Hadronic contributions to the anomalous magnetic moment of the muon

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ZPW2023: Recent highlights across phenomenology

University of Zurich[UZH]

PAUL SCHERRER INSTITUT PSI

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Magnetic moment

- relation of spin and magnetic moment of a lepton:

$$\vec{\mu}_\ell = g_\ell \frac{e}{2m_\ell} \vec{s}$$

  g_ℓ: Landé factor, gyromagnetic ratio

- Dirac's prediction: $g_e = 2$
- anomalous magnetic moment: $a_\ell = (g_\ell - 2)/2$
- helped to establish QED and QFT as the framework for elementary particle physics
- today: probing not only QED but entire SM

Electron vs. muon magnetic moments

- influence of heavier virtual particles of mass M scales as

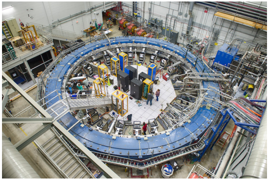
$$\frac{\Delta a_\ell}{a_\ell} \propto \frac{m_\ell^2}{M^2}$$

- $(m_\mu/m_e)^2 \approx 4 \times 10^4 \Rightarrow$ muon is much more sensitive to **new physics**, but also to **EW and hadronic contributions**
- a_τ experimentally not yet known precisely enough

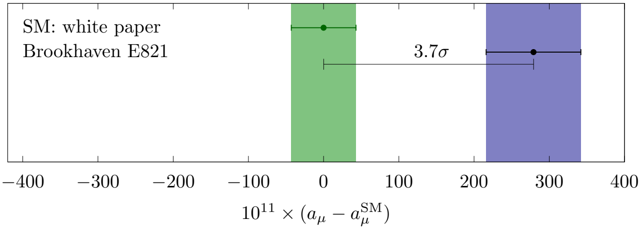
Muon anomalous magnetic moment $(g-2)_\mu$

recent and future experimental progress:

- FNAL will improve precision further: **factor of 4 wrt E821**
- theory still needs to reduce **SM uncertainty**!



muon $g-2$ discrepancy



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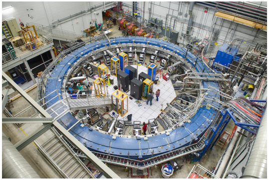
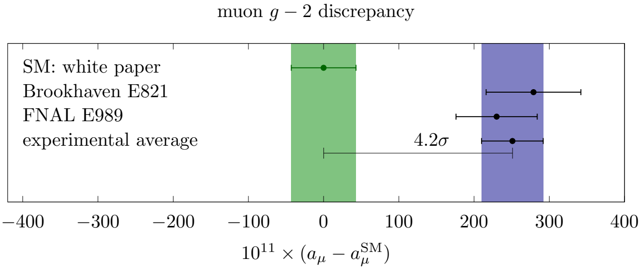


Photo: Glukicov (License: CC-BY-SA-4.0)



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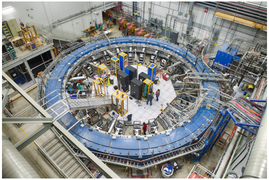

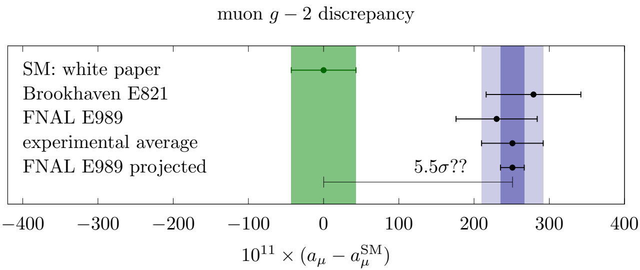

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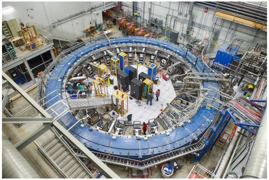

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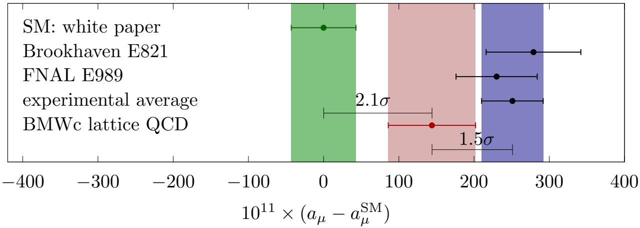


SM theory white paper

→ T. Aoyama *et al.* (Muon $g-2$ Theory Initiative), Phys. Rept. **887** (2020) 1-166

- community white paper on status of **SM calculation**
- new consensus on SM prediction, used for **comparison with FNAL 2021 result**
- many improvements on **hadronic contributions**
- since 2020: significant new developments
  ⇒ **white-paper update** in spring 2023

$(g-2)_\mu$: theory vs. experiment

- discrepancy between SM theory white paper and experiment 4.2σ
- hint of new physics?
- size of discrepancy points at **electroweak scale** $\Rightarrow$ heavy new physics needs enhancement
- theory error completely dominated by **hadronic effects**
- **tension** emerging between **lattice QCD** and **hadronic cross-section data**

