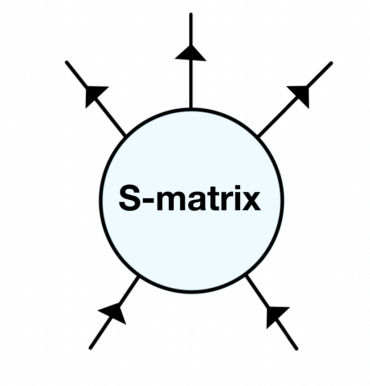
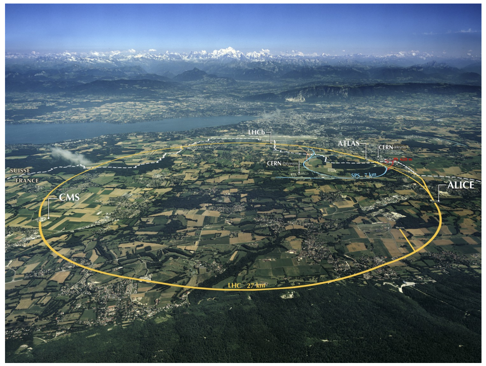
Recent Progress with Two-Loop QCD Scattering Amplitudes

Harald Ita

Laboratory for Particle Physics
Theory Group
Paul Scherrer Institut

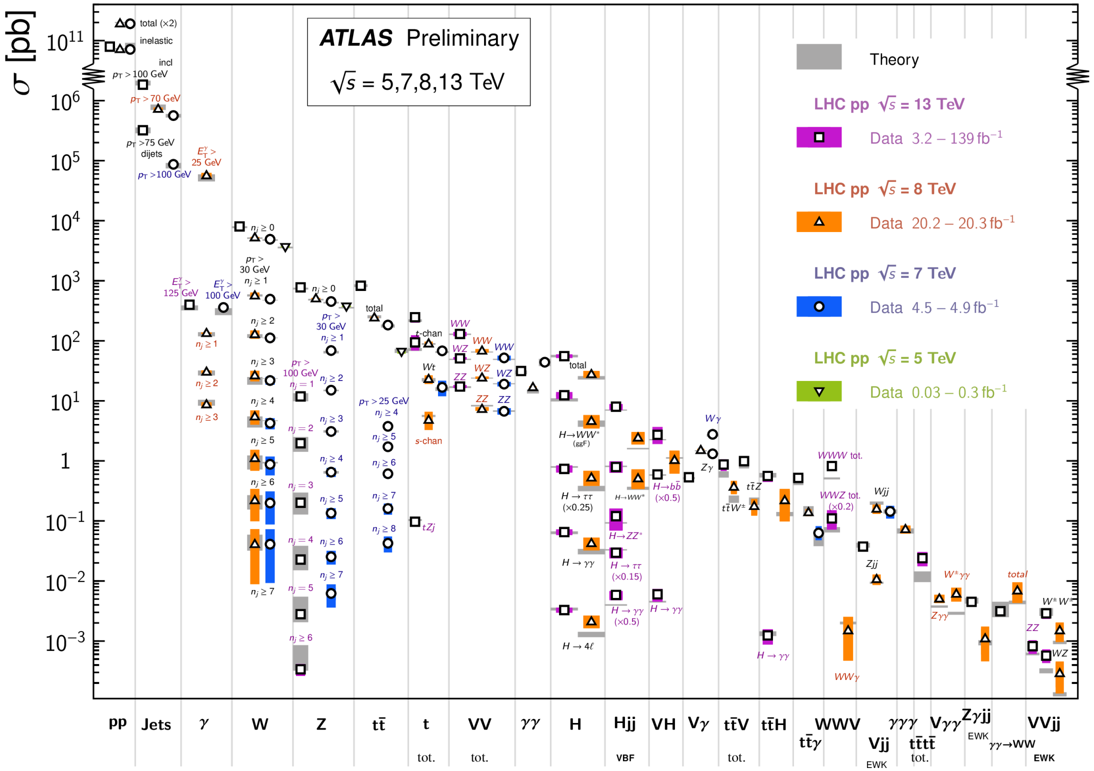




Zurich Phenomenology Workshop
11th of January 2023

Motivation

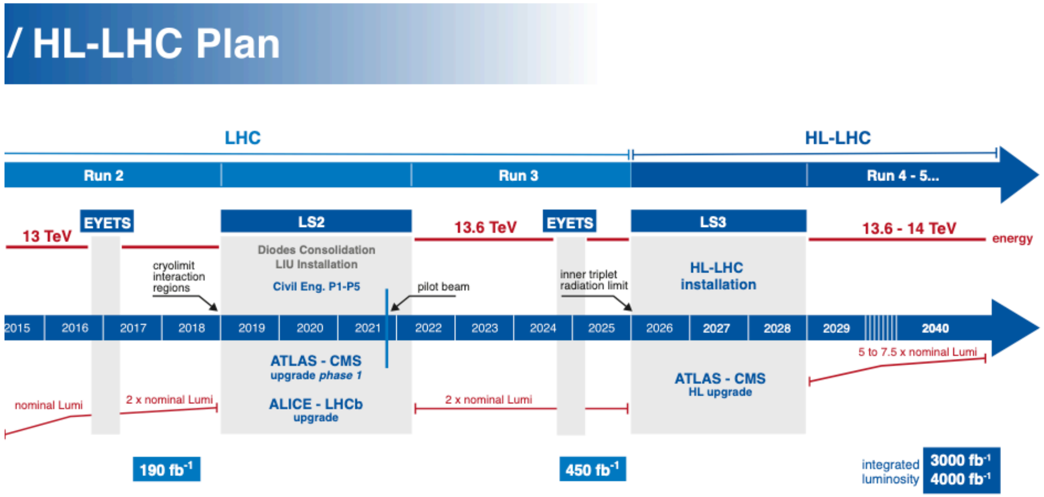
- Impressive understanding of Standard Model at high-energy collisions



[ATL-PHYS-PUB-2022-009, February 2022]

