EXPLORING QCD AMPLITUDES UP TO 3 LOOPS

Zurich Phenomenology Workshop 2023 University of Zurich - 12/01/2023

Lorenzo Tancredi - Technical University Munich

Technische Universität München

$$
\chi = -\frac{1}{4}F_{11}F_{21}^{\prime\prime}
$$

+ i\overrightarrow{y}y
+ i\overrightarrow{y}y
+ i\overrightarrow{y}y
+ |y_{1}y|^{2}-V(\phi)

AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

QCD makes modelling of collisions very complicated

AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

.

 $\begin{array}{cccccccccccccc} \bullet & \bullet \end{array}$

QCD makes modelling of collisions very complicated

.

$$
\sigma_{q\bar{q}\rightarrow gg} = \int [\text{dPS}] \, |\mathcal{M}_{q\bar{q}\rightarrow gg}|^2 \underbrace{\left\{\begin{array}{c}\n\delta \\
\delta\n\end{array}\right\}}_{\text{p}} \underbrace{\left\
$$

AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

QCD makes modelling of collisions very complicated

WHY ANALYTIC CALCULATIONS?

$$
\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)
$$

In which sense do we call this an **analytic result**?

$$
\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)
$$

In which sense do we call this an analytic result? **Written in terms of known**

functions!

$$
\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\sqrt{2\pi\sqrt{2}}\ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)
$$

In which sense do we call this an analytic result? **Written in terms of known**

functions!

Functional relations under control: No hidden zeros!

$$
\log 1/x + \log x = 0
$$

$$
\mathcal{W} = \frac{1}{\sqrt{s(s-4m^2)}} \ln \left(\frac{\sqrt{s-4m^2} + \sqrt{s}}{\sqrt{s-4m^2} - \sqrt{s}} \right)
$$

In which sense do we call this an analytic result? **Written in terms of known**

functions!

Functional relations under control: No hidden zeros!

 $\log 1/x + \log x = 0$

Branch cuts under control, $\log(x \pm i\epsilon) = \log x \pm i\pi$

$$
\sqrt{2\sqrt{3(s-4m^2)}} \ln \left(\frac{\sqrt{s-4m^2} + \sqrt{s}}{\sqrt{s-4m^2} - \sqrt{s}} \right)
$$

In which sense do we call this an analytic result?

Functional relations under control.

No hidden zeros!

$$
\log 1/x + \log x = 0
$$

Branch cuts under control,

$$
\log(x \pm ie) = \log x \pm i\pi
$$

Output

Argument transformation and Series
expansion for numerical evaluation

$$
\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)
$$

ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the g-2 of the electron

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

QED Mass-independent term: 2-loop contribution The "most famous calculation" in pQFT: the **g-2 of the electron**

+ ...

+ C³

 $C_1 =$

:
:

"3

 $+ + + +$

!
! a a i

"4

 \cdot + + + + \cdot

. . .

. . .

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

 $= +0.50000000...$

 Γ be "most femous coloulation" in nOFT, the σ 2 of the electron The "most famous calculation" in pQFT: the **g-2 of the electron**

+ ...

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

 $= +0.50000000...$

~~

 $\overline{}$

 \sim

+ C³

 $C_1 =$

:
:

"3

 $+ + + +$

!
! a a i

"4

 \cdot + + + + \cdot

. . .

. . .

7 diagrams $= -0.328478965...$

QED Mass-independent term: 2-loop contribution calculation in p Q F i: the g - z of the electron Γ be "most femous coloulation" in Γ The "most famous calculation" in pQFT: the **g-2 of the electron**

+ ...

+ C³

:
:

"3

 $+ + + +$

!
! a a i

"4

 \cdot + + + + \cdot

. . .

. . .

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

\n
$$
C_1 = \bigcup_{\substack{e = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0}} \mathbb{A} \qquad \text{where } e = 0.500000000...
$$

\n
$$
C_2 = \bigcup_{\substack{e = 1, 0, 1, 0, 0, 0}} \mathbb{A} \qquad \text{where } e = -1.181241456...
$$

\n
$$
C_3 = \bigcup_{\substack{e = 1, 0, 1, 0, 0, 0}} \mathbb{A} \qquad \text{where } e = -1.912245764...
$$

\n
$$
C_4 = \text{lets of Feynman diagrams}
$$

\n
$$
C_5 = \bigcup_{\substack{e = 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}} \mathbb{A} \qquad \text{therefore numerical} \qquad \text{therefore numerical} \qquad \text{equations by Kinoshita et al}
$$

QED Mass-independent term: 2-loop contribution calculation in p Q F i: the g - z of the electron Γ be "most femous coloulation" in Γ The "most famous calculation" in pQFT: the **g-2 of the electron**

+ ...

 $C_5 =$

+ C³

:
:

 $S_{\mathcal{S}} = -2,$ Inspired by precision, Bologna, Bologna, $2,$ Dec 2021 Page 70.

"3

 $+ + + +$

!
! a a i

"4

 \cdot + + + + \cdot

. . .

. . .

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

\n
$$
C_1 = \bigcup_{\substack{z \in \mathbb{Z} \\ \text{even } z}} \bigotimes_{z = 1} \bigotimes_{\substack{z \in \mathbb{Z} \\ \text{even } z}} \bigotimes_{z = 1.181241456...}
$$

\n
$$
C_2 = \bigotimes_{z \in \mathbb{Z} \\ \text{odd } z} \bigotimes_{z = 1.181241456...}
$$

\n
$$
C_3 = \bigotimes_{z \in \mathbb{Z} \\ \text{odd } z} \bigotimes_{z = 1.912245764...}
$$

\n
$$
C_4 = \bigotimes_{z \in \text{even } z} \bigotimes_{z \in \text{odd } z} \bigotimes_{z \in \text{even } z} \bigotimes_{z \in \text{odd } z} \bigotimes_{z \in \text{even } z} \bigotimes_{z \in \text{odd } z} \bigotimes_{z \in \text{even } z} \big
$$

 $C_5 =$ = $+6.737(159)$ converge nicely once multiplied by 1/137 :-)

QED Mass-independent term: 1-loop contribution QED Mass-independent term: 1-loop contribution **ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"**

!α

"4

!α

"5

QED Mass-independent term: 3-loop contribution

!
!
!

"3

QED Mass-independent term: 2-loop contribution calculation in p Q F i: the g - z of the electron $\tau_{\rm b}$ (most femous coloulation) in nOFT the τ) of the electron The "most famous calculation" in pQFT: the g-2 of the electron

"2

+ ...