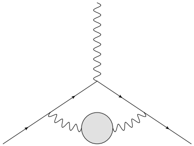
QED and electroweak contributions

- full $\mathcal{O}(\alpha^5)$ calculation by Kinoshita et al. 2012 (involves 12672 diagrams)
- EW contributions (EW gauge bosons, Higgs) calculated to two loops (three-loop terms negligible)

	$10^{11} \cdot a_\mu$	$10^{11} \cdot \Delta a_\mu$
QED total	$116\,584\,718.931$	0.104
EW	153.6	1.0
theory total	$116\,591\,810$	43

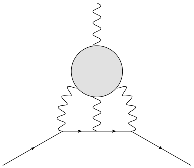
Hadronic contributions

- quantum corrections due to the strong nuclear force
- much smaller than QED, but **dominate uncertainty**



- hadronic vacuum polarization (HVP)

$$a_\mu^{\text{HVP}} = 6845(40) \times 10^{-11}$$



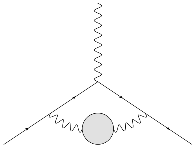
- hadronic light-by-light scattering (HLbL)

$$a_\mu^{\text{HLbL}} = 92(18) \times 10^{-11}$$

Theory vs. experiment

	$10^{11} \cdot a_\mu$	$10^{11} \cdot \Delta a_\mu$
QED total	$116\,584\,718.931$	0.104
EW	153.6	1.0
HVP	$6\,845$	40
HLbL	92	18
SM total (white paper 2020)	$116\,591\,810$	43
experiment (E821+E989)	$116\,592\,061$	41
difference exp−theory	251	59

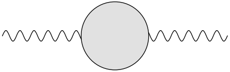
Hadronic vacuum polarization (HVP)



- at present evaluated via **dispersion relations** and cross-section input from $e^+e^- \to$ hadrons
- intriguing discrepancies between e^+e^- experiments $\Rightarrow$ treated as additional systematic uncertainty
- lattice QCD making fast progress
- $\mathbf{2.1\sigma}$ **tension** between dispersion relations and BMWc lattice results $\to$ S. Borsanyi *et al.*, Nature (2021)

Hadronic vacuum polarization (HVP)

photon HVP function:



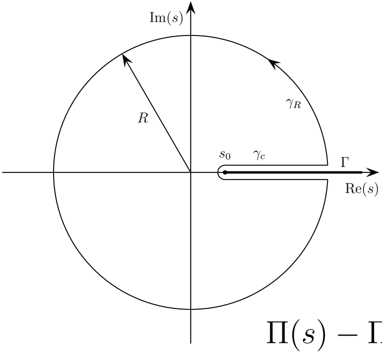
$$= i(q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)$$

unitarity of the S-matrix implies the optical theorem:

$$\mathrm{Im}\Pi(s) = \frac{s}{e(s)^2}\sigma(e^+e^- \to \mathrm{hadrons})$$

Dispersion relation

causality implies **analyticity**:



Cauchy integral formula:

$$\Pi(s) = \frac{1}{2\pi i} \oint_\gamma \frac{\Pi(s')}{s' - s} ds'$$

deform integration path:

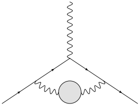
$$\Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\Pi(s')}{(s' - s - i\epsilon)s'} ds'$$

HVP contribution to $(g-2)_\mu$

$$a_\mu^{\text{HVP}} = \frac{m_\mu^2}{12\pi^3} \int_{s_{\text{thr}}}^{\infty} ds\, \frac{\hat{K}(s)}{s}\, \sigma\big(e^+e^- \to \text{hadrons}(+\gamma)\big)$$

- basic principles: unitarity and analyticity
- direct **relation to data**: total hadronic cross section $\sigma\big(e^+e^- \to \text{hadrons}(+\gamma)\big)$
- dedicated e^+e^- program (BaBar, Belle, BESIII, CMD3, KLOE, SND)

Hadronic vacuum polarization



- final white-paper number: data-driven evaluation

$$a_\mu^{\text{LO HVP, pheno}} = 6\,931(40) \times 10^{-11}$$

- white-paper 2020 average of published lattice results

$$a_\mu^{\text{LO HVP, lattice average}} = 7\,116(184) \times 10^{-11}$$

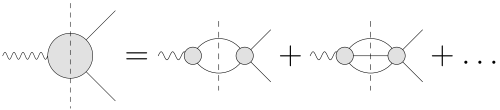
- newest complete lattice-QCD result by BMWc

