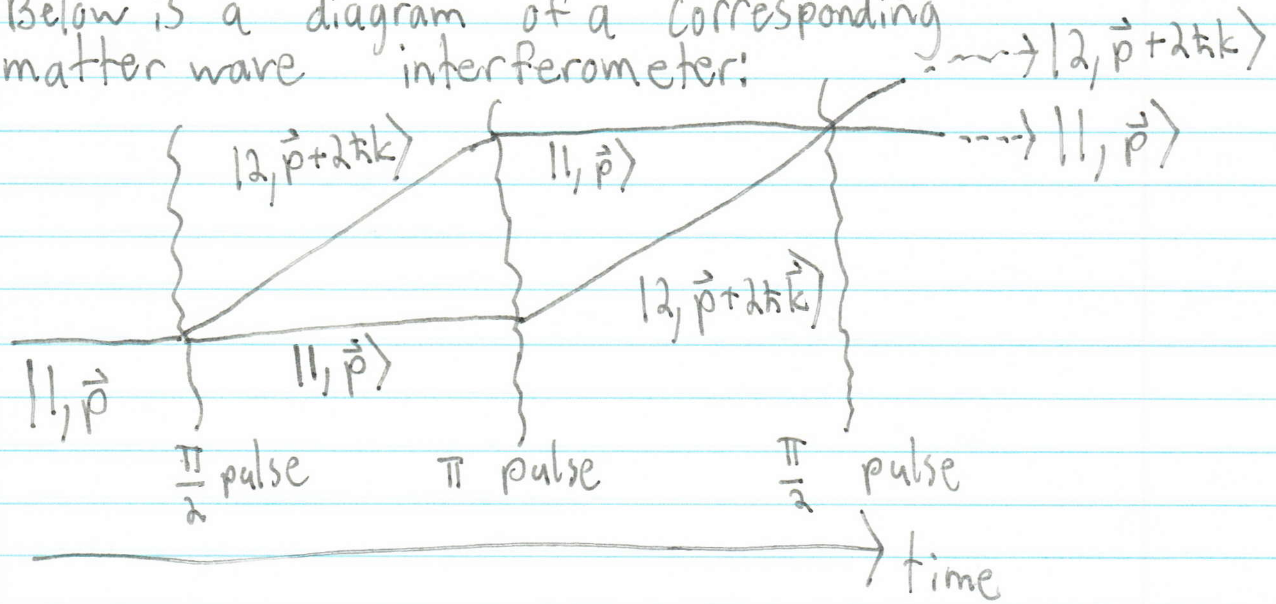
and $\phi \equiv \phi_1 - \phi_2$ where

$\phi_i = \vec{k}_i \cdot \vec{r} - w_i t_0$ is the phase of beam i at time t_0, where $\vec{r}$ is the position of the atom/particle at time t_0.

$\Rightarrow \phi = 2\vec{k} \cdot \vec{r} - (w_1 - w_2) t_0$

Also, $\Omega_{eff} = \dfrac{|\Omega_1 \Omega_2|}{2\Delta}$, where Ω_1 and Ω_2 are the Rabi frequencies of the two beams.

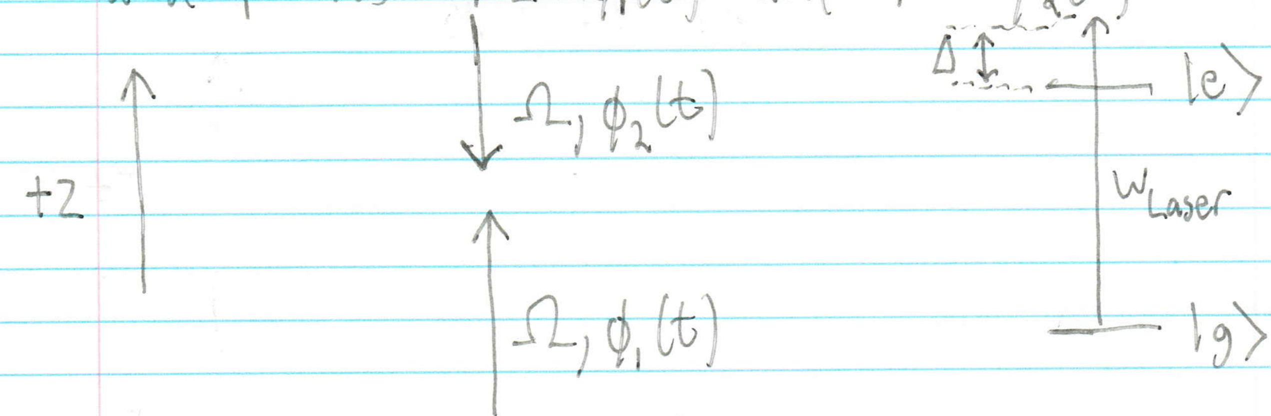
We can implement $\frac{\pi}{2}$ or π pulses with this two-level system to serve as matter wave beam splitters or mirrors, respectively. Below is a diagram of a corresponding matter wave interferometer:



The phase difference between the two interfering paths will determine the probability of ending up in $|1, \vec{p}\rangle$ or $|2, \vec{p} + 2\hbar \vec{k}\rangle$ after the final beam splitter.

Phase Gratings from Optical Standing Waves

Let us consider two counterpropagating beams in the z-direction with wave numbers k, individual Rabi frequencies Ω, and frequencies and phases $kz-\phi_1(t)$ and $-kz-\phi_2(t)$.



Both beams are taken to have detuning Δ from an excited state. For large Δ, we will approximate spontaneous emission as small and an associated potential energy from the AC Stark shift as:

$$V(z) = \frac{\hbar}{4\Delta}\left|\Omega e^{i(kz-\phi_1(t))} + \Omega e^{i(-kz-\phi_2(t))}\right|^2$$

$$= \frac{\hbar}{4\Delta}|\Omega|^2\, 2\left(1+\cos\left[2\left(kz-\frac{1}{2}(\phi_1(t)-\phi_2(t))\right)\right]\right)$$

where the expression was simplified in Mathematica.