Motivation

- Ten-fold increase in data at LHC experiments

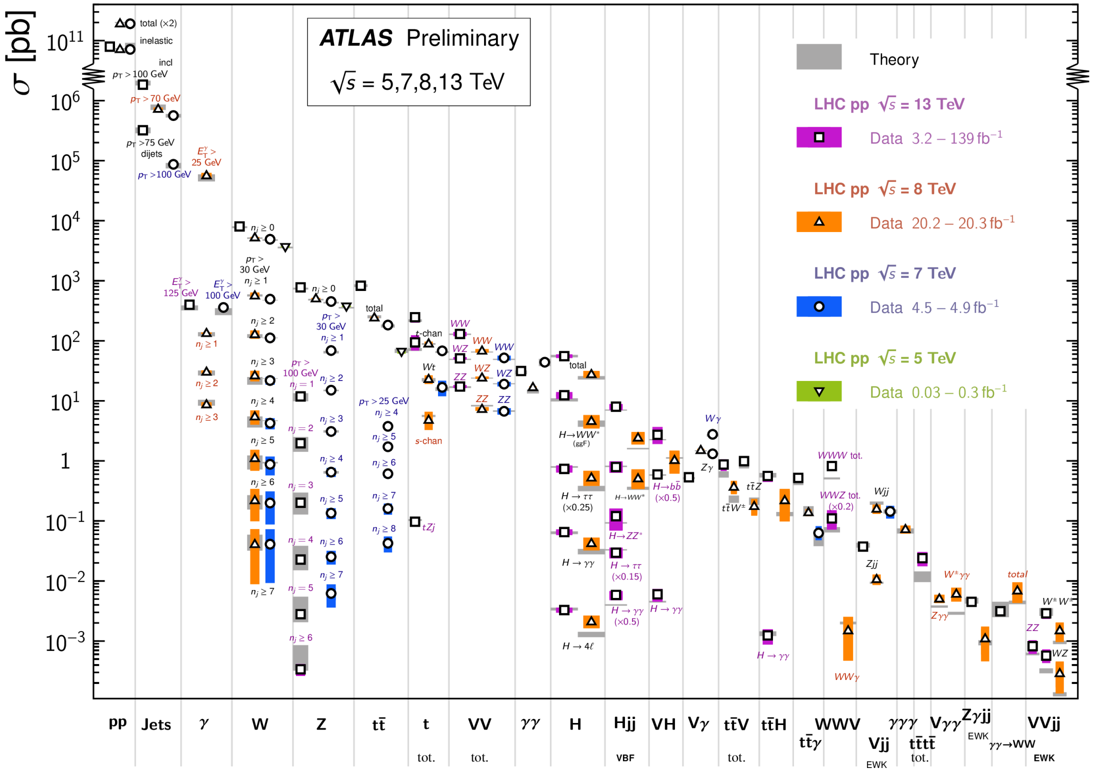


- Physics goal: Higgs couplings 2-4%, W mass, top mass, $\sin\theta_w$, multi W/Z,…

- Theory goal: few-percent precision for many observables:

$$\sigma = \sigma_{LO} + \alpha_s\, \Delta\sigma^{qcd}_{nlo} + \alpha_s^2\, \Delta\sigma^{qcd}_{nnlo} + \alpha_f\, \Delta\sigma^{ew}_{nlo} + \ldots$$

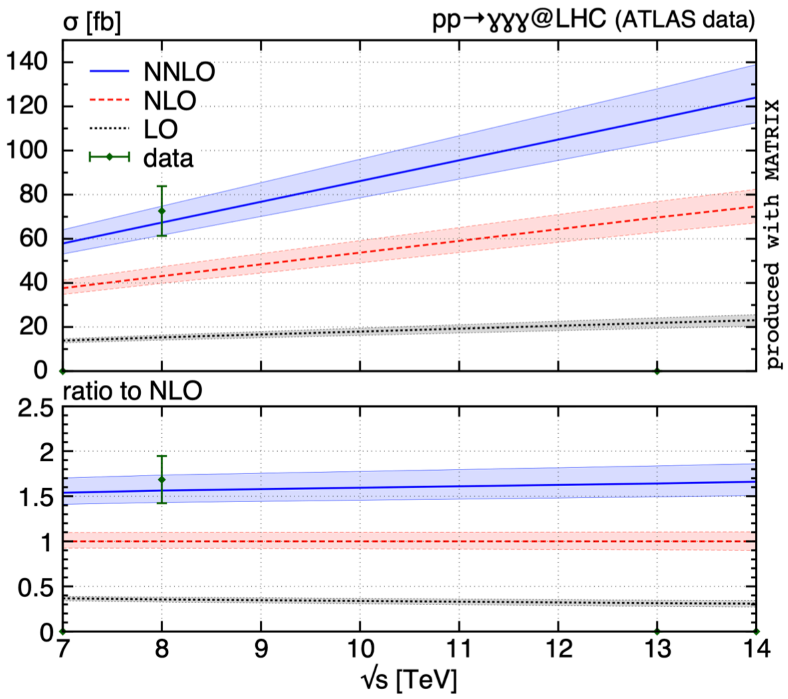
$\mathcal{O}(10\%)$ $\quad$ $\mathcal{O}(1-5\%)$ $\quad$ $\mathcal{O}(1\%)$

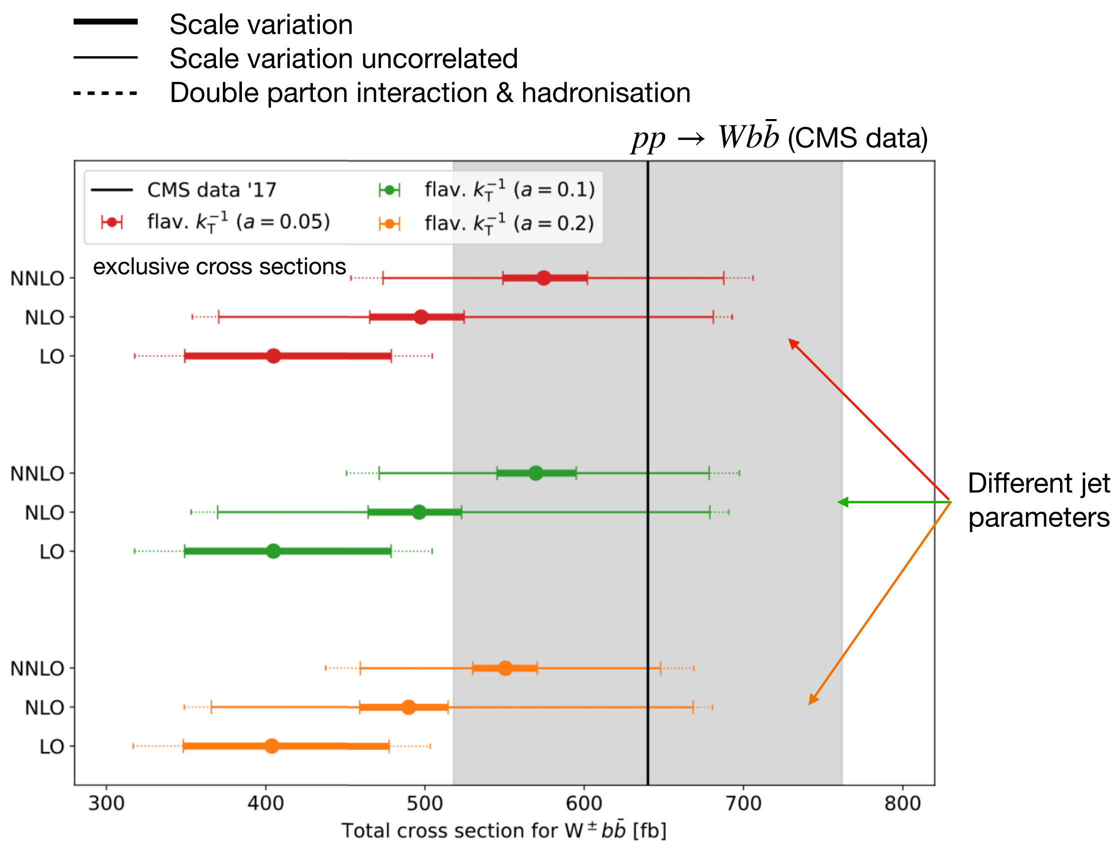


[ATL-PHYS-PUB-2022-009, February 2022]