→ S. Borsanyi *et al.*, Nature (2021)

$$a_\mu^{\text{LO HVP, BMWc}} = 7\,075(55) \times 10^{-11}$$

Two-pion contribution to HVP

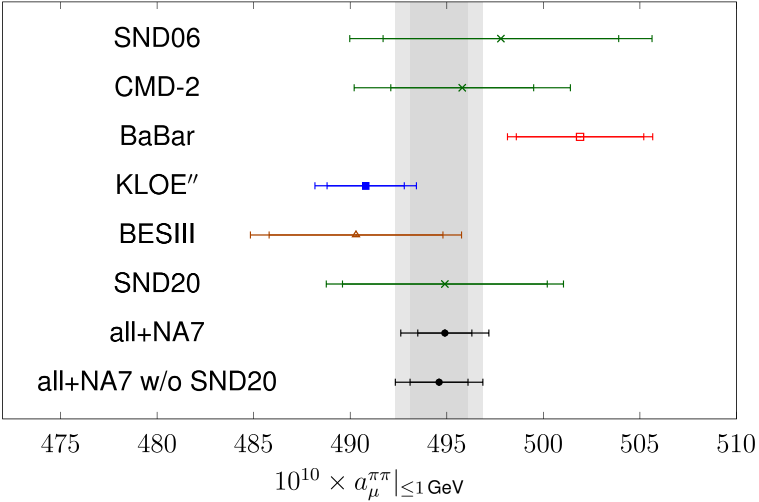
- $\pi\pi$ contribution amounts to more than 70% of HVP contribution
- responsible for a similar fraction of HVP uncertainty
- can be expressed in terms of **pion vector form factor** $\Rightarrow$ constraints from analyticity and unitarity



$\rightarrow$ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006

Result for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006
Colangelo, Hoferichter, Kubis, Stoffer, JHEP **10** (2022) 032



Tension between R-ratio and lattice

- **2.1σ tension** between R-ratio and BMWc lattice-QCD for HVP
- increases to **3.7σ for intermediate Euclidean window**
- recent results from ETMC, Mainz, RBC/UKQCD confirm BMWc intermediate window
- motivates **ongoing scrutiny** of R-ratio results

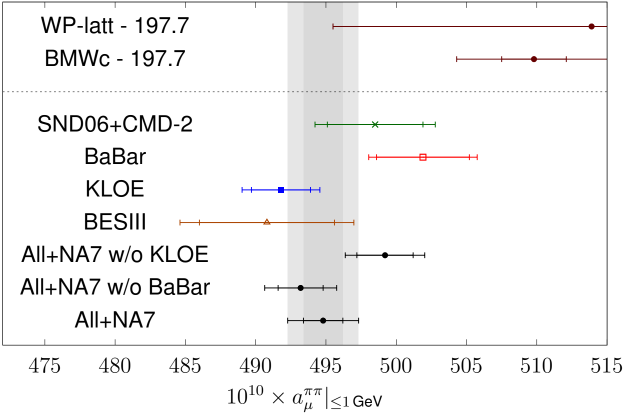
Tension with lattice QCD

$\rightarrow$ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

- implications of changing HVP?
- modifications at high energies affect **hadronic running of $\alpha_{\mathrm{QED}}^{\mathrm{eff}}$** $\Rightarrow$ clash with global EW fits

  $\rightarrow$ Passera, Marciano, Sirlin (2008), Crivellin, Hoferichter, Manzari, Montull (2020), Keshavarzi, Marciano, Passera, Sirlin (2020), Malaescu, Schott (2020)
- lattice studies point at region < 2 GeV
- **$\pi\pi$ channel** dominates
- relative changes in other channels would need to be huge

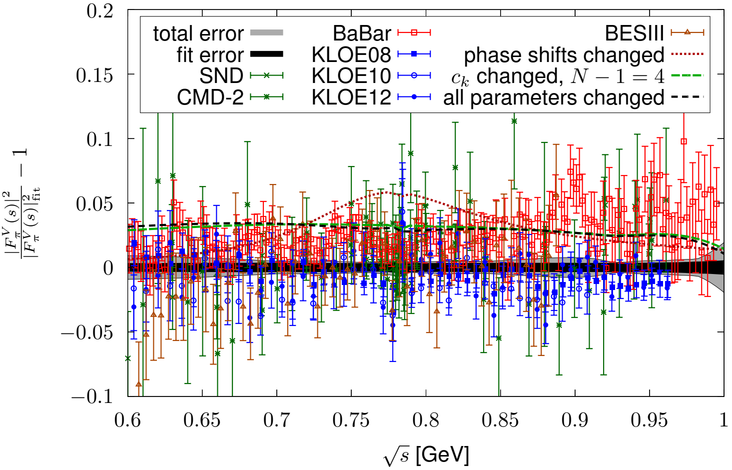
Result for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV



Assumption: suppose all changes occur in $\pi\pi$ channel < 1 GeV

$$\Rightarrow a_\mu^{\mathrm{total}}[\mathrm{WP20}] - a_\mu^{2\pi,<1\,\mathrm{GeV}}[\mathrm{WP20}] = 197.7 \times 10^{-10}$$