If we define an effective Rabi frequency

$$\Omega_{eff} = \frac{|\Omega|^2}{2\Delta}$$

by analogy to the case of Raman transitions, we obtain:

$$V(z) = \hbar\Omega_{eff} + \hbar\Omega_{eff}\cos\left[2\left(kz - \frac{1}{2}\left(\phi_1(t) - \phi_2(t)\right)\right)\right]$$

This corresponds to a constant offset AC Stark shift $\hbar\Omega_{eff}$ added to a sinusoidal potential from the spatially varying AC Stark shift.

Such a potential is sometimes called an optical lattice. We note that the positions of the maxima/minima of the cosine shift with $\phi(t) \equiv \phi_1(t) - \phi_2(t)$ as:

$$d(t) = \frac{\phi(t)}{2k}$$

If $\phi_1(t) = w_1 t$ and $\phi_2(t) = w_2 t$, the lattice translates at a constant velocity $v = \frac{w_1 - w_2}{2k}$.

Including also kinetic energy, we obtain a Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{z})$$

It interesting to consider the dynamics governed by this Hamiltonian in momentum space. We represent the quantum state as:

$$|\Psi\rangle = \int dp\, \Psi(p,t)|p\rangle \quad \text{where} \quad \Psi(p) \equiv \langle p|\Psi\rangle$$

The state evolves according to the Schrödinger equation:

$$i\hbar \frac{\partial |\Psi\rangle}{\partial t} = \hat{H}|\Psi\rangle$$

$$\Rightarrow i\hbar \frac{\partial \Psi(p,t)}{\partial t} = \langle p|\hat{H}|\Psi\rangle = \int dp'\, \Psi(p',t)\langle p|\hat{H}|p'\rangle$$

We want to evaluate the matrix elements $\langle p|\hat{H}|p'\rangle$.

First, we note that

$$\langle p|\frac{\hat{p}^2}{2m} + \hbar\Omega_{eff}|p'\rangle = \left(\frac{(p')^2}{2m} + \hbar\Omega_{eff}\right)\langle p|p'\rangle$$

$$= \left(\frac{(p')^2}{2m} + \hbar\Omega_{eff}\right)\delta(p'-p)$$

We can write the cosine term in $V(\hat{z})$ as:

$$V_{cos}(\hat{z}) = \frac{\hbar\Omega_{eff}}{2}\left[e^{i2k\hat{z}}e^{-i\phi(t)} + e^{-i2k\hat{z}}e^{i\phi(t)}\right]$$

Recalling that $e^{i\frac{p_0}{\hbar}\hat{z}}|p\rangle = |p+p_0\rangle$,

$$\langle p|V_{cos}(\hat{z})|p'\rangle = \frac{\hbar\Omega_{eff}}{2}e^{-i\phi(t)}\underbrace{\langle p|p'+2\hbar k\rangle}_{\delta(p'-(p-2\hbar k))}$$

$$+ \frac{\hbar\Omega_{eff}}{2}e^{i\phi(t)}\underbrace{\langle p|p'-2\hbar k\rangle}_{\delta(p'-(p+2\hbar k))}$$

$$\Rightarrow i\hbar\frac{\partial\Psi(p,t)}{\partial t} = \left[\frac{p^2}{2m}+\hbar\Omega_{eff}\right]\Psi(p,t)$$

$$+ \frac{\hbar\Omega_{eff}}{2}\left[e^{-i\phi(t)}\Psi(p-2\hbar k,t) + e^{i\phi(t)}\Psi(p+2\hbar k,t)\right]$$

We note that a given momentum state $|p\rangle$ is only coupled to momentum states $|p \pm 2\hbar k\rangle$.

This makes intuitive sense, since we can think of momentum transfer arising solely from two-photon processes in which a photon is absorbed from one beam and emitted into the other, counterpropagating beam — transferring momentum $\pm 2\hbar k$.

Therefore, discrete families of momentum states are coupled together:

$$|p_0 + 2n\hbar k\rangle \quad \text{for integer } n$$

For a given p_0, we then have the following coupled set of differential equations:

$$i\hbar \frac{\partial}{\partial t} \Psi(p_0 + 2n\hbar k, t) =$$

$$= \left[\frac{(p_0 + 2n\hbar k)^2}{2m} + \hbar\Omega_{eff}\right] \Psi(p_0 + 2n\hbar k, t) +$$

$$+ \frac{\hbar\Omega_{eff}}{2}\left[e^{-i\phi(t)}\Psi(p_0 + 2(n-1)\hbar k, t) + e^{i\phi(t)}\Psi(p_0 + 2(n+1)\hbar k, t)\right]$$

Diffraction at Short Interaction Times: the Raman-Nath Regime

Let us imagine that the optical lattice is pulsed on for a short time τ. The associated spread of the optical frequencies for such a pulse is:

$$\Delta \omega \sim \frac{1}{\tau}$$

Therefore, each photon that is absorbed or emitted has an energy uncertainty

$$\Delta E \sim \hbar \Delta \omega \sim \frac{\hbar}{\tau}$$

We note that this is closely related to the energy-time uncertainty principle. Let us choose a frame of reference in which $p_0 = 0$ and say that our initial state is the $0\hbar k$ momentum state:

$$\Psi(2n\hbar k, 0) = \delta_{n,0}$$

If $\Delta E \sim \frac{\hbar}{\tau}$ is much greater than the kinetic energy of the system, then we can make the approximation of neglecting the kinetic energy contributions

to the time evolution.

We will also neglect for the moment the energy shift $\hbar\Omega_{eff}$ that affects all states equally.
We then have the equations:

$$i\hbar \frac{\partial \Psi(2n\hbar k, t)}{\partial t} = \frac{\hbar\Omega_{eff}}{2}\Big[e^{-i\phi(t)}\Psi(p_0 + 2(n-1)\hbar k, t) + e^{i\phi(t)}\Psi(p_0 + 2(n+1)\hbar k, t)\Big]$$

The solution to this discrete set of equations can be expressed in terms of Bessel functions, so that the populations in different momentum states evolve as:

$$|\Psi(2n\hbar k, t|^2 = |J_n(\Omega_{eff} t)|^2$$

As t increases, momentum will spread out into higher momentum states, and the system will eventually leave the Raman-Nath regime if the pulse length τ is extended.