Motivation

- NNLO QCD calculations: challenging but important





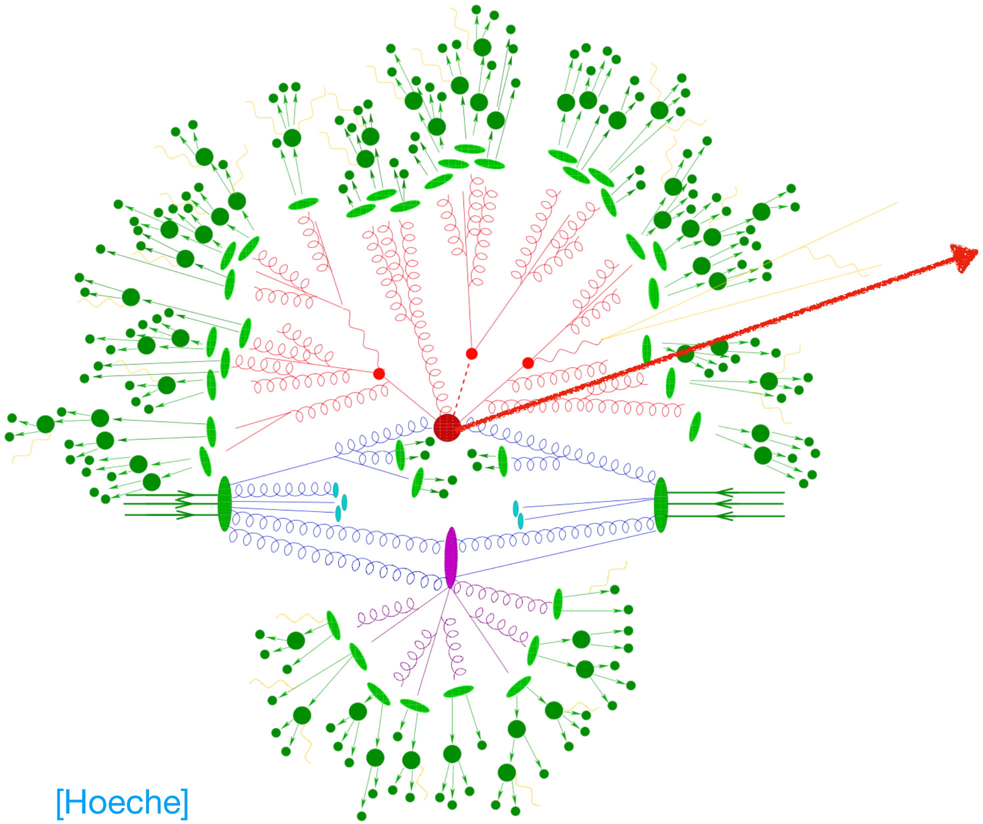
[Kallweit, Sotnikov, Wiesemann, 20]

[see also Chawdhry, Czakon, Mitov, Poncelet,19]

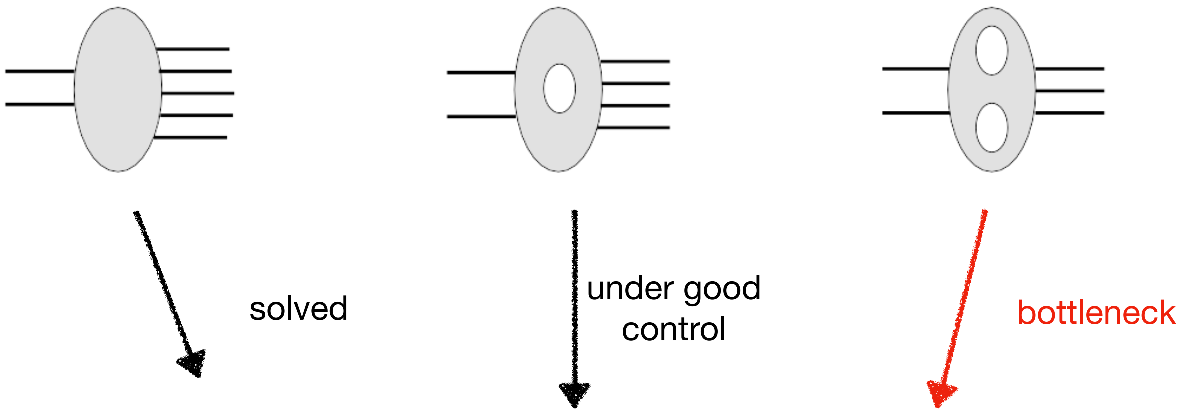
[Hartanto, Poncelet, Popescu, Zoia 22]

Motivation

- Scattering amplitudes for NNLO QCD cross sections to five-point processes:



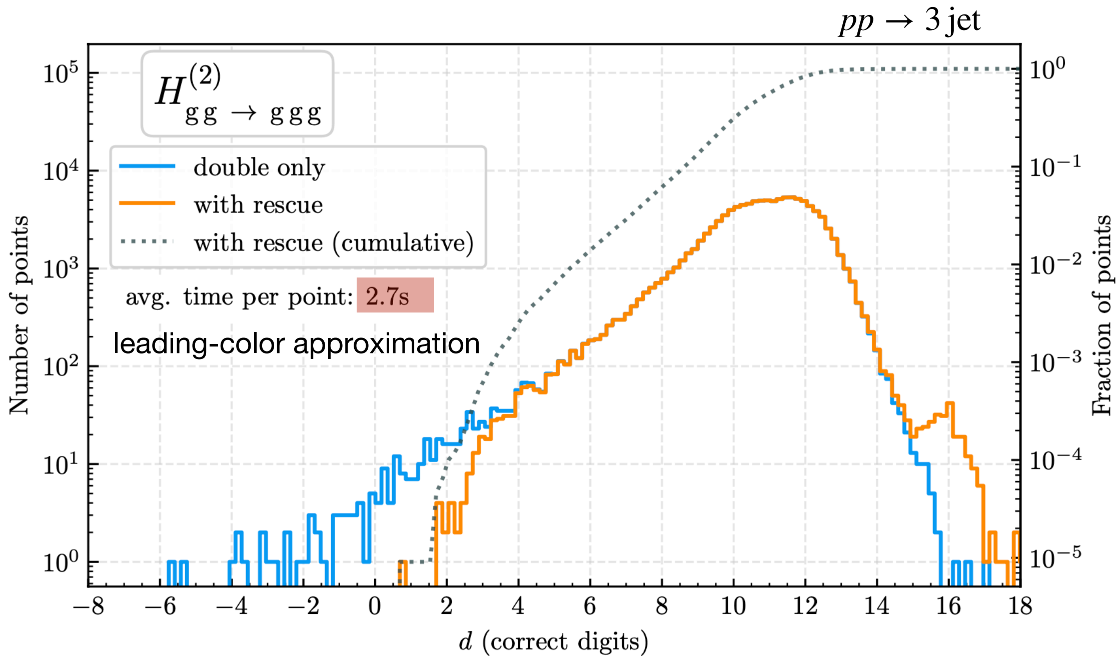
[Hoeche]