!α

+ C³

:
:

"3

 $+ + + +$

 $\ddot{}$

!
! a a i

"4

 \cdot + + + + \cdot

!!!

. . .

"

. . .

$$
a_e^{QED} = C_1 \left(\frac{\alpha}{\pi}\right) + C_2 \left(\frac{\alpha}{\pi}\right)^2 + C_3 \left(\frac{\alpha}{\pi}\right)^3 + C_4 \left(\frac{\alpha}{\pi}\right)^4 + C_5 \left(\frac{\alpha}{\pi}\right)^5 + \dots
$$

\n
$$
C_1 = \bigcup_{\substack{1 \text{Schwinger '48]}}}
$$
\n
$$
C_2 = \bigcup_{\substack{1 \text{Schwinger '48]}}}
$$
\n
$$
C_3 = \bigcup_{\substack{1 \text{Sawinger '48}}} \bigotimes_{\substack{1 \text{Sawinger '48}}} \frac{197}{12} + \frac{1}{12} \pi^2 - \frac{1}{2} \pi^2 \ln 2 + \frac{3}{4} \zeta(3) \quad \text{[Petermann, Sommerfield '57]}}
$$
\n
$$
C_4 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n
$$
C_5 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n
$$
C_6 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n
$$
C_7 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n
$$
C_8 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n
$$
C_9 = \bigoplus_{\substack{1 \text{Sawier}}}
$$
\n<math display="block</math>

But if we look at *analytic results*, some pattern starts to emerge: Γ two-loop computed analytically by Γ But if we look at *analytic results*, some pattern starts to emergence and the particular and the partition and the partition But if we lo
P∷rmanus =ata and k at *analytic results, s⊾*
موجود موجود موجود _ime pattern sta rts to emerg \mathcal{L}_{tot} if we look at *gralutic woults* some pettern starts to emerge. But if we look at analytic results, some pattern starts to emerge:

rational numbers, Riemann zeta values, rational numbers, Riemann zeta values, …, in general **multiple polylogarithms** evaluated at special (rational) points

SUCCESS OF THE PAST 20 YEARS: MULTIPLE POLYLOGARITHMS

Iterated integrals of **rational functions on the Riemann Sphere**

Multiple PolyLogarithms (MPLs)

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}
$$

SUCCESS OF THE PAST 20 YEARS: MULTIPLE POLYLOGARITHMS

Iterated integrals of **rational functions on the Riemann Sphere**

Multiple PolyLogarithms (MPLs)

$$
G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)
$$

=
$$
\int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}
$$

Provided us with the right language to make sense of a lot of the structure in scattering amplitudes

- leading singularities and dlog forms (local integrals) **[Arkani-Hamed et al '10]**
- differential equations in canonical form **[Henn '13]**
- hint towards generalisations (elliptic multiple polylogs, more general diff forms, Calabi-Yau geometries etc)

TOWARDS A NNLO REVOLUTION (?)

$$
\left| \mathcal{M}_{q\bar{q}\to gg} \right|^2 = \left| \mathcal{M}_{q\bar{q}\to gg}^{LO} \right|^2 + \left(\frac{\alpha_s}{2\pi} \right) \left| \mathcal{M}_{q\bar{q}\to gg}^{NLO} \right|^2 + \left(\frac{\alpha_s}{2\pi} \right)^2 \left| \mathcal{M}_{q\bar{q}\to gg}^{NNLO} \right|^2 + \dots
$$

By understanding analytic structure of amplitudes + how to handle and subtract IR divergences, past 2 decades have seen the beginning of a NNLO revolution…

BEYOND NNLO FOR 2 \rightarrow 2 THERE IS STILL A LOT TO LEARN

.

We are just scratching the surface…!

BEYOND NNLO FOR $2 \rightarrow 2$ **There is still a lot to LEARN**

We are just scratching the surface...!

[Duhr, Dulat, Mistlberger '20]

Non trivial uncertainty patterns observed going from NNLO to N3LO for W,y Drell-Yan

We are far from being able to do N^3LO pheno for generic processes...

BEYOND NNLO FOR 2 -> 2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!

We are far from being able to do N3 LO **pheno for generic processes…**

IR singularities and new sources for (bootstrap?)... possible **factorisation breaking** (di-jet / $t\bar{t} \otimes N^3LO...$)

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities

BEYOND NNLO FOR 2 2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!

We are far from being able to do N3 LO **pheno for generic processes…**

IR singularities and new sources for (bootstrap?)... possible **factorisation breaking** (di-jet / $t\bar{t} \otimes N^3LO...$)

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities

Particularly interesting

di-jet production @ N3LO!

TOWARDS DI-JET AT N3LO

.

First step is **3 loop scattering amplitudes**:

- Informs on **complexity of functions involved**
- Informs on **IR structure** in three-loop QCD: **quadrupole correlations!**
- Informs on **all-order structure of QCD**: High Energy limit, **Regge factorisation etc**

TOWARDS DI-JET AT N3LO

First step is **3 loop scattering amplitudes**:

- Informs on **complexity of functions involved**
- Informs on **IR structure** in three-loop QCD: **quadrupole correlations!**
- Informs on **all-order structure of QCD**: High Energy limit, **Regge factorisation etc**

3 main channels:
$$
gg \rightarrow gg
$$
, $q\bar{q} \rightarrow gg$, $q\bar{q} \rightarrow Q\bar{Q}$

We will focus mainly on the most complicated one: $gg \rightarrow gg$

Scattering Amplitudes: flashing through standard approach **[See Harald Ita's talk]**

$$
\sim \qquad \mathcal{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \bar{\nu}(q) \Gamma_{\mu_1, \dots, \mu_n} u(p)
$$

Scattering Amplitudes: flashing through standard approach **[See Harald Ita's talk]**

(Scalar) Feynman Integrals

$$
\mathcal{F} = \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \dots S_m^{b_m}}{D_1^{a_1} \dots D_n^{a_n}}
$$

with
$$
S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}
$$

Scattering Amplitudes: flashing through standard approach **[See Harald Ita's talk]**

In reality, for $gg \to gg$ @ 3 loops

+500 more pages

(\sim 50000 Feynman diagrams -10^7 integrals!!)

So we need a way to organise this mess…

Scattering Amplitudes: flashing through standard approach **[See Harald Ita's talk]**

Scattering Amplitudes: flashing through standard approach **[See Harald Ita's talk]**

TENSOR DECOMPOSITION

$$
= \int \prod_{i=1}^{L} d^D k_i R_i(k_1,\ldots,k_L,p_1,\ldots,p_E,m_j)
$$

First step: Strip it of Lorentz and Dirac structures

$$
\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k - p_2)^2 (k - p_2 - p_3)^2 (k - p_1 - p_2 - p_3)^2} = \boxed{\phantom{\frac{1}{(2\pi)^d}}}
$$

Scalar Feynman Integrals are what we know how to compute

.