Bragg Diffraction

For longer pulse times, the kinetic energy term can no longer be neglected.
Again neglecting the overall energy shift $\hbar\Omega_{eff}$ and choosing a reference frame in which $p_0 = 0$, our dynamical equations are:

$$i\hbar \frac{\partial \Psi(2n\hbar k, t)}{\partial t} = 4\hbar n^2 w_r \Psi(2n\hbar k, t), \quad w_r \equiv \frac{\hbar k^2}{2m}$$

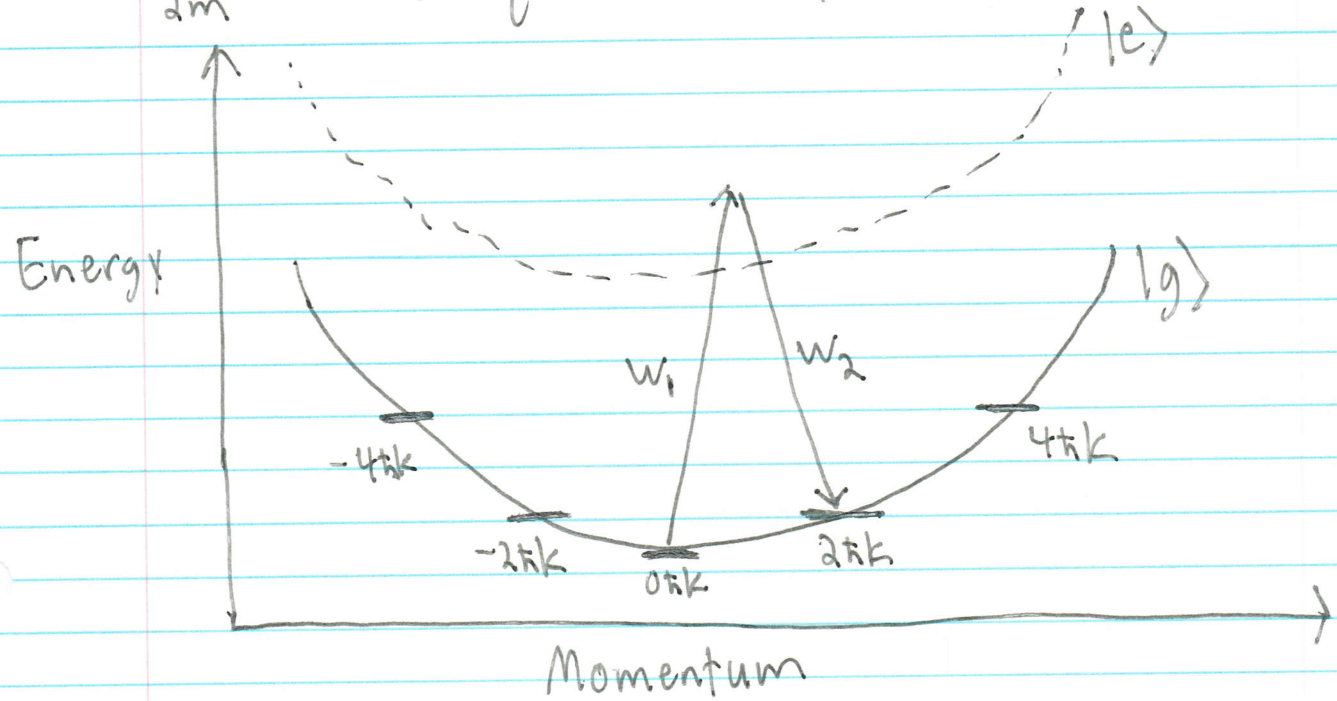
$$+ \frac{\hbar\Omega_{eff}}{2}\left[e^{-i\phi(t)}\Psi(2(n-1)\hbar k, t) + e^{i\phi(t)}\Psi(2(n+1)\hbar k, t)\right]$$

If the upward propagating laser has frequency w_1 and the downward propagating laser has frequency w_2, we will express:

$$\phi(t) = \phi_1(t) - \phi_2(t) = (w_1 - w_2)t$$

It is often useful to solve these equations numerically, but we can gain helpful intuition about the time evolution through looking at the energies of the various coupled momentum states and considering conservation of energy and momentum when photons are absorbed and emitted.

For the momentum eigenstates, the energy $\frac{p^2}{2m}$ has a quadratic dependence on momentum:



We can think of the counterpropagating beams as driving Raman transitions between different momentum states, but without the internal state changing.

Under certain conditions, we can realize an approximate two-level system coupling two momentum states $|p_0\rangle$ and $|p_0 + 2\hbar k\rangle$.

To transition from $|p_0\rangle$ to $|p_0 + 2\hbar k\rangle$, a photon is absorbed from beam 1 and emitted into beam 2, transferring momentum $2\hbar k$. The inverse of this process transitions from $|p_0 + 2\hbar k\rangle$ to $|p_0\rangle$. For the $|p_0\rangle \rightarrow |p_0 + 2\hbar k\rangle$ transition, the atom (or other particle) gains energy $\hbar(w_1 - w_2)$ from the light field. By conservation of energy, for this two-level system to be resonantly coupled, we need the change ΔKE in kinetic energy to be equal to the energy gained from the light field:

$$\hbar(w_1 - w_2) = \Delta KE = \frac{1}{2m}\left[(p_0 + 2\hbar k)^2 - p_0^2\right]$$

$$\Rightarrow w_1 - w_2 = 2\frac{p_0}{\hbar} v_r + 4 w_r \quad (|p_0\rangle \leftrightarrow |p_0 + 2\hbar k\rangle)$$

for $v_r = \frac{\hbar k}{m}$, $\quad w_r = \frac{\hbar k^2}{2m}$

We also note, however, that $|p_0 + 2\hbar k\rangle$ will be coupled to $|p_0 + 4\hbar k\rangle$. The resonance condition for this transition is

$$w_1 - w_2 = 2\frac{p_0}{\hbar}v_r + \underbrace{4\hbar k v_r}_{8w_r} + 4w_r$$

$$= \left(2\frac{p_0}{\hbar}v_r + 4w_r\right) + 8w_r \quad \left(|p_0 + 2\hbar k\rangle \longleftrightarrow |p_0 + 4\hbar k\rangle\right)$$

This transition is detuned by $8w_r$ from the resonance of our desired $|p_0\rangle \longleftrightarrow |p_0 + 2\hbar k\rangle$ coupling. Similarly, the resonance condition for the neighboring $|p_0 - 2\hbar k\rangle \longleftrightarrow |p_0\rangle$ transition is:

$$w_1 - w_2 = \left(2\frac{p_0}{\hbar} + 4w_r\right) - 8w_r$$

so this transition is detuned by $-8w_r$ from the resonance of our desired transition.