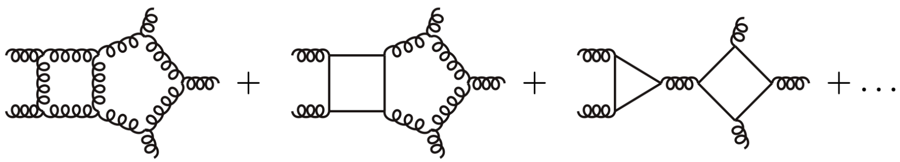
$$\sigma_{nnlo}^{qcd} = \sigma_{rr} + \sigma_{rv} + \sigma_{vv}$$

Motivation

- Bottleneck: fast and numerically stable evaluation of two-loop amplitudes



[Abreu, Febres Cordero, Ita, Page, Sotnikov, 21]



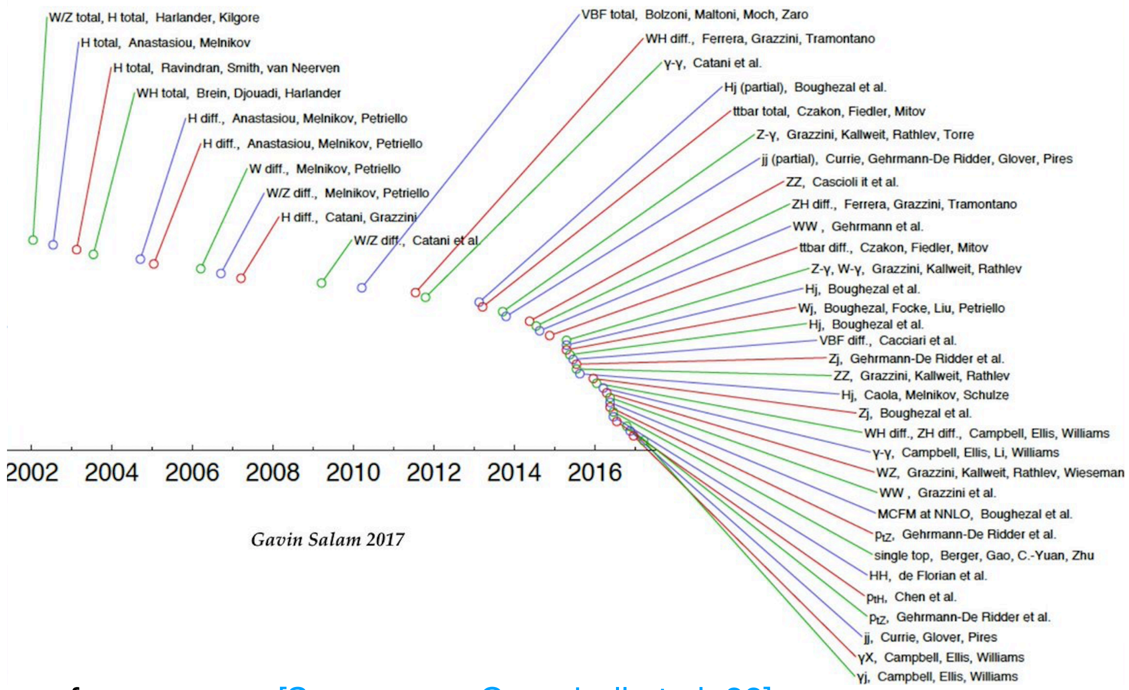
✓ σ_{VV} for 3-jet production at NNLO QCD

✓ σ_{RVV} for 2-jet production at N^{3}LO QCD

Status: NNLO QCD @ LHC

- Many available $2 \to 2$ processes
- Progress due to advances in handling infrared singularities [Boughezal, Cacciari, Campbell, Catani, Chen, Czakon, Caola, Dreyer, Focke, Gaunt, Gehrmann-De Ridder, Gehrmann, Giele, Glover, Grazzini, Gaunt, Huss, Ellis, Kallweit, Karlberg, Liu, Melnikov, Petriello, Salam, Stahlhofen, Tackmann, Walsh, Wiesemann, Williams, Zanderighi…]
- Multi-scale $2 \to 2$ processes
- Current frontier $2 \to 3$ processes: jjj,Wjj, Zjj, ttH, γjj,…
  - Multi-particle couplings
  - Standard candles

(see talks by Heinrich, Tancredi, Zanderighi)



For references see [Snowmass; Campbell et al. 22]

Higgs	SM candles	Jets	Other
H [71, 72, 84, 85]	$W^{\pm}$ [74, 78, 105, 106]	dijets [131–134]	single top [147, 148]
[74, 80, 86, 87]	Z [72, 74, 78, 106, 107]	3 jets [135]	$t\bar{t}$ [149–155]
$W^{\pm}H$ [88–90]	$\gamma\gamma$ [78, 108–111]	$W^{\pm}$+jet [73, 136–138]	$b\bar{b}$ [156]
ZH [90, 91]	$W^{\pm}\gamma$ [78, 112–114]	Z+jet [139–141]	$H \to b\bar{b}$ [157–159]
H (VBF) [76, 92]	$Z\gamma$ [78, 115, 115]	γ+jet [142, 143]	t decay [148, 160, 161]
HH [93]	W^+W^- [78, 116–120]	$Z+b$ [144]	$e^+e^- \to 3j$ [162, 163]
HHH [94]	WZ [121, 122]	$W^{\pm}c$ [145]	DIS (di-)jets [164, 165]
H+jet [95–102]	ZZ [78, 123–128]	$\gamma\gamma$+jet [146]	
$W^{\pm}H$+jet [103]	$\gamma\gamma\gamma$ [129, 130]		
ZH+jet [104]			

Status: Five-Point Two-Loop Amplitudes

	Comment	Complete analytic results	Public numerical code	Cross sections
$pp \to jjj$	l.c.	[9]	[9]	[1, 2]
$pp \to \gamma\gamma j$	l.c.*	[10, 11]	[10]	[12]
$pp \to \gamma\gamma\gamma$		[13, 14]	[13]	[15, 16]
$pp \to \gamma\gamma j$		[3]		
$gg \to \gamma\gamma g$	NLO loop induced	[4]	[4]	[17]
$pp \to Wb\bar{b}$	l.c.*, on-shell W	[18, 19]		
$pp \to W(l\nu)b\bar{b}$	l.c.	[20, 21]		[21]
$pp \to W(l\nu)jj$	l.c.	[20]		
$pp \to Z(l\bar{l})jj$	l.c.*	[20]		
$pp \to W(l\nu)\gamma j$	l.c.*	[22]		
$pp \to Hb\bar{b}$	l.c., b-quark Yukawa	[23]		
$pp \to Ht\bar{t}$	approx. 2-loop ampl.			[24]

Table 1: Known two-loop QCD corrections for five-point scattering processes at hardon colliders. "l.c." refers to the calculations in the leading-color approximation; "l.c.*" means that in addition non-planar l.c. contributions are omitted. All public codes employ `PentagonFunctions++` [25, 26] for numerical evaluation of special functions.