.

▶ Pick your favourite process (for example $q\bar{q}$ → Zg or $q\bar{q}$ → $Q\bar{Q}$)

- ► Pick your favourite process (for example $q\bar{q}$ → Zg or $q\bar{q}$ → $Q\bar{Q}$)
- ➤ Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$
\mathscr{A} = \sum_{j=1}^n F_j T_j
$$

.

- ► Pick your favourite process (for example $q\bar{q}$ → Zg or $q\bar{q}$ → $Q\bar{Q}$)
- ➤ Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$
\mathscr{A} = \sum_{j=1}^n F_j T_j
$$

➤ Derive **projectors operators** *(dual vectors)* to single out corresponding form factors:

$$
M_{ij} = \sum_{pol} T_i^{\dagger} T_j \qquad \mathscr{P}_j = \sum_k (M^{-1})_{jk} T_k^{\dagger}
$$

$$
\mathscr{P}_j \mathscr{A} = F_j
$$

- ► Pick your favourite process (for example $q\bar{q}$ → Zg or $q\bar{q}$ → $Q\bar{Q}$)
- ➤ Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$
\mathscr{A} = \sum_{j=1}^n F_j T_j
$$

➤ Derive **projectors operators** *(dual vectors)* to single out corresponding form factors:

$$
M_{ij} = \sum_{pol} T_i^{\dagger} T_j \qquad \mathcal{P}_j = \sum_k (M^{-1})_{jk} T_k^{\dagger}
$$

$$
\mathcal{P}_j \mathcal{A} = F_j
$$

➤ Apply these projectors on **Feynman diagrams** (or any other representation of the scattering amplitude) —> obtain combination of scalar integrals

FROM "TENSORS" TO HELICITY AMPLITUDES

Ultimately, we are interested in **helicity amplitudes**

(minimal, physical objects which retain full physical information on final states)

FROM "TENSORS" TO HELICITY AMPLITUDES

Ultimately, we are interested in **helicity amplitudes**

(minimal, physical objects which retain full physical information on final states)

Fix helicities (assuming that external states are in $d = 4$ dimensions)

This allows us to have the full structure of the amplitude under control True at every number of loops!

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

Typical case 4 quark scattering $q(p_2) + \bar{q}(p_1) \rightarrow Q(p_2) + \bar{Q}(p_4)$

 $D_i \sim \bar{u}(p_1)\Gamma^{\mu_1,...,\mu_n}u(p_2) \bar{u}(p_3)\Gamma_{\mu_1,...,\mu_n}u(p_4)$

Infinite number of tensor structures in *d* dimensions

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

Typical case 4 quark scattering $q(p_2) + \bar{q}(p_1) \rightarrow Q(p_2) + \bar{Q}(p_4)$

$$
D_i \sim \bar{u}(p_1)\Gamma^{\mu_1,\dots,\mu_n}u(p_2) \,\bar{u}(p_3)\Gamma_{\mu_1,\dots,\mu_n}u(p_4)
$$
\nInfinite number of tensor structures in *d* dimensions

up to 2 loops!

- $\mathcal{D}_1 = \bar{u}(p_1)\gamma_{\mu_1}u(p_2) \bar{u}(p_3)\gamma_{\mu_1}u(p_4),$
- $\mathcal{D}_2 = \bar{u}(p_1)p_3u(p_2) \bar{u}(p_3)p_1u(p_4),$
- $\mathcal{D}_3 = \bar{u}(p_1) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} u(p_2) \bar{u}(p_3) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} u(p_4),$
- $\mathcal{D}_4 = \bar{u}(p_1)\gamma_{\mu_1} p_3\gamma_{\mu_3} u(p_2) \bar{u}(p_3)\gamma_{\mu_1} p_1\gamma_{\mu_3} u(p_4),$
- $\mathcal{D}_5 = \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_2) \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_4),$
- $\mathcal{D}_6 = \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\rlap/p_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\rlap/p_1\gamma_{\mu_4}\gamma_{\mu_5}u(p_4).$

TENSOR DECOMPOSITION: UPGRADE IN THV

Improvements in **'t Hooft - Veltman (tHV)** scheme **[Peraro, Tancredi '19,'20]**

2 independent helicity configurations: $q(p_2) + \bar{q}(p_1) \rightarrow Q(p_2) + \bar{Q}(p_4)$

→ only **two "tensors" are linearly independent** if external states are in *d* = 4

$$
\mathcal{D}_1 = \bar{u}(p_1)\gamma_{\mu_1}u(p_2) \bar{u}(p_3)\gamma_{\mu_1}u(p_4), \n\mathcal{D}_2 = \bar{u}(p_1)\cancel{p}_3u(p_2) \bar{u}(p_3)\cancel{p}_1u(p_4),
$$

TENSOR DECOMPOSITION: UPGRADE IN THV

Improvements in **'t Hooft - Veltman (tHV)** scheme **[Peraro, Tancredi '19,'20]**

2 independent helicity configurations: $q(p_2) + \bar{q}(p_1) \rightarrow Q(p_3) + \bar{Q}(p_4)$

→ only **two "tensors" are linearly independent** if external states are in *d* = 4

$$
\mathcal{D}_1 = \bar{u}(p_1)\gamma_{\mu_1}u(p_2) \bar{u}(p_3)\gamma_{\mu_1}u(p_4), \n\mathcal{D}_2 = \bar{u}(p_1)p_3u(p_2) \bar{u}(p_3)p_1u(p_4),
$$

4 dimensional tensors alone are enough to obtain full result in 't Hooft-Veltman scheme and also <u>the finite remainder in CDR!</u>

Used successfully for $pp\to pp\text{ }\mathscr{Q}$ 3 loops [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21,'22] And first **full colour** calculation for a 2 → 3 amplitude: *qq*¯ → *γγj* at 2 loops in QCD [Agarwal, Buccioni, Manteuffel, Tancredi '21] $\text{S}\text{C}\text{C}\text{C}\text{C}\text{C}\text{C}$ and $\text{C}\text{C}\text{D}\text{C}\text{D}\text{D}\text{D}$ $\text{C}\text{C}\text{D}\text{D}$ $\text{C}\text{C}\text{D}\text{D}$ and $\text{C}\text{C}\text{D}\text{D}$ and $\text{C}\text{C}\text{C}\text{D}$ and $\text{C}\text{C}\text{D}\text{D}$ and $\text{C}\text{C}\text{D}\text{D}$ and $\text{C}\text{D}\text{D$ dimentially colour calculation for a $2 \rightarrow 3$ amplitude: $a\bar{a} \rightarrow vvi$ at 2 loops in OCD

Let's see how this works for chiral theories

Consider the decay of a Z-boson and to three jet

 $Z(p_4) \rightarrow q(p_1) + \bar{q}(p_2) + g(p_3)$

.