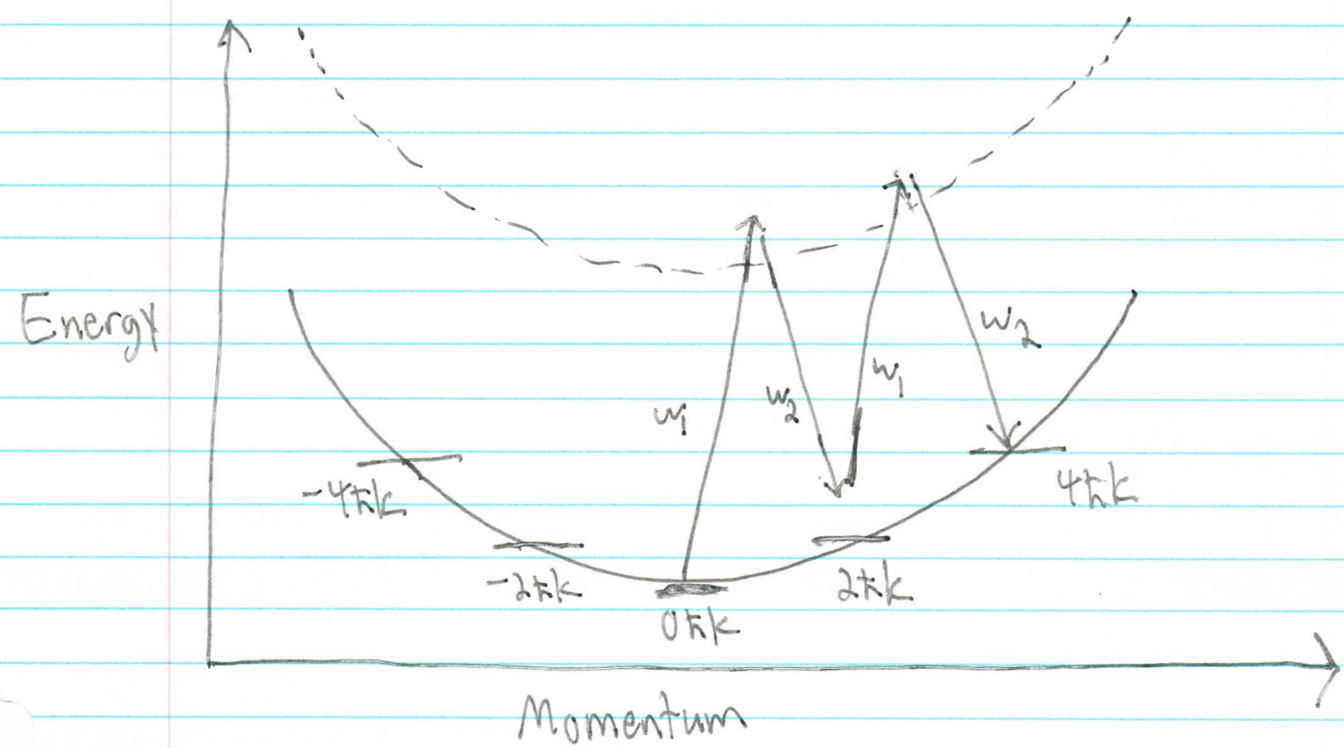
If $\Omega_{eff} << 8\omega_{r1}$ these neighboring transitions can be considered far off-resonance, and we have an approximately pure two-level system coupling $|p_0\rangle$ and $|p_0 + 2\hbar k\rangle$. For larger Ω_{eff}, the dynamics become more complicated, involving more than two states.

Transfer between the different momentum states arising interaction of a matter wave with an optical lattice is closely analogous of Bragg diffraction (into different diffraction orders) of light interacting with a crystal.

Higher order Bragg transitions

While the presence of neighboring transitions can be viewed as an undesirable complication in some cases, it can also have advantages.

For example, it is possible to drive higher order Bragg transitions that in a single pulse transfer a larger amount of momentum. An nth order Bragg transition transfers $2n\hbar k$ of momentum and corresponds to the absorbtion of n photons from one beam and the emission of n photons into the other beam:



In the example shown above, $|0\hbar k\rangle$ and $|4\hbar k\rangle$ are coupled, with $|2\hbar k\rangle$ serving as a detuned intermediate state.

For coupling between $|p_0\rangle$ and $|p_0 + 2n\hbar k\rangle$, the resonance condition becomes:

$$n(w_1 - w_2) = 2n\frac{p_0}{\hbar}v_r + 4n^2 w_r$$

$$\Rightarrow (w_1 - w_2) = 2\frac{p_0}{\hbar}v_r + 4n w_r$$

Calculating the Phase Shift in Matter Wave Interferometers

In using matter wave interferometers for measurements, it is important to be able to calculate how the phase shift between the interferometer arms depends on external influences one might want to measure, such as gravity or rotations, for example. We will study a formalism for carrying out these calculations, focusing on the case of atom optics based on pulses of light (e.g., Raman or Bragg transitions).

We divide the calculation into periods of light-free evolution in which the atoms external degrees of freedom evolve according to a Hamiltonian:

$$\hat{H}_{ext} = H(\hat{\vec{r}}, \hat{\vec{p}})$$

and periods of interaction with the light.

Propagation Phase

We will first consider the phase associated with propagation of the matter waves between the times at which the atom optics laser pulses occur. The external quantum state along a given interferometer path, represented by $|\psi\rangle$, evolves as:

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = H(\hat{\vec{r}}, \hat{\vec{p}})|\psi\rangle$$

$H(\hat{\vec{r}}, \hat{\vec{p}})$ may include the influence of factors such as the gravitional potential, Coriolis and centrifugal forces associated with rotations, and potentials induced by external electric or magnetic fields.