8

Adapted from [Sotnikov '22; Abreu '22]

See also Les Houches Standard-Model Precision Wishlist [Huss, Huston, Jones, Pellen '22]

- Demand: multi-scale processes with 5-10 kinematic scales

  [5 x 4 (mom.) + 5 (masses) -

  - 5 (mass shell) - 4 (mom. cons.) - 6 (Lorentz) = 10]

Status: Five-Point Two-Loop Amplitudes

[1] M. Czakon, A. Mitov and R. Poncelet, *Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC*, Phys. Rev. Lett. **127** (2021) 152001 [2106.05331]. (pages 2 and 3)

[2] X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, *Automation of antenna subtraction in colour space: gluonic processes*, 2203.13531. (pages 2 and 3)

[3] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-Loop Helicity Amplitudes for Diphoton Plus Jet Production in Full Color*, Phys. Rev. Lett. **127** (2021) 262001 [2105.04585]. (pages 2 and 3)

[4] S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H.B. Hartanto, J. Henn et al., *Virtual QCD corrections to gluon-initiated diphoton plus jet production at hadron colliders*, JHEP **11** (2021) 083 [2106.08664]. (pages 2, 3, and 5)

[9] S. Abreu, F.F. Cordero, H. Ita, B. Page and V. Sotnikov, *Leading-color two-loop QCD corrections for three-jet production at hadron colliders*, JHEP **07** (2021) 095 [2102.13609]. (pages 3 and 7)

[10] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-loop leading colour QCD corrections to $q\bar{q} \to \gamma\gamma g$ and $qg \to \gamma\gamma q$*, JHEP **04** (2021) 201 [2102.01820]. (page 3)

[11] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-colour QCD helicity amplitudes for two-photon plus jet production at the LHC*, JHEP **07** (2021) 164 [2103.04319]. (page 3)

[12] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to diphoton production with an additional jet at the LHC*, JHEP **09** (2021) 093 [2105.06940]. (page 3)

[13] S. Abreu, B. Page, E. Pascual and V. Sotnikov, *Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders*, JHEP **01** (2021) 078 [2010.15834]. (page 3)

[14] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-color helicity amplitudes for three-photon production at the LHC*, JHEP **06** (2021) 150 [2012.13553]. (page 3)

[15] H.A. Chawdhry, M.L. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, JHEP **02** (2020) 057 [1911.00479]. (page 3)

[16] S. Kallweit, V. Sotnikov and M. Wiesemann, *Triphoton production at hadron colliders in NNLO QCD*, Phys. Lett. B **812** (2021) 136013 [2010.04681]. (page 3)

[17] S. Badger, T. Gehrmann, M. Marcoli and R. Moodie, *Next-to-leading order QCD corrections to diphoton-plus-jet production through gluon fusion at the LHC*, Phys. Lett. B **824** (2022) 136802 [2109.12003]. (page 3)

[18] S. Badger, H.B. Hartanto and S. Zoia, *Two-Loop QCD Corrections to Wbb¯Production at Hadron Colliders*, Phys. Rev. Lett. **127** (2021) 012001 [2102.02516]. (pages 3 and 7)

[19] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *Flavour anti-k_T algorithm applied to $Wb\bar{b}$ production at the LHC*, 2209.03280.

[20] S. Abreu, F. Febres Cordero, H. Ita, M. Klinkert, B. Page and V. Sotnikov, *Leading-color two-loop amplitudes for four partons and a W boson in QCD*, JHEP **04** (2022) 042 [2110.07541]. (pages 3 and 5)

[21] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *NNLO QCD corrections to $Wb\bar{b}$ production at the LHC*, 2205.01687. (pages 3 and 4)

[22] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading colour helicity amplitudes for $W\gamma + j$ production at the LHC*, JHEP **05** (2022) 035 [2201.04075]. (page 3)

[23] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading-colour QCD helicity amplitudes for Higgs boson production in association with a bottom-quark pair at the LHC*, JHEP **11** (2021) 012 [2107.14733]. (page 3)

[24] S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli and C. Savoini, *$t\bar{t}H$ production in NNLO QCD*, 2210.07846. (page 3)

Used Feynman-Integral package

[25] D. Chicherin and V. Sotnikov, *Pentagon Functions for Scattering of Five Massless Particles*, JHEP **12** (2020) 167 [2009.07803]. (pages 2, 3, 6, 7, and 8)

[26] D. Chicherin, V. Sotnikov and S. Zoia, *Pentagon functions for one-mass planar scattering amplitudes*, JHEP **01** (2022) 096 [2110.10111]. (pages 3, 6, and 9)

Amplitude Computation

- Feynman diagrams:

$$A = \sum_{i \in \text{all integrals}} I_i(\epsilon, \vec{p}) \,,$$

$$I_i(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{n_i(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2}$$

- Integration-by-parts relations (IBP): [Chetyrkin, Tkachov 81; Laporta 00]

$$\int d^D\ell \, d^D\tilde{\ell} \, \frac{\partial}{\partial \ell_\mu} \left[\frac{v^\mu(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2} \right] = 0$$

$$\to \sum_{i \in \text{all integrals}} b_i(\epsilon, \vec{p}) \, I_i(\epsilon, \vec{p}) = 0$$

↪ find basis of integrals by solving linear system

- Sum of master integrals:

$$A = \sum_{i \in \text{basis}} c_i(\epsilon, \vec{p}) \, I_i(\epsilon, \vec{p})$$

- Integration:

$$I_i(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{n_i(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2}$$

↪ gives 11-dimensional integrals at two loop five-point

- Differential equations method: [Kotikov 91; Bern, Dixon, Kosower 93; Remiddi 97, Gehrmann, Remiddi 99,…]

$$\frac{\partial}{\partial s_{ij}} I_k(\epsilon, \vec{p}) = \int d^D\ell \, d^D\tilde{\ell} \, \frac{\partial}{\partial s_{ij}} \left[\frac{n_k(\ell, \tilde{\ell})}{\ell^2(\ell - p_1)^2 \cdots (\tilde{\ell} - p_1 - \ldots p_n)^2} \right] = \sum_{k,j} m_{kj}(\epsilon, \vec{p}) I_j(\epsilon, \vec{p})$$