Let's see how this works for chiral theories

Consider the decay of a Z-boson and to three jet

$$
Z(p_4) \to q(p_1) + \bar{q}(p_2) + g(p_3)
$$

Status:

Pheno @ NNLO including **only vector-like** couplings of singlet type

Amplitudes [Garland, Gerhmann et al '02] Pheno [Gehrmann-De Ridder et al '17, '18] etc etc

One issue for **axial couplings** is **evanescent structures in chiral tensor**

TENSOR DECOMPOSITION: CHIRAL THEORIES (*F*˜*i*) and 6 parity-odd (*G*˜*i*) structures. We stress once more that this simple separation is

One issue for **axial couplings** is **evanescent structures in chiral tensor** *A*, *a*, *only in the second that* are particularly useful to get rid of the possible ambiguity in th

In our approach, only **tensors in** $d = 4$ are relevant, we can span amplitude with a basis of vectors in $d = 4$: $p_1^{\mu}, p_2^{\mu}, p_3^{\mu}$, plus the fourth **parity-odd one** *n s* the fourth parity-odd one <u>د</u>

$$
\epsilon_{\nu\rho\sigma\mu}p_1^\nu p_2^\rho p_3^\sigma=\epsilon^{p_1p_2p_3\mu}=v_A^\mu
$$

 \overline{W} timese, a possible basis call be written as: (could be further optimised for singlet contributions
i ^A. This implies With these, a possible basis can be written as: (could be further optimised for singlet contributions)

$$
A^{\mu\nu} = \bar{u}(p_2)\rlap/p_3u(p_1)\Big[F_1p_1^{\mu}p_1^{\nu} + F_2p_2^{\mu}p_1^{\nu} + F_3g^{\mu\nu} + G_1p_1^{\mu}v_A^{\nu} + G_2p_2^{\mu}v_A^{\nu} + G_3v_A^{\mu}p_1^{\nu}\Big] + \bar{u}(p_2)\gamma^{\nu}u(p_1)\Big[F_4p_1^{\mu} + F_5p_2^{\mu}\Big] + \bar{u}(p_2)\gamma^{\mu}u(p_1)F_6p_1^{\nu} + \bar{u}(p_2)\rlap/v_Au(p_1)\Big[G_4p_1^{\mu}p_1^{\nu} + G_5p_2^{\mu}p_1^{\nu}\Big] + G_6\Big[\bar{u}(p_2)\gamma^{\mu}u(p_1)v_A^{\nu} + \bar{u}(p_2)\gamma^{\nu}u(p_1)v_A^{\mu}\Big]
$$

We can start from eq. (2) and try to add the corresponding axial part. One has to pay attention here **[Gehrmann, Peraro, Tancredi '22]**

^Aµ⌫ = ¯*u*(*p*2)*p/*3*u*(*p*1) h *F*1*p^µ* 1 *p*⌫ ¹ ⁺ *^F*2*p^µ* 2 *p*⌫ ¹ ⁺ *^F*3*gµ*⌫ ⁺ *^G*1*p^µ* 1 *v*⌫ *^A* ⁺ *^G*2*p^µ* 2 *v*⌫ *^A* ⁺ *^G*3*v^µ Ap*⌫ 1 i + ¯*u*(*p*2)⌫*u*(*p*1) h *F*4*p^µ* ¹ ⁺ *^F*5*p^µ* 2 i + ¯*u*(*p*2)*µu*(*p*1)*F*6*p*⌫ 1 + ¯*u*(*p*2)*v /Au*(*p*1) h *G*4*p^µ* 1 *p*⌫ ¹ ⁺ *^G*5*p^µ* 2 *p*⌫ 1 i + *G*⁶ h *u*¯(*p*2)*µu*(*p*1)*v*⌫ *^A* + ¯*u*(*p*2)⌫*u*(*p*1)*v^µ A* i

The counting is straightforward. The counting is straightforward:

- **2 helicities** for the $q\bar{q}$ line (massless) **a** Gives a total of = **12** Gives a total of = **12**
	- ▶ 2 helicities for the (physical) gluon *^p*1*p*2*p*3*^µ* = *v^µ*
- ► 3 helicities for the (physical) Z boson **are in this vector we are in** *defension* tensors and form factors time dimensions, we can make a choice and use as independent vectors *p^µ*

helicity amplitudes Gives a total of = 12

helicity amplitudes

matched by the number of

tensors and form factors

Note that manipulations are done in **tHV / Larin scheme**

$$
p_i \cdot v_A = 0, \qquad v_A \cdot v_A = \epsilon^{p_1 p_2 p_3 \mu} \epsilon^{p_1 p_2 p_3 \mu} = \frac{d-3}{4} s_{12} s_{13} s_{23}
$$

We can start from eq. (2) and try to add the corresponding axial part. One has to pay attention here **[Gehrmann, Peraro, Tancredi '22]** because, since we have an extra independent vector at disposal, the countries of degrees of degrees of the coun
In the countries of the countries of degrees of the countries of freedom has to free down has to free down has

THE CASE OF 4-GLUON SCATTERING *f* the UASE of 4-gluun sualiering *s >* 0*, t<* 0*,u<* 0; 0 *<x<* 1*.* (5)

COLOR AND LORENTZ DECOMPOSITION

Applied all these ideas to $gg \to gg$ [Caola, Cha 8 helicity amplitudes \sim 8 form factors for each colour ordered amplitude 8 helicity amplitudes \sim 8 form factors for each colour ordered amplitude de and the state of the sta Applied all these ideas to $gg \rightarrow gg$ [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21] *^A^a*1*a*2*a*3*a*⁴ = 4⇡↵*s,b* ^X

dence with the tensors in eq. (9), such that *Pⁱ*

^P *· ^T^j* ⁼

$$
\mathcal{A}^{a_1 a_2 a_3 a_4} = 4\pi \alpha_{s,b} \sum_{i=1}^6 \mathcal{A}^{[i]} \mathcal{C}_i \qquad \longrightarrow \qquad \mathcal{A} = \sum_{j=1}^8 \mathcal{F}_i T_i
$$

