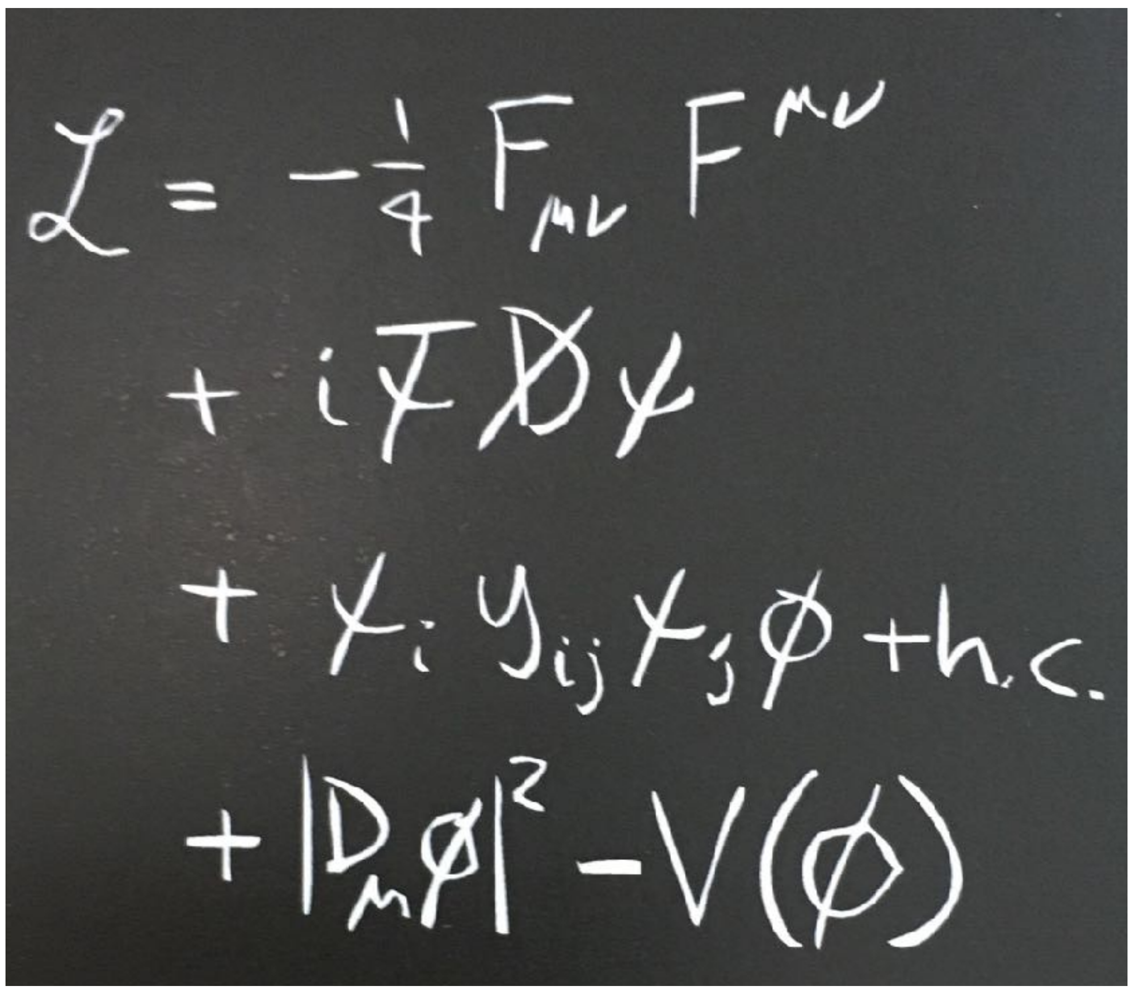
EXPLORING QCD AMPLITUDES UP TO 3 LOOPS