↪ IBP reduction

↪ solve multi-variate differential equation & boundary conditions

- Integral functions in Laurent expansion in ϵ :

$$I_i(\epsilon, \vec{p}) = \sum \epsilon^k h_{ik}(\vec{p})$$

$h_i(\vec{p}) \in \{1, \ln(s_{12}), \ldots\}$ … integral functions

- Integrated amplitude:

$$A = \sum_{i \in \text{basis}} \epsilon^j \, d_{ij}(\vec{p}) \, h_{ij}(\vec{p})$$

Amplitude Computation

- Computational steps well established, but very complex:

  - number of diagrams/terms

  - number of variables in linear system: momenta & masses

  - multi-dimensional integration

- Keys to progress:

  - advance methods, new ideas

  - examples and structural understanding

- Simplicity of analytic results:

  - indicates mathematical & physical properties of amplitudes, which may lead to better ways to compute

Feynman diagrams

Integral reduction using integration by parts (IBP)

↓

Sum of master integrals

$$A = \sum_{i \in basis} c_i(\epsilon, \vec{p}\,)\, I_i(\epsilon, \vec{p}\,)$$

Differential equations (DE) or numerical integration

↓

Integrated amplitude

$$A = \sum_{i,j} \epsilon^j d_{ij}(\vec{p}\,)\, h_{ij}(\vec{p}\,)$$

Numerical Amplitude Computation

- Numerical evaluations avoid problems of manipulating multi-variate expressions

  - Numerical algorithms for one-loop amplitudes during `NLO revolution' [Blackhat, GoSam, Recola, OpenLoops, NJet, Recola,…]

- Challenges:

  - Numerical instabilities in integral reduction

  - Dimension dependence

- Solution:

  - Exact rational arithmetic $\mathbb{Q}$ instead of floating point $\mathbb{R}$ (actually: finite-field arithmetic $\mathbb{F}$) [$\mathbb{Q}$: commonly for checks; $\mathbb{F}$: vManteuffel, Schabinger 15]

  - Focus on simple rational functions

  - Functional reconstruction of analytic expressions [Peraro 16]

$$\boxed{\vec{p} \to \mathbb{Q}\,, \quad \epsilon \to \mathbb{Q}}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

? Laurent expansion in ϵ

Numerical Amplitude Computation

- Rationality of integral coefficients in ϵ

[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng 17]

$$c(\epsilon) = \frac{n_0 + n_1\,\epsilon + n_2\,\epsilon^2 + \ldots}{d_0 + d_1\,\epsilon + d_2\,\epsilon^2 + \ldots}$$

similar to [Giele, Kunst, Melnikov 08]

- Rational function reconstructed in finite number of evaluations in ϵ:

- Linear system for unknowns $\{n_i, d_i\}$:

$$c(\epsilon_1)(d_0 + d_1\,\epsilon_1 + d_2\,\epsilon_1^2 + \ldots) = n_0 + n_1\,\epsilon_1 + n_2\,\epsilon_1^2 + \ldots$$

$$c_i(\epsilon_2)(d_0 + d_1\,\epsilon_2 + d_2\,\epsilon_2^2 + \ldots) = n_0 + n_1\,\epsilon_2 + n_2\,\epsilon_2^2 + \ldots$$

$$\ldots$$

- Efficiency depends on number of evaluations, which in turn depends on degree of rational function

$$\boxed{\vec{p} \to \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

✓ Laurent expansion in ϵ

Amplitude Reconstruction

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

- Rationality of integral coefficients in momenta

[Peraro 16; Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng, 17]

$$c(\epsilon, \vec{p}) = \frac{n_0(\vec{p}) + n_1(\vec{p})\,\epsilon + n_2(\vec{p})\,\epsilon^2 + \ldots}{d_0(\vec{p}) + d_1(\vec{p})\,\epsilon + d_2(\vec{p})\,\epsilon^2 + \ldots}$$

$$c(\epsilon, \vec{p})(d_0(\vec{p}) + d_1(\vec{p})\,\epsilon + d_2(\vec{p})\,\epsilon^2 + \ldots) = n_0(\vec{p}) + n_1(\vec{p})\,\epsilon + n_2(\vec{p})\,\epsilon^2 + \ldots$$

$$n_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}}\,(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \ldots)\,, \quad s_{ij} = (p_i + p_j)^2,$$

similar for $d_i(\vec{p})$

↪ linear systems for numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $c_i(\epsilon, \vec{p})$

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}\,, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}$$

↪ Efficient and numerically stable analytic forms of amplitudes

Feynman diagrams

Integral reduction using integration by parts (IBP)

↓

Sum of master integrals

$$A = \sum_i c_i(\epsilon, \vec{p})\, I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

↓

Integrated amplitude

$$A = \sum_{i,k} \epsilon^i\, d_{ik}(\vec{p})\, h_{ik}(\vec{p})$$

Laurent expansion in ϵ

Differential-Equation Approach

- How to compute master integrals $I_i(\epsilon, \vec{p}) = \sum \epsilon^k h_{ik}(\vec{p})$?

  - Identify independent transcendental functions $h_{ij}(\vec{p}) \in \{1, \ln(s_{12}), \ln(s_{23}), \ldots\}$

  - Find representation for fast and stable numerical evaluation

- Solution: differential equation in canonical form [Henn '13]

$$d\vec{I}(\epsilon, \vec{p}) = M(\epsilon, \vec{p})\,\vec{I}(\epsilon, \vec{p}) \longrightarrow \boxed{\begin{aligned} d\vec{I}(\epsilon, \vec{p}) &= \epsilon\, M(\vec{p})\,\vec{I}(\epsilon, \vec{p}) \\ M(\vec{p}) &= \sum_i M_i\, d \ln \mathrm{W}_i(\vec{p}) \end{aligned}}$$

$$\vec{I} \to A(\epsilon, \vec{p}) \cdot \vec{I}$$

  - Find special 'pure' basis

  - Determine expressions $d \ln \mathrm{W}_i \;\; \to \;\; \mathrm{W}_i$ are 'symbol alphabet'; determines analytic properties of integrals

  - Determine the rational matrices M_i

Numerical Differential Equation

$$d\vec{I}(\epsilon, \vec{p}) = \epsilon\, M(\vec{p}) \cdot \vec{I}(\epsilon, \vec{p}), \qquad M(\vec{p}) = \sum_i M_i\, d\ln \mathrm{W}_i(\vec{p})$$

[Abreu, Page, Zeng '19;
Abreu, Ita, Moriello, Page, Tschernow, Zeng '20]

- Finding pure basis:
  - Several semi-automated methods, but not yet systematic
  - Maximal cuts/leading singularities, dimension shifting, ... [e.g.]
  - Still important bottleneck, often based on trial an error
- Finding symbol alphabet:
  - Alphabet marks distinguished points in phase space: Landau conditions
  - Educated guesses & tests
- Finding rational matrices M_i :
  - Trivial once alphabet and pure basis is known