 $\mathcal{C}_1 = \text{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}] + \text{Tr}[T^{a_1}T^{a_4}T^{a_3}T^{a_2}]$ etc etc…

f the UASE of 4-gluun sualiering *s >* 0*, t<* 0*,u<* 0; 0 *<x<* 1*.* (5) **THE CASE OF 4-GLUON SCATTERING**

COLOR AND LORENTZ DECOMPOSITION

Applied all these ideas to $g\overline{g} \rightarrow g\overline{g}$ [Caola, Cha 8 helicity amplitudes \sim 8 form factors for each colour ordered amplitude de and the state of the sta Applied all these ideas to *gg* → *gg* [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21] *^A^a*1*a*2*a*3*a*⁴ = 4⇡↵*s,b* ^X 8 helicity amplitudes \sim 8 form factors for each colour ordered amplitude

$$
\mathcal{A}^{a_1 a_2 a_3 a_4} = 4\pi \alpha_{s,b} \sum_{i=1}^6 \mathcal{A}^{[i]} \mathcal{C}_i \qquad \longrightarrow \qquad \mathcal{A} = \sum_{j=1}^8 \mathcal{F}_i T_i
$$

$$
\mathcal{C}_1 = \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \text{Tr}[T^{a_1} T^{a_4} T^{a_3} T^{a_2}] \qquad \text{etc...}
$$

C Helicity amplitudes $\mathscr{A}_{\lambda} = s_{\lambda} H_{\lambda}$ pend on the Mandelstam invariants and *s* is a phase that Helicity amplitudes $\mathscr{A}_{\lambda} = s_{\lambda} H_{\lambda}$ $\mathcal{A} = \mathfrak{c}$ *H*

$$
mplitudes \t\t \mathscr{A}_{\lambda} = s_{\lambda} H_{\lambda}
$$

 $\{\, + + + +, - + + +, + - + +, \textit{etc}\,\}$ identities \mathcal{S}^{max} $R_{\lambda} = S_{\lambda} H_{\lambda}$ where $\lambda = \{ + + + +, - + + +, + - + +, etc \}$ $\begin{array}{ccc} -\int \bot \bot \bot \bot & -\bot \bot \bot & \bot \bot \bot \bot & \theta t c \end{array}$ ten as a linear combination of the form factors *^F*[*j*]

+

[*i|^µ|ⁱ* + 1ⁱ

^p2h*ⁱ* + 1*|i*ⁱ

dence with the tensors in eq. (9), such that *Pⁱ*

^P *· ^T^j* ⁼

$$
s_{+++} = \frac{\langle 12 \rangle \langle 34 \rangle}{[12][34]}, \qquad s_{+++} = \frac{\langle 12 \rangle \langle 14 \rangle [24]}{\langle 34 \rangle \langle 23 \rangle \langle 24 \rangle}
$$
 For example:
\n
$$
s_{+++} = \frac{\langle 21 \rangle \langle 24 \rangle [14]}{\langle 34 \rangle \langle 13 \rangle \langle 14 \rangle}, \qquad s_{+++} = \frac{\langle 32 \rangle \langle 34 \rangle [24]}{\langle 14 \rangle \langle 21 \rangle \langle 24 \rangle}
$$

\n
$$
s_{+++} = \frac{\langle 42 \rangle \langle 14 \rangle [12]}{\langle 13 \rangle \langle 23 \rangle \langle 12 \rangle}, \qquad s_{+++} = \frac{\langle 12 \rangle [34]}{[12] \langle 34 \rangle}, \qquad H_{+++} = t^2 \left(\frac{\mathcal{F}_8}{su} - \frac{\mathcal{F}_3}{2s} + \frac{\mathcal{F}_6}{2u} - \frac{\mathcal{F}_1}{4} \right)
$$

\n
$$
s_{+-+} = \frac{\langle 13 \rangle [24]}{[13] \langle 24 \rangle}, \qquad s_{+---} = \frac{\langle 14 \rangle [23]}{[14] \langle 23 \rangle}.
$$

REDUCTION TO MASTER INTEGRALS

Path to get there extremely complicated, became possible thanks to new mathematical tools

[See Harald's talk]

THE CASE OF 4-GLUON SCATTERING

Three-loop calculation is very non-trivial, it took "20 more years"!

Many master integrals (\sim but only 500 vs 10^7 integrals before reduction!)

 Approached by **differential equations method [Kotikov '97; Remiddi '99; Gehrmann Remiddi '00]** Figure 1. The nine integral families needed to describe all master integrals for three-loop massless

$$
d\vec{I} = \epsilon A(x)\vec{I}
$$
 [Arkani-Hamed '10; Kotikov '07 '10; Henn '13, Lee '15]

Finding a so-called "canonical basis" is very non-trivial [Henn, Mistlberger, Smirnov, Wasser, 2020] integrals contributing to four-particle scattering. We denote the momenta of the four

Result can be written in terms of **simple functions**: (harmonic) **multiple polylogarithms** *p* θ = 0 (2.1) *is said be written in terms of omipre remembre, marmoine,*

$$
G(a_1, \ldots, a_n; x) = \int_0^x \frac{dt_1}{t_1 - a_1} G(a_2, \ldots, a_n; t_1), \quad a_j = \{0, 1\}, \qquad G(0, \ldots, 0, x) = \frac{1}{n!} \log^n x
$$

[Remiddi, Vermaseren '99]

INFRA-RED STRUCTURE

IR singularities are known to **factorise in gauge theories**

Tejeda-Yeomans, Mert Aybat, Almelid, Duhr, Gardi, Ferroglia, Czakon, Mitov, … many others …]*J*¹ *J*¹ *S H S H* σ \overline{O} 20000 *J*² **Picture from Agarwal, Magnea, Signorile-Signorile, Tripathi '21**

[Becher, Neubert, Dixon, Magnea, Sterman,

INFRA-RED STRUCTURE with respect to one of the hard parton directions. For fixed-angle angle angle angle angle amplitudes, such gluons essen-angle amplitudes, such gluons essen-angle amplitudes, such gluons essen-angle angle angle angle angle

IR singularities are known to factorise in gauge theories tially have vanishing longitudinal components, and are usually called *Glauber* gluons. Glauber