It is useful to boost into the center of mass frame of the wave packet, in which

$$\langle \hat{\vec{r}} \rangle = \langle \hat{\vec{p}} \rangle = 0$$

Where $\vec{r}_c$ and $\vec{p}_c$ correspond the classical trajectory of the wave packet in the lab frame, and L_c is the Lagrangian in the lab frame evaluated along this trajectory (note that $\vec{r}_c$, $\vec{p}_c$, and L_c will in general be time dependent), we can write a unitary transformation $\hat{U}_c$ that boosts between the center of mass (COM) frame and the lab frame:

$$\hat{U}_c = \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} L_c \, dt'\right] \exp\left[-\frac{i}{\hbar}\vec{r}_c \cdot \hat{\vec{p}}\right] \exp\left[\frac{i}{\hbar}\vec{p}_c \cdot \hat{\vec{r}}\right]$$

such that:

$$\underbrace{|\Psi_{Lab}\rangle}_{\text{lab frame state}} = \hat{U}_c \underbrace{|\Psi_{com}\rangle}_{\text{COM frame state}}$$

Let examine why this is the case. First, we note that:

$$\exp\left[-\frac{i}{\hbar}\vec{r}_c\cdot\hat{\vec{p}}\right]|\vec{r}\rangle = |\vec{r}+\vec{r}_c\rangle$$

$$\exp\left[\frac{i}{\hbar}\vec{p}_c\cdot\hat{\vec{r}}\right]|\vec{p}\rangle = |\vec{p}+\vec{p}_c\rangle$$

$$\Rightarrow \hat{U}_c^\dagger\hat{\vec{r}}\,\hat{U}_c|\vec{r}\rangle = \exp\left[\frac{i}{\hbar}\vec{r}_c\cdot\hat{\vec{p}}\right]\hat{\vec{r}}\exp\left[-\frac{i}{\hbar}\vec{r}_c\cdot\hat{\vec{p}}\right]|\vec{r}\rangle$$

$$= \exp\left[\frac{i}{\hbar}\vec{r}_c\cdot\hat{\vec{p}}\right]\hat{\vec{r}}|\vec{r}+\vec{r}_c\rangle$$

$$= (\vec{r}+\vec{r}_c)|\vec{r}\rangle$$

for any position eigenstate $|\vec{r}\rangle$. This implies that

$$\hat{U}_c^\dagger\hat{\vec{r}}\,\hat{U}_c = \hat{\vec{r}}+\vec{r}_c$$

By an analogous argument,

$$\hat{U}_c^\dagger\hat{\vec{p}}\,\hat{U}_c = \hat{\vec{p}}+\vec{p}_c$$

$$\Rightarrow \hat{U}_c^\dagger H(\hat{\vec{r}},\hat{\vec{p}})\hat{U}_c = H(\hat{\vec{r}}+\vec{r}_c, \hat{\vec{p}}+\vec{p}_c)$$

Time evolution in the COM frame is given by:

$$i\hbar \frac{\partial |\Psi_{COM}\rangle}{\partial t} = i\hbar \frac{\partial}{\partial t}\left(\hat{U}_c^\dagger |\Psi_{Lab}\rangle\right)$$

$$= i\hbar\left(\frac{\partial}{\partial t} U_c^\dagger\right)|\Psi_{Lab}\rangle + \hat{U}_c^\dagger i\hbar \frac{\partial}{\partial t}|\Psi_{Lab}\rangle$$

$$= i\hbar\left(\frac{\partial}{\partial t}\hat{U}_c^\dagger\right)|\Psi_{Lab}\rangle + \hat{U}_c^\dagger H(\hat{\vec{r}}, \hat{\vec{p}})|\Psi_{Lab}\rangle$$

$$= \left[\hat{U}_c^\dagger H(\hat{\vec{r}}, \hat{\vec{p}})\hat{U}_c + i\hbar\left(\frac{\partial}{\partial t}\hat{U}_c^\dagger\right)\hat{U}_c\right]|\Psi_{COM}\rangle$$

$$= \left[H(\hat{\vec{r}} + \vec{r}_c, \hat{\vec{p}} + \vec{p}_c) + \dot{\vec{p}}_c \cdot \hat{\vec{r}} - (\hat{\vec{p}} + \vec{p}_c)\cdot \dot{\vec{r}}_c + L_c\right]|\Psi_{COM}\rangle$$

We can Taylor expand $H(\hat{\vec{r}} + \vec{r}_c, \hat{\vec{p}} + \vec{p}_c)$ about the classical trajectory $\vec{r}_c$ and $\vec{p}_c$:

$$H(\hat{\vec{r}} + \vec{r}_c, \hat{\vec{p}} + \vec{p}_c) = H(\vec{r}_c, \vec{p}_c) + \nabla_{\hat{\vec{r}}} H(\hat{\vec{r}}, \hat{\vec{p}})\Big|_{\vec{r}_c, \vec{p}_c} \cdot \hat{\vec{r}}$$
$$+ \nabla_{\hat{\vec{p}}} H(\hat{\vec{r}}, \hat{\vec{p}})\Big|_{\vec{r}_c, \vec{p}_c} \cdot \hat{\vec{p}} + \hat{H}_2$$

where $\hat{H}_2$ includes all terms 2nd order or higher in $\hat{\vec{r}}$ and $\hat{\vec{p}}$.

So:

$$i\hbar \frac{\partial|\Psi_{com}\rangle}{\partial t} = \Big[\left(H_c - \vec{p}_c\cdot\dot{\vec{r}}_c + L_c\right) + \left(\nabla_{\vec{r}_c} H_c + \dot{\vec{p}}_c\right)\cdot\hat{\vec{r}}$$
$$+ \left(\nabla_{\vec{p}_c} H_c - \dot{\vec{r}}_c\right)\cdot\hat{\vec{p}} + \hat{H}_2\Big]|\Psi_{com}\rangle$$

where $H_c \equiv H(\vec{r}_c, \vec{p}_c)$, is the Hamiltonian evaluated along the classical trajectory. From classical mechanics, we recall that:

$$\dot{\vec{r}}_c = \nabla_{\vec{p}_c} H_c$$

$$\dot{\vec{p}}_c = -\nabla_{\vec{r}_c} H_c \quad \left(\text{where } \vec{p}_c = \frac{\partial L}{\partial \dot{\vec{r}}_c}\right)$$

$$L_c = \dot{\vec{r}}_c\cdot\vec{p}_c - H_c$$

Therefore, all terms cancel except for the $\hat{H}_2$ term, leaving us with:

$$i\hbar \frac{\partial|\Psi_{com}\rangle}{\partial t} = \hat{H}_2|\Psi_{com}\rangle$$