All invesitgations done with
finite-field numerical evaluations:
$\vec{p} \in \mathbb{F}, \quad \epsilon \in \mathbb{F}$
[Schabinger, von Manteuffel '20; Peraro '16]

Tools: [FiniteFlow, FIRE, Kira, Reduce,…]

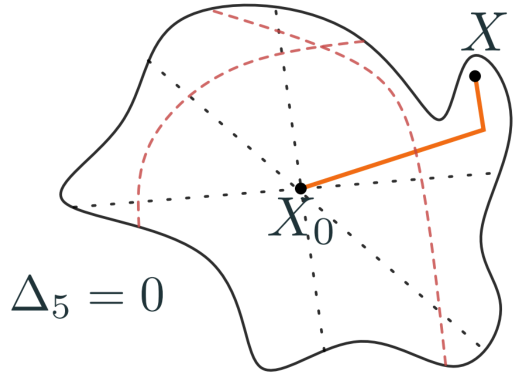
↪ $\mathcal{O}(100)$ numerical evaluations of IBP reduction sufficient for obtaining the differential equation

Integration

$$\boxed{d\vec{I}(\epsilon,\vec{p}) = \epsilon\, M(\vec{p})\cdot \vec{I}(\epsilon,\vec{p}),\quad M(\vec{p}) = \sum_i M_i\, d\ln \mathrm{W}_i(\vec{p})}$$

- Structure of equations implies order-by-order solution on a line in phase space

$$\vec{I}(\epsilon,\vec{p}) = \sum_j \epsilon^j\, \vec{h}_j(\vec{p})\,,\quad \vec{h}_j(\vec{p}) = \int_0^1 M[\vec{p}(t)]\cdot \vec{h}_{j-1}[\vec{p}(t)] \;\text{ with }\; \vec{p}(t) = t(\vec{p}-\vec{p}_0)+\vec{p}_0$$



- Multiple integration steps: @ two-loop: j=0,1,2,3,4

- Challenging integration because of holes in phase space given by alphabet: $d\ln \mathrm{W}_i(\vec{p})$

- Multiple polylogarithms

✓ Chen iterated integrals: function algebra and dedicated codes for evaluation [Chicherin, Sotnikov '20; Chicherin, Sotnikov, Zoia '21]

- Numerical alternatives:

  - pySecDec [Heinrich et al.]
  - Diffexp [Moriello '19; Hidding '20]
  - AMFlow [Liu, Ma, (Wang), '17,'21,'22]

Integrals — State of the Art

- Five-point two-loop massless integrals

  [Papadopoulos, Tommasini, Wever, 15] [Gehrmann, Henn, Lo Presti, 18]

  [Abreu, Page, Zeng, 18]

  [Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser,18]

  [Abreu, Dixon, Herrmann, Page, Zeng, 18]

  [Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 18]

- Five-point two-loop one-mass integrals

  [Abreu, Ita, Moriello, Page, Tschernow, Zeng, 20]

  [Abreu, Ita, Page, Tschernow, 21]

  [Papadopoulos, Tommasini, Wever, 15] [Papadopoulos, Wever, 19]

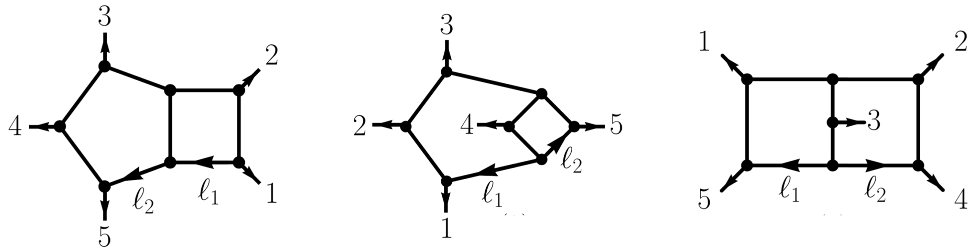
  [Canko, Papadopoulos, Syrrakos, 20]

  [Kardos, Papadopoulos, Smirnov, Syrrakos, Wever, 19]

- Public library: https://gitlab.com/pentagon-functions/PentagonFunctions-cpp

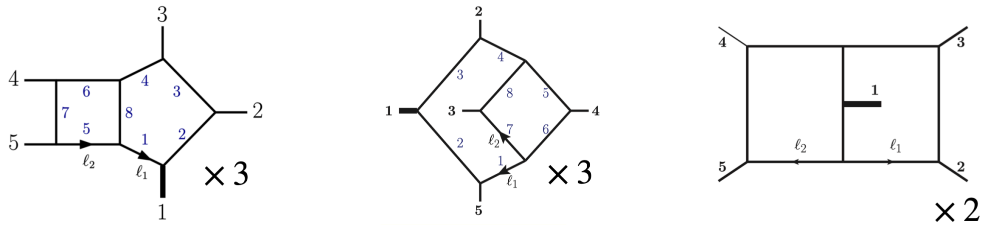
- Five-point two-loop two-mass integrals

  [Badger, Becchetti, Chaubey, Marzucca, 22]



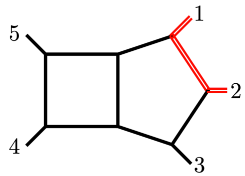
[Gehrmann, Henn, Lo Presti, 18]

[Chicherin, Sotnikov, 20]



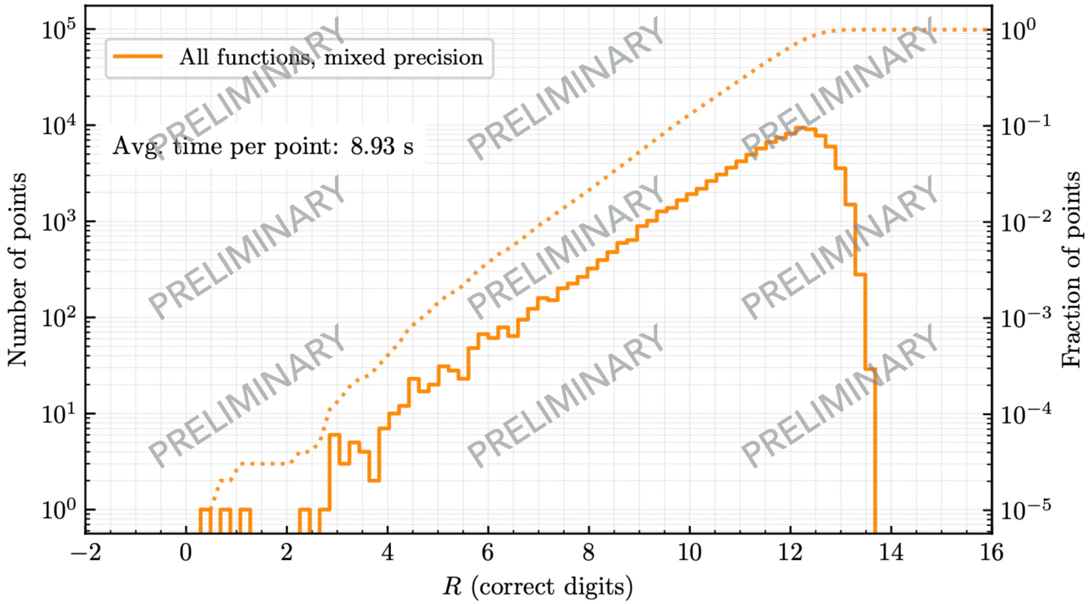
[Chicherin, Sotnikov, Zoia, 21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear]