*S H J*¹ **2020** *S H J*¹ *J*² \mathcal{A}_{α} $\left(\frac{p_i}{\alpha} \mathcal{A}_{\alpha}(\mu)\right)$ ϵ = $\prod_{i=1}^n \frac{\mathcal{J}_i\left(\frac{\mu}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\alpha_i(\mu^2)}$ $\left\{ \mu \right\} \left\{ \mu \right\}$ \sum_{s} $\binom{R_{i}, R_{i}, \ldots, R_{i}}{q_{i}}$ $\binom{p_{i} \cdot p_{j}}{q_{i}}$ $\binom{p_{i} \cdot p_{j}}{q_{i}}$ $\alpha \in \mathcal{A}_n(\rho_i \cdot \rho_j, \alpha_s(\mu_j), \epsilon) \nrightarrow \mathcal{A}_n$ $\left(\frac{\mu^2}{\mu^2}, \frac{\mu^2}{\mu^2}\right)$ Picture from Agarwal, Magnea, Signorile-Signorile, Tripathi '21 Example 2011 [Becher, Neubert, Dixon, Magnea, Sterman, Tejeda-Yeomans, Mert Aybat, Almelid, Duhr, Gardi, Ferroglia, Czakon, Mitov, ... many **others …]** variants become space-like [207]. For fixed-angle scattering amplitudes, however, Glauber gluons do not contribute at least β with the second propose a generalisation of Eq. (4.12) for multi-particle and the equation of Eq. (4.12) for multi-particle and the experimental model of Eq. (4.12) for multi-particle and the experimental model of Eq. (4.1 \mathcal{H} $\mathcal{V} \vee \vee$ and \mathcal{S} is that collinear dynamics will be collinear dynamics with \mathcal{S} \mathscr{A} out gluons, on the other hand, can connect any pair of hand, can connect any particles, and they are the second \mathcal{I}_2 colour of the colour of the colour of the \mathcal{I}_2 the colour indices of all external particles. The emergent form of soft-collinear factorisation for fixed-angle multi-particle scattering amplitudes in massless gauge theories is then \mathcal{A}_n $\int p_i$ *µ* $,\alpha_s(\mu),\epsilon$ ◆ $=$ Π *n i*=1 *Ji* $\left(\frac{(p_i \cdot n_i)^2}{n_i}\right)$ $\frac{(\rho_i\cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon$ \setminus $\overline{\mathcal{J}_{E,i}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2}\right)}$ $\overline{n^2_i}$ $,\alpha_s(\mu^2),\epsilon$ $\overline{\mathcal{N}}$ \times \mathcal{S}_n $(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)$ H_n $\sqrt{2}$ $p_i \cdot p_j$ $\frac{r_J}{\mu^2},$ $(p_i \cdot n_i)^2$ $\frac{n_i^2\mu^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon$!

INFRA-RED STRUCTURE coloured participate to the scattering participate to the scattering process \mathbb{R}^n . In particular, it has been seen it has been seen in particular, it has been seen in particular, it has been seen in particular, it ha S shown that IS shown that IS shown that IS shown that S is operator S of operator S or S or S matrix elements in SCET provides in SCET provides in SCET (See Therefore, UV responses to the extension in SCET
.

The can be "multiplicatively renormalised away" similarly to UV divergences IR san be "multiplicatively nepermediced every" similarly to IIV divergences ne can be inuitiplicatively renormalised away similarly to OV divergences

$$
\mathcal{H}_{i, \text{ fin}}(\epsilon, \{p\}) = \lim_{\epsilon \to 0} \mathcal{Z}^{-1}(\epsilon, \{p\}, \mu) \; \mathcal{H}_{i, \text{ ren}}(\epsilon, \{p\})
$$
\n
$$
\mathcal{Z}(\epsilon, \{p\}, \mu) = \mathbb{P} \exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \Gamma(\{p\}, \mu')\right]
$$

Z acts on the amplitude as a matrix in colour space, non-trivial correlations among partons

INFRA-RED STRUCTURE coloured participate to the scattering participate to the scattering process \mathbb{R}^n . In particular, it has been seen it has been seen in particular, it has been seen in particular, it has been seen in particular, it ha S shown that IS shown that IS shown that IS shown that S is operator S of operator S or S or S matrix elements in SCET provides in SCET provides in SCET (See Therefore, UV responses to the extension in SCET
. is to be thought of as a soft gluon being exchanged by the hard legs (in black).

The can be "multiplicatively renormalised away" similarly to UV divergences IR san be "multiplicatively nepermediced every" similarly to IIV divergences ne can be inuitiplicatively renormalised away similarly to OV divergences **5. The IR structure**

$$
\mathcal{H}_{i,\text{ fin}}(\epsilon,\{p\}) = \lim_{\epsilon \to 0} \mathcal{Z}^{-1}(\epsilon,\{p\},\mu) \ \mathcal{H}_{i,\text{ ren}}(\epsilon,\{p\})
$$
\n
$$
\mathcal{Z}(\epsilon,\{p\},\mu) = \mathbb{P} \exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \mathbf{\Gamma}(\{p\},\mu')\right]
$$

n the amplitude as a matrix in colour space, non-trivial correlations among partons values of *µ*0 . Following the notation of $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathcal{S})$ first computed up to the loops, where loops, where loops, where loops, where $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathcal{S})$ and $\mathcal{B}(\mathcal{S})$ and $\mathcal{B$ than dimension operator for 4 coloured external particles is written as well as well as well as well as well as *n*=0 The anomalous dimension Γ is fully known up to three loops *Z* acts on the amplitude as a matrix in colour space, non-trivial correlations among partons itude as a matrix in colour space, non-trivial correlations among partons form is a matter in term of the soft and the soft and the soft $\frac{1}{2}$

The adronatotes dimension
$$
\mathbf{r}
$$
 is tail, shown up to two loops
\n
$$
\mathbf{r}(\{p\}, \mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\}, \mu) + \dots
$$
\n
$$
\text{Cusp anomalous dimension} \qquad \text{anomalous} \qquad \text{anomalous} \qquad \text{dimensions}
$$
\n
$$
\text{soft-collinear double poles} \qquad \text{dimensions}
$$
\n
$$
\mathbf{\Gamma}_{\text{dip}} = \sum_{1 \le i < j \le 4} \mathbf{T}_i^a \mathbf{T}_j^a \gamma^K \ln \left(\frac{\mu^2}{-s_{ij} - i\delta} \right) + \sum_i \gamma^i
$$

INFRA-RED STRUCTURE $A_{\rm{max}}$ dipole represents the well known dipole colour colour colour colour colour colour colour colour colour where *z* is a color matrix that acts on the color basis \overline{C}

At three loops we see for the first time quadrupole correlations $\overline{\text{ions}}$ *<u>ze</u>* the quadrupole correlation