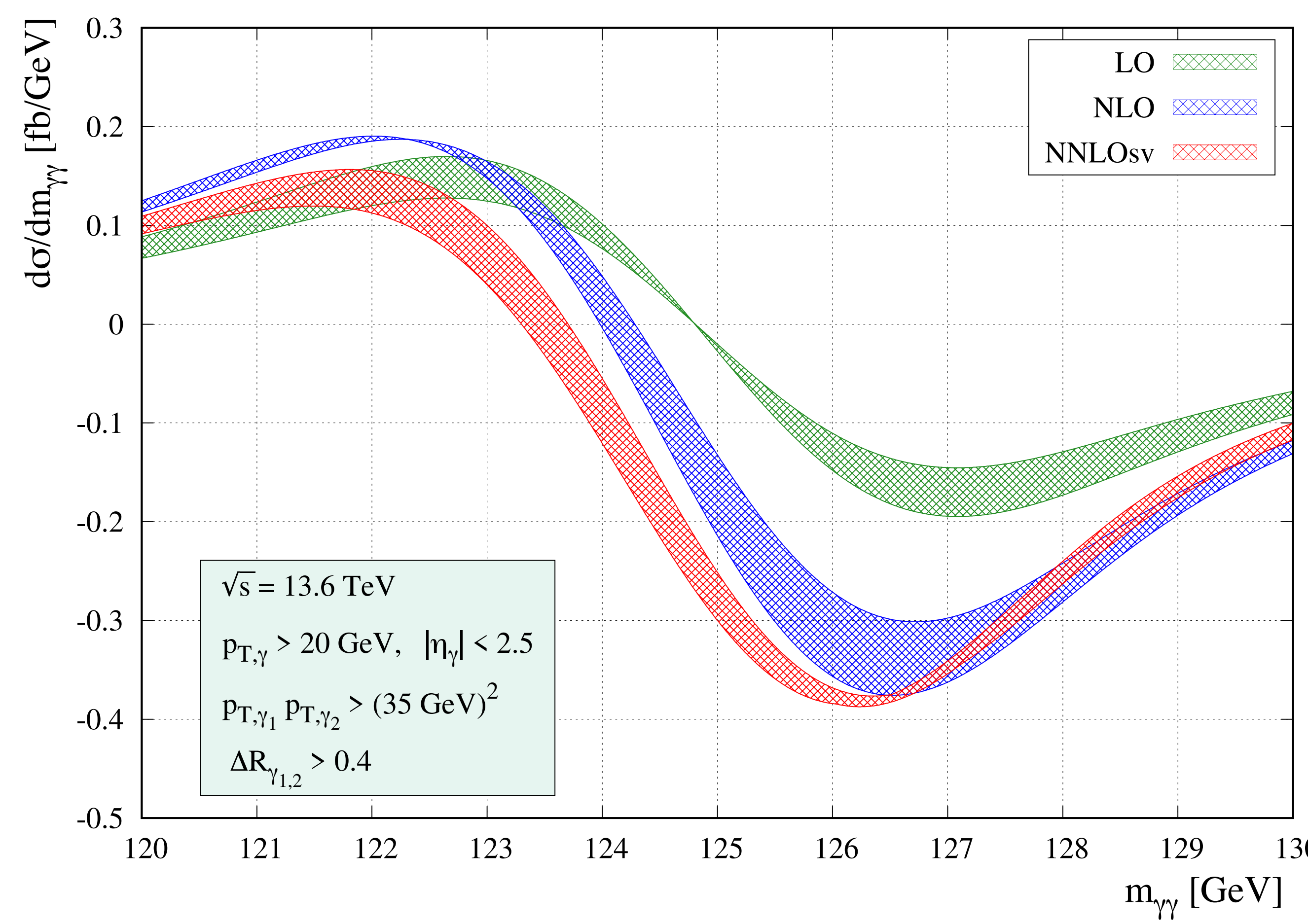
Zurich Phenomenology Workshop 2023
University of Zurich - 12/01/2023