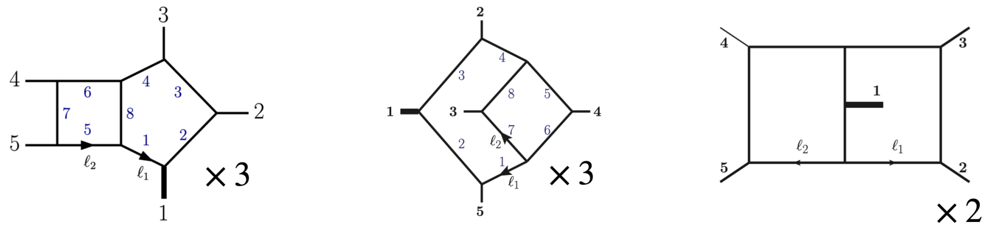


Integrals — State of the Art

- Five-point two-loop one-mass integrals
  - $\mathcal{O}(200)$ letters $W_i(\vec{p})$
  - $\mathcal{O}(1250)$ integral functions ($\mathcal{O}(1000)$ at weight 4 functions)
- All the functions required for five-point one-mass processes at two loops



[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear]



Structure — `Good' Integral Bases

$$\vec{p} \to \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \dots\} \in \mathbb{Q}$$

- Analytic properties yield crucial simplifications in expressions:

  - Good integral bases lead to factorisation

$$c(\epsilon, \vec{p}) = \frac{\mathrm{poly}_1(\epsilon, \vec{p})}{\mathrm{poly}_2(\epsilon, \vec{p})} = \frac{\mathrm{poly}_1(\epsilon, \vec{p})}{\mathrm{poly}_2(\epsilon)\ \mathrm{poly}_3(\vec{p})}$$

[Usovitsch 20; Smirnov, Smirnov 20]

↪ universal $\mathrm{poly}_2(\epsilon)$ simplifies reconstruction

  - Canonical kinematic denominators:

$$c(\epsilon, \vec{p}) = \frac{\mathrm{poly}_1(\epsilon, \vec{p})}{\mathrm{poly}_2(\epsilon)\ \prod_i W_i^{m_i}(\vec{p})}$$

[Abreu, Dormans, Febres Cordero, Ita, Page '18]

$W_i(\vec{p})$ … `letters' associated to integral

↪ denominators require to obtain integer exponents m_i

- Factorisation properties simplify reconstruction and improve numerical stability

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i \frac{\tilde{c}_i(\epsilon, \vec{p})}{\hat{c}_i(\epsilon) \prod_j W_j(\vec{p})} I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

Laurent expansion in ϵ

$$A = \sum_{i,k} \epsilon^i d_{ik}(\vec{p}) h_{ik}(\vec{p})$$

Structure — Function Bases

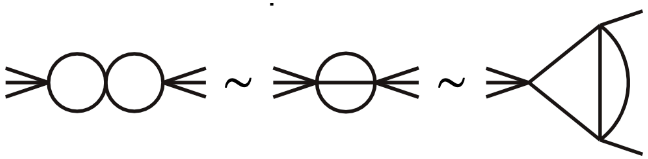
- Integral coefficients very complicated

  - only finite orders in ϵ-expansion needed

  - subtract universal IR/UV poles and reconstruct finite remainders

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_i}(\vec{p})} h_i(\vec{p})$$

- Relations after ϵ expansion



$$\sim r_0 + r_1 \epsilon \ln(s) + r_2 \epsilon^2 \ln^2(s) + \ldots$$

↪ cancellations and simplification

- Reconstruct polynomials $e_i(\vec{p})$

$$\boxed{\vec{p} \to \{\vec{p}_1, \vec{p}_2, \ldots\} \in \mathbb{Q}, \quad \epsilon \to \{\epsilon_1, \epsilon_2, \ldots\} \in \mathbb{Q}}$$

Feynman diagrams

Integral reduction using integration by parts (IBP)

Sum of master integrals

$$A = \sum_i \frac{\tilde{c}_i(\epsilon, \vec{p})}{\hat{c}_i(\epsilon) \prod_j W_j(\vec{p})} I_i(\epsilon, \vec{p})$$

Differential equations (DE) or numerical integration

Integrated amplitude

Laurent expansion in ϵ

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_i}(\vec{p})} h_i(\vec{p})$$

Structure — Regularity

$$A \to R = \sum_i \frac{e_i(\vec{p})}{\prod_j W_j^{m_j}(\vec{p})} h_i(\vec{p})$$

- Simplifications:

  - Regularity of amplitudes/remainders in phase space

  - Many of poles in $W_j(\vec{p}) = 0$ unphysical and cancel $\Longrightarrow$ correlations between numerator polynomials $e_j(\vec{p})$

- Many advanced ideas for reconstruction:

  - Univariate slices

  - Univariate/multivariate partial fractions [Badger, Hartanto, Zoia, 21]

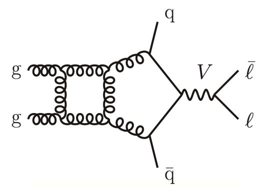
  - Choice of variables, e.g. spinor helicity

  - p-adic numbers [Page, De Laurentis 22]

  - Reconstruction programs: FireFly [Klappert, Lange 19]

- Example: planar two-loop four-parton + W-boson amplitudes [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 22]

  - Observe factor-50 reduction of needed evaluations



Start

Factor-50 reduction

			Max Ansatz Size		Max Non-Zero Terms
$\mathcal{R}_g$	$p_5 \parallel p_i$	$\lvert K \rvert$	Common Denominator	Partial Fractioning	Result
	1	58	5500 k	180 k	37 k
$+-\ N_f^0$	2	67	7000 k	480 k	110 k
	3	67	5900 k	380 k	90 k

Conclusions

- Real demand for precision predictions for ongoing and future LHC physics program
- Discussed status of NNLO five-point processes and some key methods
- Progress relies on advancing analytic understanding: differential equation method, integral evaluation, amplitude computation, integral reduction
- Key recent methods: exact numerical evaluations (finite fields), functional reconstruction & understanding of integral functions; apply broadly in many fields of physics
- One-mass five point processes almost ready
- New challenges will appear to handle more masses ($t\bar{t}H$, $t\bar{t}j$,...)

  - Elliptic functions, numerical integration approaches,…
- New amplitudes computations and new formal developments are the way to go for broad availability of NNLO results.