Finding Theories II and Theories Four-Point Scattering Almelid, Duhr, Gardi '15] ns [Becher, Neubert '13
[Almelid, Duhr, Gar] *. (Almelid, Duhr, Gare* **[Becher, Neubert '13] [Dixon, Gardi, Magnea '11]**

⁸ } (6) and can be written in exponential

e external legislation en elegant legislation en elegant legislation en elegant legislation en elegant legisla
Description

 $\mathbf{\Gamma}(\{p\},\mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) + \mathbf{\Delta}_4(\{p\})$... $\mathbf{\Delta}_4(\{p\}) = \sum$ $\frac{\infty}{\sqrt{2}}$ *L*=3 $\sqrt{\alpha_s}$ 4π $\int_{-a}^{b} \Delta_4^{(L)}(\{p\})$ $\frac{1}{\sqrt{2}}$ $L=3$ represents the $L=3$ ∞ is instead due to exchanges of color charges of color charges σ $\mathbf{e}(\mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\}, \mu) + \mathbf{\Delta}_4(\{p\})$ and $\mathbf{\Delta}_4(\{p\}) = \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{k}\right)^k \mathbf{\Delta}_4^{(L)}(\{p\})$

$$
\Delta_4^{(3)} = f_{abe} f_{cde} \left[-16C \sum_{i=1}^4 \sum_{\substack{1 \le j < k \le 4 \\ j,k \ne i}} \{ \mathbf{T}_i^a, \mathbf{T}_i^d \} \, \mathbf{T}_j^b \mathbf{T}_k^c + 128 \left[\mathbf{T}_1^a \mathbf{T}_2^c \mathbf{T}_3^b \mathbf{T}_4^d D_1(x) - \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d D_2(x) \right] \right],
$$

HIGH ENERGY LIMIT (REGGE FACTORIZATION) produced in the Supplemental Material for convenience. *ⁱ* , which only act on the *i*-th exter-*ⁱ ^T^bⁱ* ⁼ *ifabiciT^cⁱ* = [*T^bⁱ , T ^a*]. $\mathbf{r} = \mathbf{r}$, $\mathbf{$ *x* = *t/s*. Figure 1: Tree level and interference in the square square square square square square square square square sq tree level with *L* = 1*,* 2*,* 3 loop amplitudes in dependence of **INSERTION OPERATORS TRAIT ACT ONLY A PARTIES** niun civerui liiviii (Reuue f *{Ci}* is defined as ^T*^a x* **0017**

Calculation elucidates general structures in QCD nal color integration of the color case the color case of the c
 $\frac{1}{2}$ *ⁱ ^Tbⁱ* ⁼ *ifabiciTcⁱ* = [*Tbⁱ , T ^a*]. **incide Calculation elucidates general stri** four external legs. It becomes relevant for the first time $\frac{1}{2}$ $\frac{1}{2}$ For studying the studying the split scatter \mathbf{F}

Verified all-order structures in Regge kinematics $\overline{\text{C}}$ Verified all-order structures in Regge ki

egge kinematics $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

terms of the variables introduced in this letter, this limit

four external legs. It becomes relevant for the first time *^s* (*n*) Define even and odd amplitude Define even and odd amplitude under $(s \leftrightarrow u)$ And the even "large logarithm" *j,k*6=*i*

x = *t/s*.

$$
\mathcal{H}_{\text{ren},\pm} = \frac{1}{2} \left[\mathcal{H}_{\text{ren}}(s, u) \pm \mathcal{H}_{\text{ren}}(u, s) \right] \qquad L = -
$$

 $\Gamma(s \leftrightarrow u)$ and the even "large logarithm" And the even "large logarithm"

$$
L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2} \left(\ln\left(\frac{-s - i\delta}{-t}\right) + \ln\left(\frac{-u - i\delta}{-t}\right)\right)
$$

$\mathcal{H}_{\mathrm{ren},\pm} =$ $\mathcal{H}_{\mathrm{ren},\pm} = \frac{1}{2}\left[\mathcal{H}_{\mathrm{ren}}(s,u) \pm \mathcal{H}_{\mathrm{ren}}(u,s)\right]$ $L =$

 $\frac{1}{2}$

At leading power in *x* and up to *next-to-leading logarithmic accuracy (NLL)* for even part and *NNLL for the odd part*, the amplitude factorises in a well understood way (LO BFKL) in terms of At leading powe *the amplitude factorises in a* $\frac{1}{2}$ At leading power in x and up to next-to-leading logarithm our notation are reported in the Supplemental Material. for the odd part, the amplitude factorises in a v $\frac{1}{2}$ *the odd part*, the amplitude factorises in a well understood way (LO BFKL) in terms of the means of "Dependence" (methinle exclusively species to R) and anti-thermal of 1 ² (T² *^s* ^T² at leading power in x and up to *next-to-leading logarithmic accuracy (NUI)* for even part and NNUI *j,k6* $\emph{ct-to-leading logarithmic accuracy (NLL)}$ for even part and $NNLL$

for the oud part, the amplitude factorises in a well understood way (LO BFKL) in terms of exchanges of "Reggeons" (multiple exchanges give rise to **Regge cut contributions**) ้
มห $ributions)$

$$
\mathcal{H}_{\mathrm{ren},\pm} = Z_g^2 \, e^{L \mathbf{T}_t^2 \boldsymbol{\tau_g}} \sum_{n=0}^3 \bar{\alpha}_s^n \sum_{k=0}^n L^k \mathcal{O}_k^{\pm,(n)} \mathcal{H}_{\mathrm{ren}}^{(0)}
$$

ture. In the strong coupling, Regge trajectory **H**_{α}*_{* β *}* δ complex angular momentum δ , this single-particle particle pa *s* **s** α is the gluon Region At least contract power in \mathcal{A} and up to the next-to-leading power in \math

HIGH ENERGY LIMIT (REGGE FACTORIZATION) Figure 1: Tree level amplitude squared and interferences of tree level with *L* = 1*,* 2*,* 3 loop amplitudes in dependence of insertion operators T*^a ⁱ* , which only act on the *i*-th external color index. In particular, in our case their action on *{Ci}* is defined as ^T*^a ⁱ ^T^bⁱ* ⁼ *ifabiciT^cⁱ* = [*T^bⁱ , T ^a*]. *x* = *t/s*. signature even/odd amplitudes at NLL/NNLL and test Regge factorisation to this accuracy is the three-loop produced in the Supplemental Material for convenience. In eq. (20) we have also introduced the standard color *ⁱ* , which only act on the *i*-th exter-Figure 1: Tree level amplitude squared and interferences of tree level with *L* = 1*,* 2*,* 3 loop amplitudes in dependence of *x* = *t/s*.