Under certain conditions, if $|\Psi_{cm}\rangle$ begins with $\langle\hat{\vec{r}}\rangle = \langle\hat{\vec{p}}\rangle = 0$, then $\langle\hat{\vec{r}}\rangle$ and $\langle\hat{\vec{p}}\rangle$ remain 0 as the system evolves. This makes the COM frame especially convenient to work with. The time evolution of these expectation values is governed by:

$$\frac{d}{dt}\langle\hat{\vec{r}}\rangle = \frac{i}{\hbar}\left\langle[\hat{H}_2, \hat{\vec{r}}]\right\rangle = \left\langle\nabla_{\hat{\vec{p}}}\hat{H}_2\right\rangle$$

$$\frac{d}{dt}\langle\hat{\vec{p}}\rangle = \frac{i}{\hbar}\left\langle[\hat{H}_2, \hat{\vec{p}}]\right\rangle = -\left\langle\nabla_{\hat{\vec{r}}}\hat{H}_2\right\rangle$$

We let $\hat{r}_i$ and $\hat{p}_i$ respectively denote the ith component of $\hat{\vec{r}}$ and $\hat{\vec{p}}$ for $i = x, y, z$.

We can then Taylor expand $\left\langle\frac{\partial}{\partial\hat{p}_i}\hat{H}_2\right\rangle$ and $\left\langle\frac{\partial}{\partial\hat{r}_i}\hat{H}_2\right\rangle$ about $\hat{r}_i = \langle\hat{r}_i\rangle$, $\hat{p}_i = \langle\hat{p}_i\rangle$:

$$\frac{d}{dt}\langle\hat{r}_i\rangle = \left\langle\frac{\partial}{\partial\hat{p}_i}\hat{H}_2\Big|_{\langle\hat{\vec{r}}\rangle,\langle\hat{\vec{p}}\rangle} + \sum_j \frac{\partial}{\partial\hat{p}_j}\frac{\partial}{\partial\hat{p}_i}\hat{H}_2\Big|_{\langle\hat{\vec{r}}\rangle,\langle\hat{\vec{p}}\rangle}\left(\hat{p}_j - \langle\hat{p}_j\rangle\right)\right.$$

$$\left. + \sum_j \frac{\partial}{\partial\hat{x}_j}\frac{\partial}{\partial\hat{p}_i}\hat{H}_2\Big|_{\langle\hat{\vec{r}}\rangle,\langle\hat{\vec{p}}\rangle}\left(\hat{x}_j - \langle\hat{x}_j\rangle\right) + \text{higher order terms}\right\rangle = 0 + \text{higher order terms}$$

The higher order terms involve three (or more) partial derivatives of $\hat{H}_2$ with respect to different combinations of $\hat{r}_i$ and $\hat{p}_i$. An example is:

$$\frac{1}{2}\left\langle \sum_{j,k} \frac{\partial}{\partial \hat{p}_j}\frac{\partial}{\partial \hat{p}_k}\frac{\partial}{\partial \hat{p}_i}\hat{H}_2\Big|_{\langle\hat{\vec{r}}\rangle,\langle\hat{\vec{p}}\rangle}\left(\hat{p}_j-\langle\hat{p}_j\rangle\right)\left(\hat{p}_k-\langle\hat{p}_k\rangle\right)\right\rangle$$

$$=\frac{1}{2}\sum_{j,k}\frac{\partial}{\partial \hat{p}_j}\frac{\partial}{\partial \hat{p}_k}\frac{\partial}{\partial \hat{p}_i}\hat{H}_2\Big|_{\langle\hat{\vec{r}}\rangle,\langle\hat{\vec{p}}\rangle}\underbrace{\left(\langle\hat{p}_j\hat{p}_k\rangle-\langle\hat{p}_j\rangle\langle\hat{p}_k\rangle\right)}_{\text{factors of this type correspond to the wave packet's width in phase space}}$$

If $\hat{H}_2$ is at most 2nd order in $\hat{\vec{r}}, \hat{\vec{p}}$, then these higher order terms exactly vanish due to having three or more partial derivatives. Even if $\hat{H}_2$ has third or higher order terms, it is often the case that the higher order terms are very small.

If we neglect the higher order terms, which amounts to a sort of semiclassical approximation,

$$\frac{d}{dt}\langle \hat{r}_i \rangle = 0$$

and so $\langle \hat{\vec{r}} \rangle$ will remain 0. An analogous argument can be used to show that $\langle \hat{\vec{p}} \rangle$ remains 0 in the semiclassical approximation. Even with $\langle \hat{\vec{r}} \rangle = \langle \hat{\vec{p}} \rangle = 0$, the spatial amplitude and phase profile of $|\psi_{com}\rangle$ will evolve under $\hat{H}_2$. This evolution could in principle be different for the two interferometer arms, which would affect the ultimate interference pattern. However, for $\hat{H}_2$ limited to 2nd order in $\hat{\vec{r}}, \hat{\vec{p}}$, $\hat{H}_2$ will be indepent of $\vec{r}_c$ and $\vec{p}_c$ and therefore identical for both interferometer arms. For higher order $\hat{H}_2$, this is no longer in general the case, but often the associated corrections are small. We will thus further approximate $\hat{H}_2$ as being the same for both interferometer arms.

The position space lab frame wave function at time t is then:

$$\Psi_{Lab}(\vec{r},t) = \langle \vec{r} | \hat{U}_c | \Psi_{com} \rangle = \int d\vec{r}\,' \langle \vec{r} | \hat{U}_c | \vec{r}\,' \rangle \langle \vec{r}\,' | \Psi_{com} \rangle$$

$$= \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} L_c\,dt'\right] \int d\vec{r}\,' \exp\left[\frac{i}{\hbar}\vec{p}_c \cdot \vec{r}\,'\right] \underbrace{\langle \vec{r} | \vec{r}\,' + \vec{r}_c \rangle}_{\delta(\vec{r}\,' - (\vec{r} - \vec{r}_c))} \Psi_{com}(\vec{r}\,',t)$$

$$= \exp\left[\frac{i}{\hbar}\int_{t_0}^{t} L_c\,dt'\right] \exp\left[\frac{i}{\hbar}\vec{p}_c \cdot (\vec{r} - \vec{r}_c)\right] \Psi_{com}(\vec{r} - \vec{r}_c, t)$$

As expected, since $\Psi_{com}(\vec{r}\,',t)$ is centered at $\vec{r}\,'=0$, the lab frame wavefunction is centered on the classical position $\vec{r}_c$.