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$



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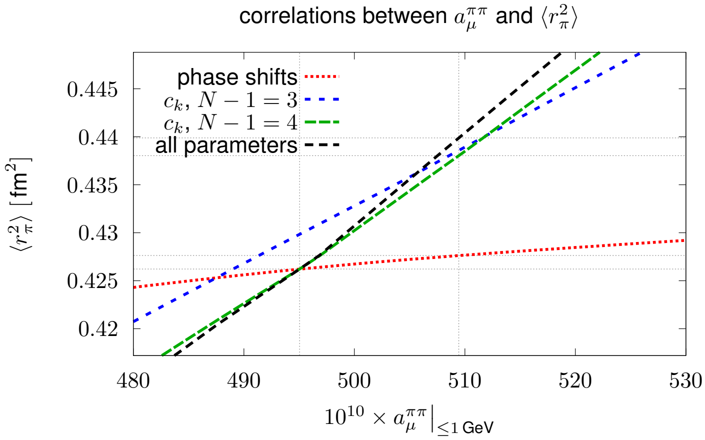
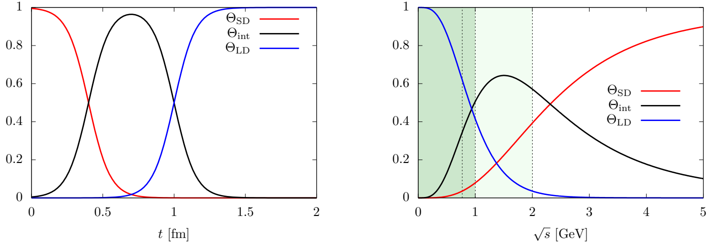


Euclidean window quantities



- smooth window weight functions in Euclidean time

  → Blum et al. [RBC/UKQCD], PRL **121** (2018) 022003

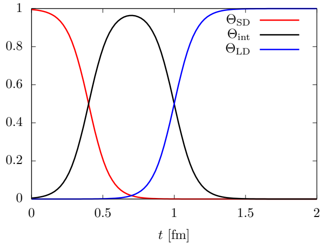
- total discrepancy:

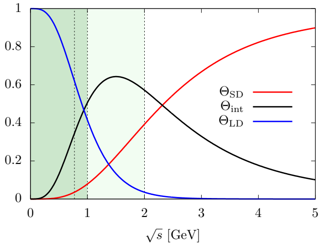
$$a_\mu[\text{BMWc}] - a_\mu[\text{WP20}] = 14.4(6.8) \times 10^{-10}$$

- intermediate window: → Colangelo et al., PLB **833** (2022) 137313

$$a_\mu^{\text{int}}[\text{BMWc}] - a_\mu^{\text{int}}[e^+e^-] = 7.3(2.0) \times 10^{-10}$$

Euclidean window quantities

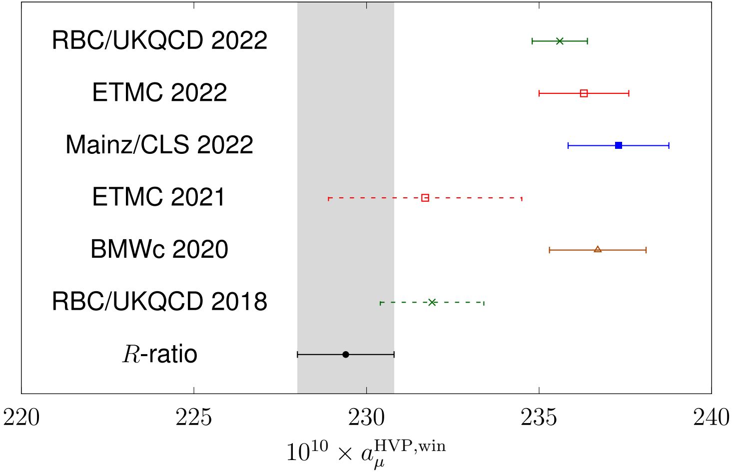




- using form of weight functions:
  at least $\sim 40\%$ from **above 1 GeV**

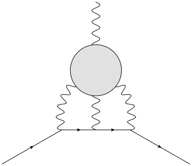
- assumptions:
  - rather uniform shifts in low-energy $\pi\pi$ region
  - no significant negative shifts

Results for intermediate window



R-ratio result: → Colangelo et al., PLB **833** (2022) 137313

Hadronic light-by-light (HLbL)



- previously based only on hadronic models
- our work: **dispersive framework** based on unitarity and analyticity, replacing hadronic models step by step
- **hadronic models** only for subdominant contributions
- matching to **asymptotic constraints**

BTT Lorentz decomposition

→ Colangelo, Hoferichter, Procura, Stoffer, JHEP **09** (2015) 074

Lorentz decomposition of the HLbL tensor:

→ Bardeen, Tung (1968) and Tarrach (1975)

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_i T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures manifestly **gauge invariant**
- scalar functions Π_i **free of kinematic singularities**
  ⇒ dispersion relation in the Mandelstam variables
- asymptotic behavior of scalar functions that avoids subtractions

Dispersive representation

→ Colangelo, Hoferichter, Procura, Stoffer, JHEP **04** (2017) 161

- write down a double-spectral (Mandelstam) representation for the HLbL tensor
- split the HLbL tensor according to the sum over intermediate (on-shell) states in unitarity relations