Calculation elucidates general structures in QCD instead for the exchange of color charge among (up to) four external legs. It becomes relevant for the first time corresponds to *|s|* ⇡ *|u| |t|*, or equivalently *x* ! 0. For studying this region it is convenient to split scatter-*N* = 4 SYM [14, 16], and in pure gluodynamics under nal color index. In particular, in our case their action on *ⁱ ^Tbⁱ* ⁼ *ifabiciTcⁱ* = [*Tbⁱ , T ^a*].

 V erified all-order structures in Regge kinematics $\overline{\text{C}}$ Verified all-order structures in Regge kinematics $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$) egge kinematics $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

1

Define even and odd amplitude

x = *t/s*.

^s (*n*)

four external legs. It becomes relevant for the first time

terms of the variables introduced in this letter, this limit

$$
u_{\pm} = \frac{1}{2} \left[\mathcal{H}_{\text{ren}}(s, u) \pm \mathcal{H}_{\text{ren}}(u, s) \right] \qquad L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2} \left(\ln\left(\frac{-s - i\delta}{-t}\right) + \ln\left(\frac{-u - i\delta}{-t}\right) \right)
$$

Define even and odd amplitude under
$$
(s \leftrightarrow u)
$$
 And the even "large logarithm"

$$
4nd the even "larea loo3r
$$

 α . Currently, it is one can also trajectory. Currently, it is only it is o

HIGH ENERGY LIMIT (REGGE FACTORIZATION) tree level with *L* = 1*,* 2*,* 3 loop amplitudes in dependence of **INSERTION OPERATORS TRAIT ACT ONLY A PARTIES** niun civerui liiviii (Reuue f *{Ci}* is defined as ^T*^a ⁱ ^T^bⁱ* ⁼ *ifabiciT^cⁱ* = [*T^bⁱ , T ^a*]. terms of the variables introduced in this letter, this limit *ⁱ* , which only act on the *i*-th exter-*x* = *t/s*. *x* **0017** produced in the Supplemental Material for convenience. **THIGH ENERGY LIMII (REGGE FACTURIZATION)** $\mathbf{r} = \mathbf{r}$, $\mathbf{$

Calculation elucidates general structures in QCD four external legs. It becomes relevant for the first time $\frac{1}{2}$ F_{F} study is convenient to split scatternal color integration of the color case the color case of the c
 $\frac{1}{2}$ *ⁱ ^Tbⁱ* ⁼ *ifabiciTcⁱ* = [*Tbⁱ , T ^a*].

 V erified all-order structures in Regge kinematics $\overline{\text{C}}$ *Verified all-order structures in Regge ki*

egge kinematics $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

Define even and odd amplitude under $(s \leftrightarrow u)$ And the even "large logarithm" *D*, *b*eine even and oud amplitude four external legs. It becomes relevant for the first time *^s* (*n*)

x = *t/s*.

And the even "large logarithm" $\Gamma(s \leftrightarrow u)$ and the even "large logarithm"

$$
\mathcal{H}_{\text{ren},\pm} = \frac{1}{2} \left[\mathcal{H}_{\text{ren}}(s, u) \pm \mathcal{H}_{\text{ren}}(u, s) \right] \qquad \qquad L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2} \left(\ln\left(\frac{-s - i\delta}{-t}\right) + \ln\left(\frac{-u - i\delta}{-t}\right) \right)
$$

At leading powe *the amplitude factorises in a* $\frac{1}{2}$ At leading power in x and up to next-to-leading logarithm for the oud part, the amplitude factorises in a well understood way (LO BFKL) in terms of exchanges of "Reggeons" (multiple exchanges give rise to **Regge cut contributions**) our notation are reported in the Supplemental Material. for the odd part, the amplitude factorises in a v $\frac{1}{2}$ *u* derstood way (LO BFKL) in terms o 1 ² (T² *^s* ^T² At leading power in *x* and up to *next-to-leading logarithmic accuracy (NLL)* for even part and *NNLL for the odd part*, the amplitude factorises in a well understood way (LO BFKL) in terms of $4 + 16$ *j,k6* $\emph{ct-to-leading logarithmic accuracy (NLL)}$ for even part and $NNLL$ ้
มห $ributions)$

To test Regge factorisation at this order last needed ingredient was the gluon Regge trajectory at 3 loops, can be extracted from any three-loop process. We found agreement between $gg \to gg$ and $qq \rightarrow QQ$: this allows us to **predict** $qq \rightarrow gg$ 3loop amplitude to NNLL accuracy! We verified this prediction to be correct by comparing to a successive explicit calculation *su* To test Regge factorisation at this order last needed ingredient was the gluon Regge trajectory at at be called from any three-loop process. $q\bar q \to \bar Q \bar Q$. This allows us to predict $q\bar q \to gg$ sloop and 3 loops, can be extracted from any three-loop process. We found agreement between $g\rho \rightarrow g\rho$ and $\alpha a \rightarrow 0$. this allows us to **predict** $\alpha a \rightarrow \alpha a$ interference with the tree level, $\frac{1}{\sqrt{2}}$ of a single to $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ interaction, whose interaction, whose interactions in the set of $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ are contributed in the set of $\frac{1}{\sqrt{2}}$ and $qq \to QQ$: this allows us to predict $qq \to gg$ sloop amplitude to invitil accuracy: We verified this prediction to be correct by comparing to a successive explicit calculation 3 loops, can be extracted from any three-loop process. We found agreement between $gg \to gg$ and $ga \to Q\Omega$; this allows us to **predict** $ga \to gg$ 3 loop amplitude to NNH L accuracy. der last needed ingredient was th $\frac{1}{\sqrt{1+\frac{1}{$ *su* $\frac{1}{2}$ $\overline{1}$

CONCLUSIONS

Multiloop amplitudes are **essential for pheno**, but they are also *a lot of fun!*

Recent developments have allowed us to push investigations up to **3 loops** for **complete QCD** $2 \rightarrow 2$ **amplitudes** — *and beyond for simpler building blocks* —

Exploring QCD amplitudes at high loops we learn about **physics** and **mathematics**

- 1. All-order results in high-energy / Regge kinematics (beyond basic BFKL)
- 2. Structure of IR singularities in non-abelian QFTs
- 3. New ways to organise amplitudes in dim-regularisation
- 4. New geometries in pQFT (CY and higher genus) (didn't talk about this here)
- 5. … and much more …

Exciting times ahead!

THANK YOU VERY MUCH!