For propagation between times t_i and t_{i+1}, the associated phase shift comes from the classical action integral:

$$\phi_{prop,i} = \frac{1}{\hbar}\int_{t_i}^{t_{i+1}} (L_c - E_{int})\,dt$$

where we also include phase evolution from the internal energy of the particle E_{int}. This will matter, for instance, if different arms of the interferometer are in different internal states.

The total propagation phase shift between the two interferometer arms is:

$$\Delta\phi_{prop} = \sum_{arm 1} \phi_{prop,i} - \sum_{arm 2} \phi_{prop,i}$$

Laser Phase

When the matter wave exchanges momentum with the laser field, it receives a phase that we refer to as the laser phase. We will consider here the example of a two-photon Bragg transition, though a similar analysis holds more generally (e.g., for Raman transitions).

Neglecting the overall light shift, the interaction is governed by the interaction Hamiltonian

$$V_{cos}(\hat{\vec{r}}) = \frac{\hbar\Omega_{eff}}{2}\left[e^{i\vec{k}\cdot\hat{\vec{r}}}e^{-i\phi(t)} + e^{-i\vec{k}\cdot\hat{\vec{r}}}e^{i\phi(t)}\right]$$

$\downarrow e^{i(\vec{k}_2\cdot\vec{r}-\phi_2(t))}$

$\uparrow e^{i(\vec{k}_1\cdot\vec{r}-\phi_1(t))}$

$$\vec{k} \equiv \vec{k}_1 - \vec{k}_2$$

$$\phi(t) \equiv \phi_1(t) - \phi_2(t)$$

In the absence of laser pulses, we can write the state in the lab frame as:

$$|\Psi_{Lab}\rangle = \int d\vec{p}\, \Psi_{Lab}(\vec{p})\, |\phi_{\vec{p}}\rangle$$

We adopt the interaction picture in which time evolution due to sources other than the laser pulses comes the time dependence of the states $|\phi_{\vec{p}}\rangle$ rather than the coefficients $\Psi_{Lab}(\vec{p})$. The laser pulse will introduce time dependence into $\Psi_{Lab}(\vec{p})$, while the time evolution of $|\phi_{\vec{p}}\rangle$ is governed by influences not including the laser pulse.

In the COM frame, momentum states evolve according to:

$$e^{-\frac{i}{\hbar}\hat{H}_2(t-t_0)}|\vec{p}\rangle,$$

where we reference to the time t_0 at which the laser pulse is applied.

$$\Rightarrow |\phi_p\rangle = \hat{U}_c(t)\, e^{-\frac{i}{\hbar}\hat{H}_2(t-t_0)}|\vec{p}\rangle$$

During the pulse, the interaction picture dynamical equations are:

$$i\hbar \frac{\partial}{\partial t} \Psi_{Lab}(\vec{p},t) = \int d\vec{p}\,' \Psi_{Lab}(\vec{p}\,') \langle \phi_{\vec{p}} | V_{cos}(\hat{\vec{r}}) | \phi_{\vec{p}\,'} \rangle$$

$$= \int d\vec{p}\,' \Psi_{Lab}(\vec{p}\,') \langle \vec{p} | e^{\frac{i}{\hbar}\hat{H}_2(t-t_0)} \hat{U}_c^\dagger(t) V_{cos}(\hat{\vec{r}}) \hat{U}_c(t) \cdot$$

$$\cdot e^{-\frac{i}{\hbar}\hat{H}_2(t-t_0)} | \vec{p}\,' \rangle$$

For a short pulse length, the $\frac{\hat{p}^2}{2m}$ term in $\hat{H}_2$ often dominates, and it is often valid to approximate $\vec{r}_c(t)$ as $\vec{r}_c(t_0)$. Then:

$$i\hbar \frac{\partial}{\partial t} \Psi_{Lab}(\vec{p},t) = \int d\vec{p}\,' \Psi_{Lab}(\vec{p}\,') \langle \vec{p} | V_{cos}(\vec{r} + \vec{r}_c(t_0)) | \vec{p}\,' \rangle \cdot$$

$$\cdot \underbrace{\exp\left[\frac{i}{\hbar}\left(\frac{p^2}{2m} - \frac{(p')^2}{2m}\right)(t-t_0)\right]}$$

This last factor relates to the resonance condition such that energy is conserved. Assuming resonance is satisfied, we will approximate $t \approx t_0$ so that this factor is 1.

assuming a short pulse so that quanties are evaluated at $t=t_0$

We note that

$$\langle \vec{p} | V_{cos}(\vec{r}+\vec{r}_c(t_0) | \vec{p}\,' \rangle = \frac{\hbar\Omega_{eff}}{2} e^{i(\vec{k}\cdot\vec{r}_c(t_0)-\phi(t_0))} \cdot \underbrace{\langle \vec{p} | \vec{p}\,'+\hbar\vec{k} \rangle}_{\delta(\vec{p}\,'-(\vec{p}-\hbar\vec{k}))} + \frac{\hbar\Omega_{eff}}{2} e^{-i(\vec{k}\cdot\vec{r}_c(t_0)-\phi(t_0))} \underbrace{\langle \vec{p}\,' | \vec{p}\,'-\hbar\vec{k} \rangle}_{\delta(\vec{p}\,'-(\vec{p}+\hbar\vec{k}))}$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi_{Lab}(\vec{p},t) = \frac{\hbar\Omega_{eff}}{2}\left[e^{i\phi_L(t_0)}\Psi_{Lab}(\vec{p}-\hbar\vec{k},t) + e^{-i\phi_L(t_0)}\Psi_{Lab}(\vec{p}+\hbar\vec{k},t)\right]$$

for $\phi_L(t_0) \equiv \vec{k}\cdot\vec{r}_c(t_0)-\phi(t_0)$.

Note that we have dropped terms related to kinetic energy, which will serve the purpose of imposing a resonance condition that determines which momentum states are resonantly coupled.