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \dots$$

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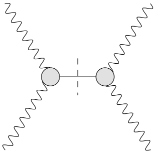
one-pion intermediate state



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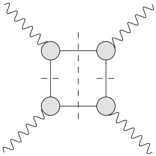
two-pion intermediate state in both channels



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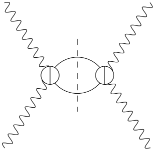
two-pion intermediate state in first channel



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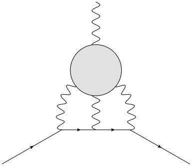
higher intermediate states

Hadronic light-by-light scattering



- dispersion relations + hadronic models (LO, without charm)

$$a_\mu^{\text{HLbL, pheno}} = 89(19) \times 10^{-11}$$

- first lattice-QCD results

$a_\mu^{\text{HLbL, lattice}} = 79(35) \times 10^{-11}$ → T. Blum *et al.*, PRL **124** (2020) 132002

$a_\mu^{\text{HLbL, lattice}} = 106.8(15.9) \times 10^{-11}$ → E.-H. Chao *et al.*, EPJC **81** (2021) 651

HLbL overview

→ T. Aoyama *et al.*, Phys. Rept. **887** (2020) 1-166

	$10^{11} \cdot a_\mu$	$10^{11} \cdot \Delta a_\mu$
π^0, η, η'-poles	93.8	4.0
pion/kaon box	-16.4	0.2
S-wave $\pi\pi$ rescattering	-8	1
scalars, tensors	-1	3
axials	6	6
light quarks, short distance	15	10
c-loop	3	1
HLbL total (LO)	92	19

HLbL: recent progress

- asymptotic constraints: OPE and two-loop QCD corrections to symmetric limit $Q_{1,2,3} \gg \Lambda_{\text{QCD}}$

  $\rightarrow$ Bijnens et al., JHEP **10** (2020) 203; JHEP **04** (2021) 240

- scalar contributions: $\pi\pi/\bar{K}K$ S-wave rescattering up to $1.3\,$GeV plus $a_0(980)$ in NWA:

$$a_\mu^{\text{HLbL}}[\text{scalars}] = -9(1) \times 10^{-11}$$

  $\rightarrow$ Danilkin, Hoferichter, Stoffer, PLB **820** (2021) 136502

- first steps towards including axials in dispersive framework: $\rightarrow$ Zanke, Hoferichter, Kubis, JHEP **07** (2021) 106, Colangelo, Hagelstein, Hoferichter, Laub, Stoffer, EPJC **81** (2021) 702

- holographic-QCD models point to rather large axial contribution $\rightarrow$ Cappiello et al., PRD **102** (2020) 016009, Leutgeb, Rebhan, PRD **101** (2020) 114015; arXiv:2108.12345 [hep-ph]

Summary

- FNAL 2021 result increased tension with white-paper SM value to $\boldsymbol{4.2\sigma}$
- tension emerging between lattice HVP and R-ratio
- white-paper update to appear in spring 2023, before FNAL run-2+3 release
- final FNAL precision goal calls for **further improvement** in HLbL and HVP

Summary: HLbL

- precise **dispersive evaluations** of dominant contributions
- models reduced to sub-dominant contributions, but **dominate uncertainty**
- consistent with lattice-QCD evaluations
- recent progress on scalar contributions, ongoing work on axial-vector and tensor resonances and asymptotic matching

Summary: HVP

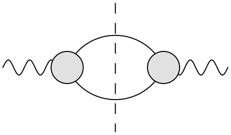
- long-standing discrepancy between BaBar/KLOE
  $\Rightarrow$ wait for new e^+e^- data
- intriguing tension with lattice-QCD
- Euclidean windows useful tools for detailed scrutiny
- unitarity/analyticity enable **independent checks** via pion VFF and $\langle r_\pi^2 \rangle$, in addition to further direct lattice results on HVP

Backup

Unitarity and analyticity

implications of unitarity (two-pion intermediate states):

1. $\pi\pi$ contribution to HVP—pion vector form factor (VFF)
2. pion VFF—$\pi\pi$ scattering
3. $\pi\pi$ scattering—$\pi\pi$ scattering

 : $\sigma(e^+e^- \to \pi^+\pi^-) \propto |F_\pi^V(s)|^2$

analyticity $\Rightarrow$ dispersion relation for HVP contribution

Unitarity and analyticity

implications of unitarity (two-pion intermediate states):

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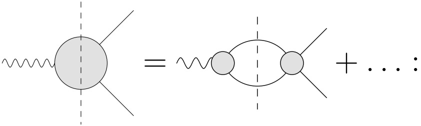

$$F_\pi^V(s) = |F_\pi^V(s)|e^{i\delta_1^1(s)+\dots}$$

analyticity $\Rightarrow$ dispersion relation for pion VFF

Unitarity and analyticity

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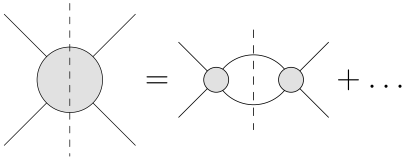
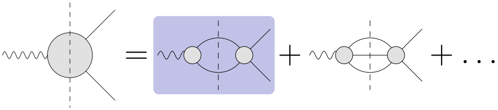


analyticity, crossing, PW expansion $\Rightarrow$ Roy equations

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\rm in}^N(s)$$

- Omnès function with elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input:

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

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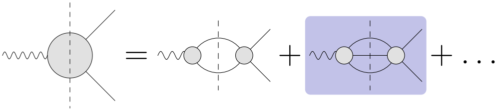

$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- isospin-breaking 3π intermediate state: negligible apart from ω resonance (ρ–ω interference effect)

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\text{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4,$$

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

Dispersive representation of pion VFF

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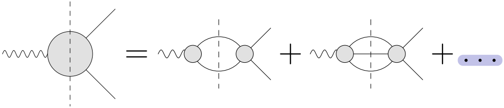


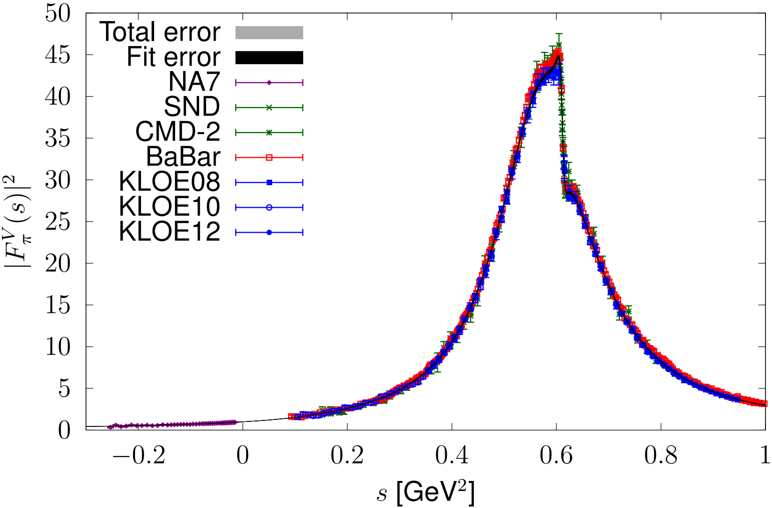
$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- heavier intermediate states: 4π (mainly $\pi^0\omega$), $\bar{K}K, \dots$
- described in terms of a conformal polynomial with cut starting at $\pi^0\omega$ threshold

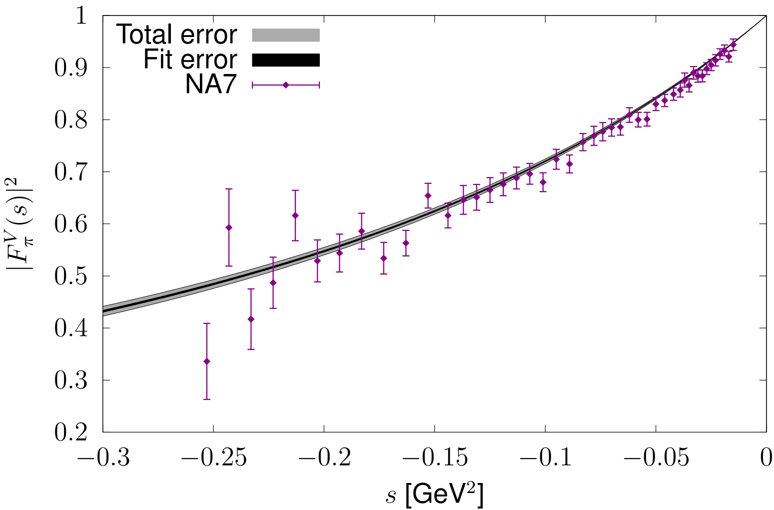
$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k \big(z^k(s) - z^k(0)\big)$$

- correct P-wave threshold behavior imposed

Fit result for the VFF $|F_\pi^V(s)|^2$



Fit result for the VFF $|F_\pi^V(s)|^2$



Contribution to $(g-2)_\mu$

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006

- low-energy $\pi\pi$ contribution:

$$a_\mu^{\text{HVP},\pi\pi}|_{\leq 0.63\,\text{GeV}} = 132.8(0.4)(1.0) \times 10^{-10}$$

- $\pi\pi$ contribution up to 1 GeV:

$$a_\mu^{\text{HVP},\pi\pi}|_{\leq 1\,\text{GeV}} = 495.0(1.5)(2.1) \times 10^{-10}$$

- enters the white-paper value in a conservative merging with direct cross-section integration