So if the matter wave gains momentum $\pm\hbar\vec{k}$ from the laser field, it receives a phase of $\pm\phi_L(t_0)$

If we apply many pulses (or a higher order Bragg pulse) to exchange momentum $\pm n\hbar k$, then the received phase is $\pm n\phi_L(t_0)$.

The total laser phase contribution to the interferometer phase shift sums up all these contributions for each arm, and then takes the difference between the two arms:

$$\Delta\phi_{Laser} = \sum_{arm 1} \pm n\phi_L(t_i) - \sum_{arm 2} \pm n\phi_L(t_i)$$

Separation Phase

After the final beam splitter pulse, we can consider the wavefunction of each arm of the interferometer in a given output port:

$$\Psi_{Lab}(\vec{r},t) = e^{i\phi_{prop,\,arm\,i}}\, e^{i\phi_{laser,\,arm\,i}} \cdot$$

$$\cdot \exp\left[\frac{i}{\hbar}\vec{p}_{c,i}\cdot(\vec{r}-\vec{r}_{c,i})\right]\Psi_{com}(\vec{r}-\vec{r}_{c,i},t)$$

where $i=1,2$ indicates arm 1 or arm 2.

In addition to $\Delta\phi_{prop}$ and $\Delta\phi_{Laser}$, we thus have another contribution to the phase shift:

$$\frac{1}{\hbar}\left[(\vec{p}_{c,1}-\vec{p}_{c,2})\cdot\vec{r} - \vec{p}_{c,1}\cdot\vec{r}_{c,1} + \vec{p}_{c,2}\cdot\vec{r}_{c,2}\right]$$

We can rewrite this in terms of $\bar{\vec{p}}_c \equiv \frac{1}{2}(\vec{p}_{c,1}+\vec{p}_{c,2})$, $\Delta\vec{p} \equiv \vec{p}_{c,1}-\vec{p}_{c,2}$, $\bar{\vec{r}}_c \equiv \frac{1}{2}(\vec{r}_{c,1}+\vec{r}_{c,2})$, and $\Delta\vec{r}_c \equiv \vec{r}_{c,1}-\vec{r}_{c,2}$:

$$-\frac{1}{\hbar}\bar{\vec{p}}_c\cdot\Delta\vec{r}_c + \Delta\vec{p}_c\cdot(\vec{r}-\bar{\vec{r}}_c)$$

The first term is called the separation phase:

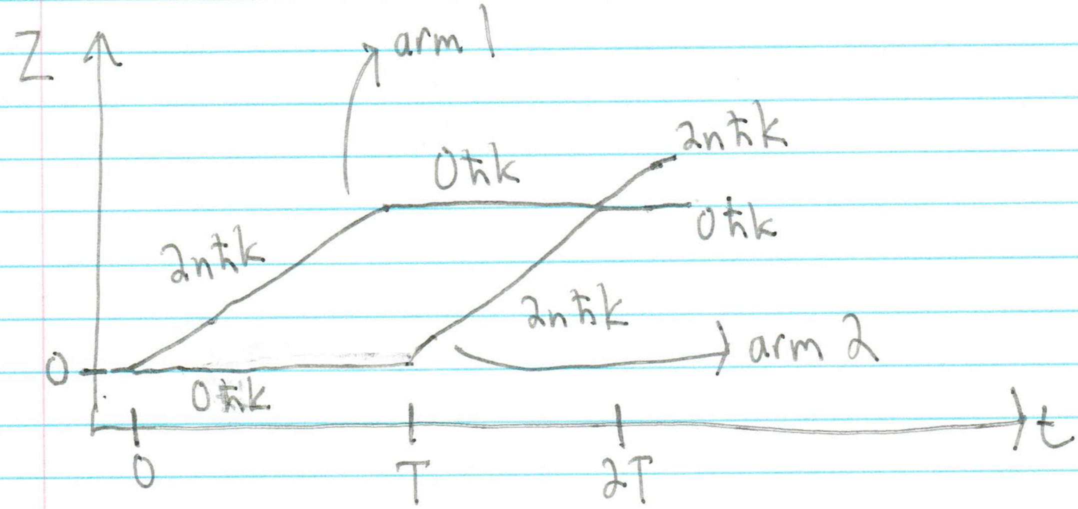
$$\Delta\phi_{sep} = -\frac{1}{\hbar}\vec{p}_c \cdot \Delta\vec{r}_c$$

The second term corresponds to a position dependent phase shift, which is often (though not always) small. For a symmetric wavefunction, this phase will average to zero. Approximating the position dependent phase as negligible, the total interferometer phase shift is:

$$\Delta\phi = \Delta\phi_{prop} + \Delta\phi_{Laser} + \Delta\phi_{sep}$$

Example: Phase Shift in a Uniform Gravitational Field

Let us consider an interferometer in a uniform gravitional field with momentum transferred in units of $2n\hbar k$ vertically. In a freely falling frame, the trajectories of the two arms look like:



For a pulse transferring momentum $\pm 2n\hbar k$ at time t_i, the imprinted laser phase will be:

$$\pm\left[2n\hbar k\, z(t_i) - \phi(t_i)\right]$$

For simplicity, let us assume that $\phi(t_i)$ is maintained at 0.

$\Delta\phi_{sep} = 0$ since the two arms overlap at the end of the interferometer. It can also be shown that $\Delta\phi_{prop} = 0$ in this case, so $\Delta\phi = \Delta\phi_{Laser}$. We note that:

$$\Delta\phi_{Laser} = 2n\,k \cdot 0 - 2n\,k\left(-\frac{1}{2}gT^2 + 2n\frac{\hbar k}{m}T\right)$$

$$-2n\,k\left(-\frac{1}{2}gT^2\right) + 2n\,k\left(-\frac{1}{2}g(2T)^2 + 2n\frac{\hbar k}{m}T\right)$$

$$= 2n\,kgT^2 - 4n\,kgT^2$$
$$= -2n\,kgT^2$$

To put in some example numbers, for $n=10$, $g = 9.8\ \frac{m}{s^2}$, $T = 1s$, and $k = \frac{2\pi}{780\ nm}$:

$$\Delta\phi = 1.6 \times 10^9 \text{ rad},$$

which is quite a large phase shift.