Tension with lattice QCD

→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

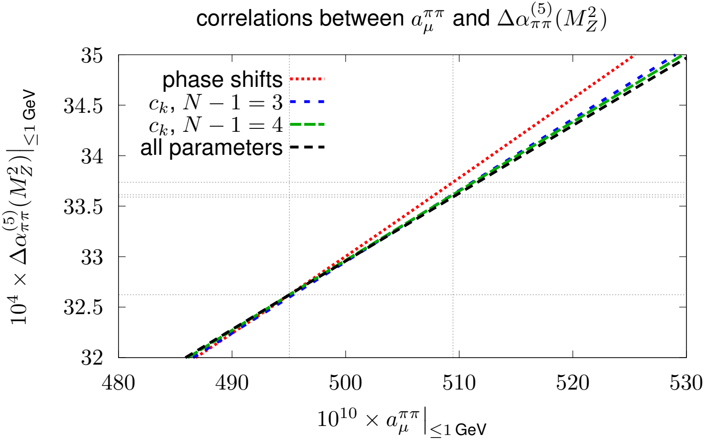
- force a different HVP contribution in VFF fits by including “lattice” datum with tiny uncertainty
- three different scenarios:
  - “low-energy” physics: $\pi\pi$ phase shifts
  - “high-energy” physics: inelastic effects, c_k
  - all parameters free
- study effects on pion charge radius, hadronic running of $\alpha_{\rm QED}^{\rm eff}$, phase shifts, cross sections

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

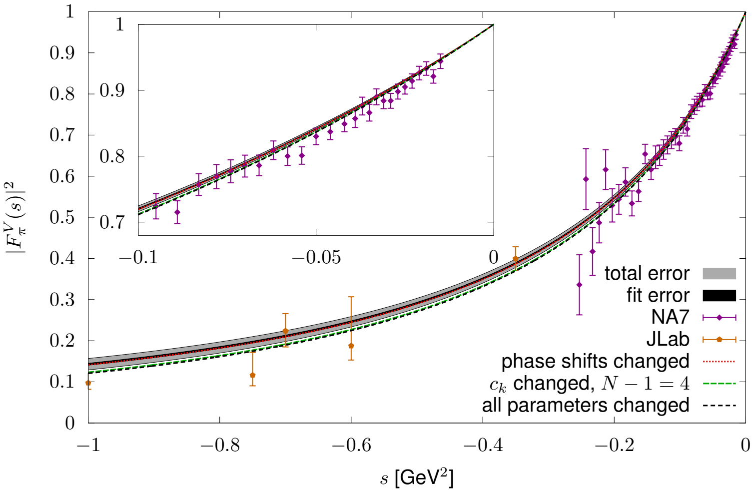
$\rightarrow$ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

- "low-energy" scenario requires large local changes in the cross section in the ρ region
- "high-energy" scenario has an impact on **pion charge radius** and the space-like VFF $\Rightarrow$ chance for independent lattice-QCD checks

Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$



Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$

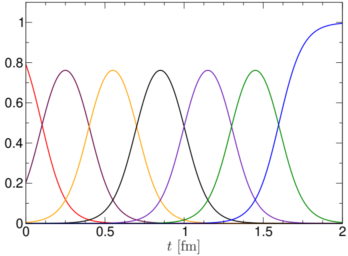


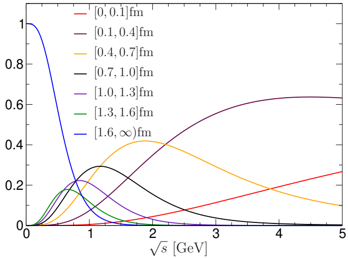
Data-driven evaluation of window quantities

→ Colangelo et al., PLB **833** (2022) 137313

- standard windows: $[0, 0.4]$ fm, $[0.4, 1.0]$ fm, $[1.0, \infty)$ fm with $\Delta = 0.15$ fm
- **additional windows**: cuts at $\{0.1, 0.4, 0.7, 1.0, 1.3, 1.6\}$ fm
- **data-driven evaluation** based on merging of KNT and CHHKS
- systematic effect due to BaBar vs. KLOE tension close to the WP estimate
- full covariance matrices for windows provided

Additional Euclidean-time windows



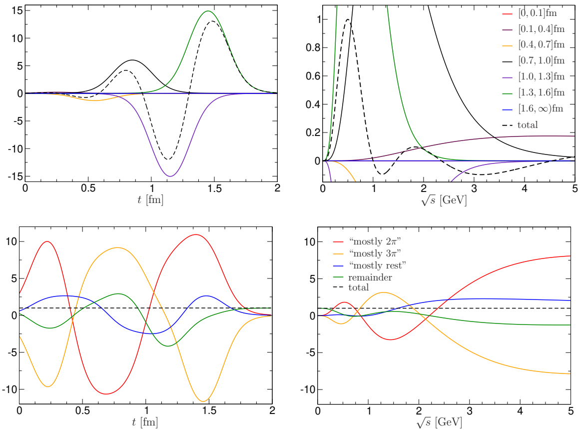


→ Colangelo et al., PLB **833** (2022) 137313

Localization in time-like region possible?

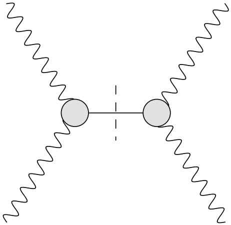
- better **localization in time-like region** could be achieved by taking linear combinations of Euclidean-time windows
- typically leads to **large cancellations** in Euclidean-time integral
- reflecting ill-posed inverse Laplace transform
- assessing usefulness requires knowledge of **full covariances**
- combinations dominated by **exclusive hadronic channels** suffer from similar problems

Localization in time-like region possible?



→ Colangelo et al., PLB **833** (2022) 137313

Pion pole



$$\bar{\Pi}_1^{\pi^0\text{-pole}} = \frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma}(q_3^2, 0)}{q_3^2 - M_\pi^2}$$

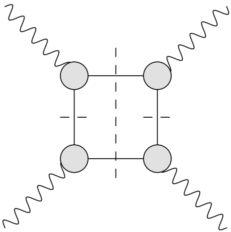
$\bar{\Pi}_2^{\pi^0\text{-pole}}$ via crossing symmetry

- input: doubly-virtual and singly-virtual pion transition form factors $\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$
- dispersive analysis of transition form factor:
  $a_\mu^{\pi^0\text{-pole}} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$

  → Hoferichter et al., PRL **121** (2018) 112002, JHEP **10** (2018) 141

Pion-box contribution

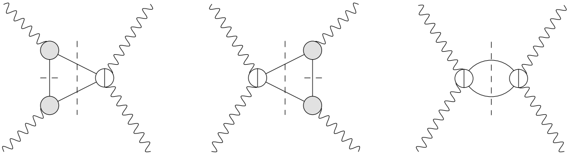
→ Colangelo, Hoferichter, Procura, Stoffer, JHEP **04** (2017) 161



- simultaneous two-pion cuts in two channels
- Mandelstam representation explicitly constructed
- q^2-dependence: pion VFF $F_\pi^V(q_i^2)$ for each off-shell photon factor out

- Wick rotation: integrate over space-like momenta
- dominated by low energies ≤ 1 GeV
- result: $a_\mu^{\pi\text{-box}} = -15.9(2) \times 10^{-11}$

Rescattering contribution



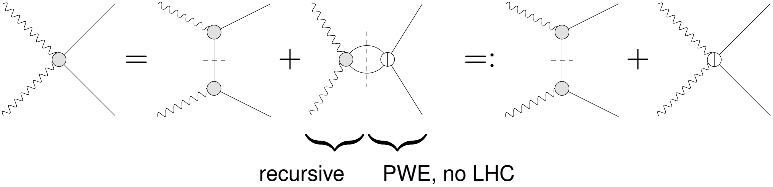
- expansion into partial waves
- unitarity gives imaginary parts in terms of **helicity amplitudes** for $\gamma^*\gamma^{(*)} \to \pi\pi$:

$$\mathrm{Im}_{\pi\pi} h^J_{\lambda_1\lambda_2,\lambda_3\lambda_4}(s) \propto \sigma_\pi(s) h_{J,\lambda_1\lambda_2}(s) h^*_{J,\lambda_3\lambda_4}(s)$$

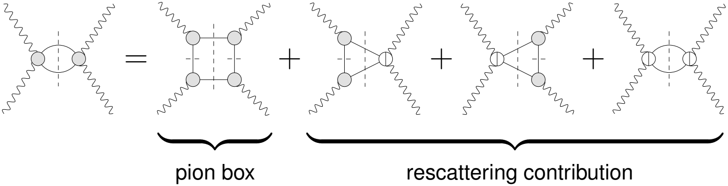
- resummation of PW expansion reproduces full result: checked for pion box

Topologies in the rescattering contribution

our S-wave solution for $\gamma^*\gamma^* \to \pi\pi$:



two-pion contributions to HLbL:



S-wave rescattering contribution

→ Colangelo, Hoferichter, Procura, Stoffer, JHEP **04** (2017) 161

- pion-pole approximation to left-hand cut
  $\Rightarrow q^2$-dependence given by F_π^V
- phase shifts based on modified inverse-amplitude method ($f_0(500)$ parameters accurately reproduced)
- result for S-waves:

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

S-wave rescattering and scalar contributions

→ Danilkin, Hoferichter, Stoffer, PLB **820** (2021) 136502

- extension to $f_0(980)$
- using coupled-channel input for $\gamma^*\gamma^* \to \pi\pi/\bar{K}K$

  → Danilkin, Deineka, Vanderhaeghen, PRD **101** (2020) 054008
- dispersive definition compared to narrow resonance:

$$a_\mu^{\text{HLbL}}[f_0(980)] = -0.2(2) \times 10^{-11}$$

- S-wave rescattering up to $1.3\,\text{GeV}$ including $a_0(980)$ in NWA:

$$a_\mu^{\text{HLbL}}[\text{scalars}] = -9(1) \times 10^{-11}$$

Extension to D-waves

→ Hoferichter, Stoffer, JHEP **07** (2019) 073

- inclusion of resonance LHC
- unitarization with Omnès methods