Lorenzo Tancredi - Technical University Munich

FROM LAGRANGIANS TO CROSS-SECTIONS…

…it's a very long way



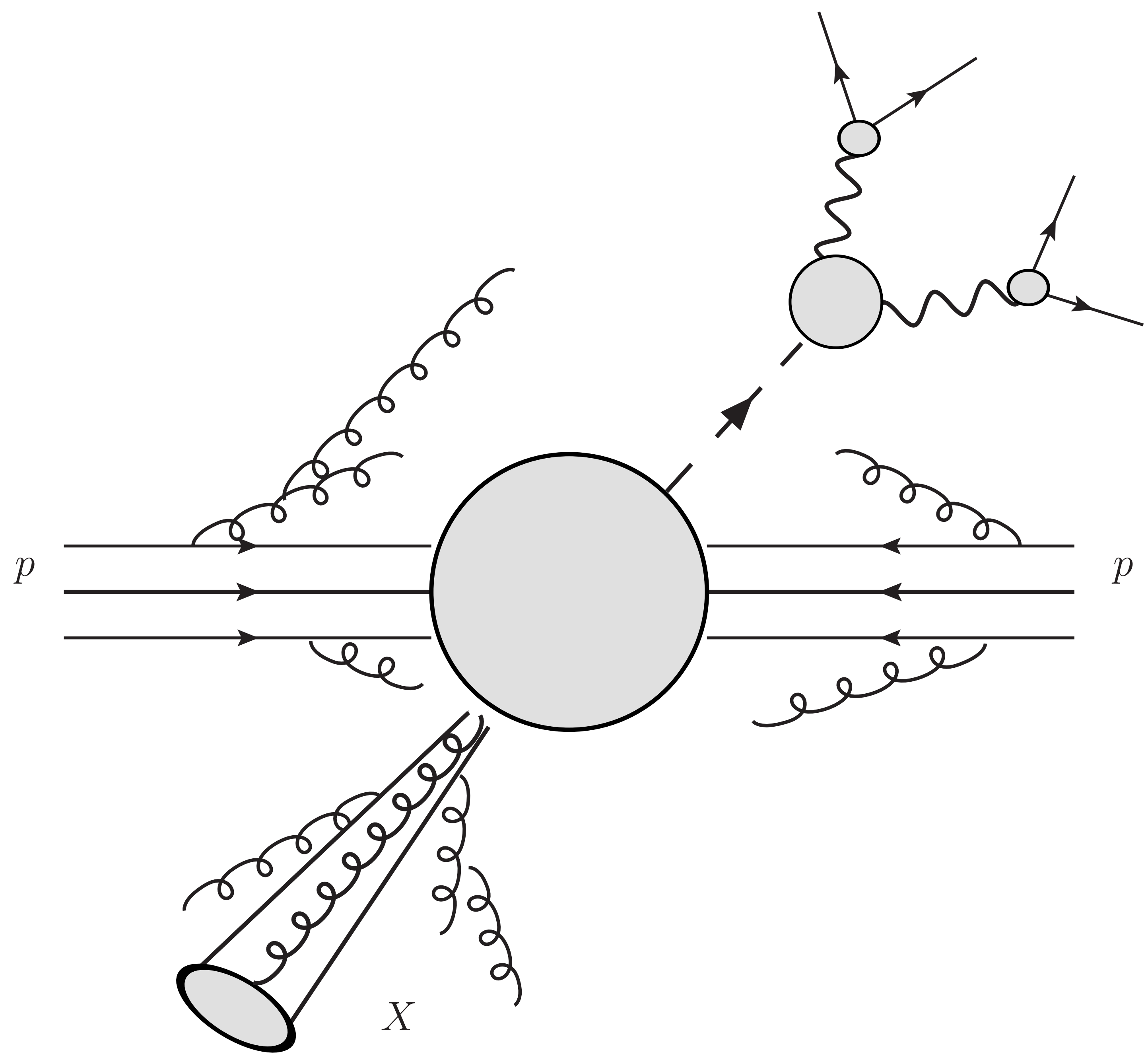


signal to bkg interference for $gg \to H \to \gamma\gamma$

[P. Bargiela et al '22]

AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

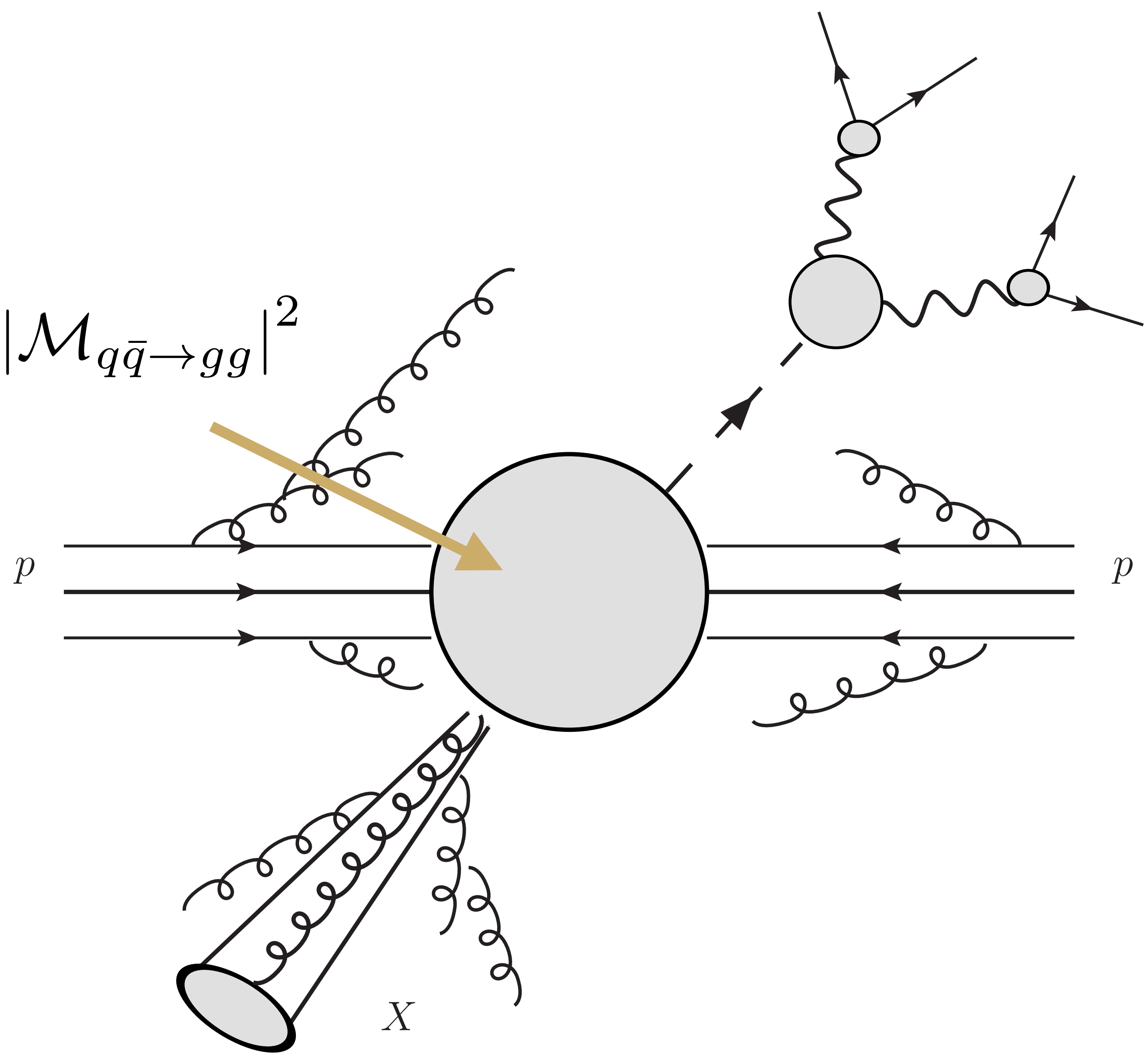
QCD makes modelling of collisions very complicated



AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

QCD makes modelling of collisions very complicated

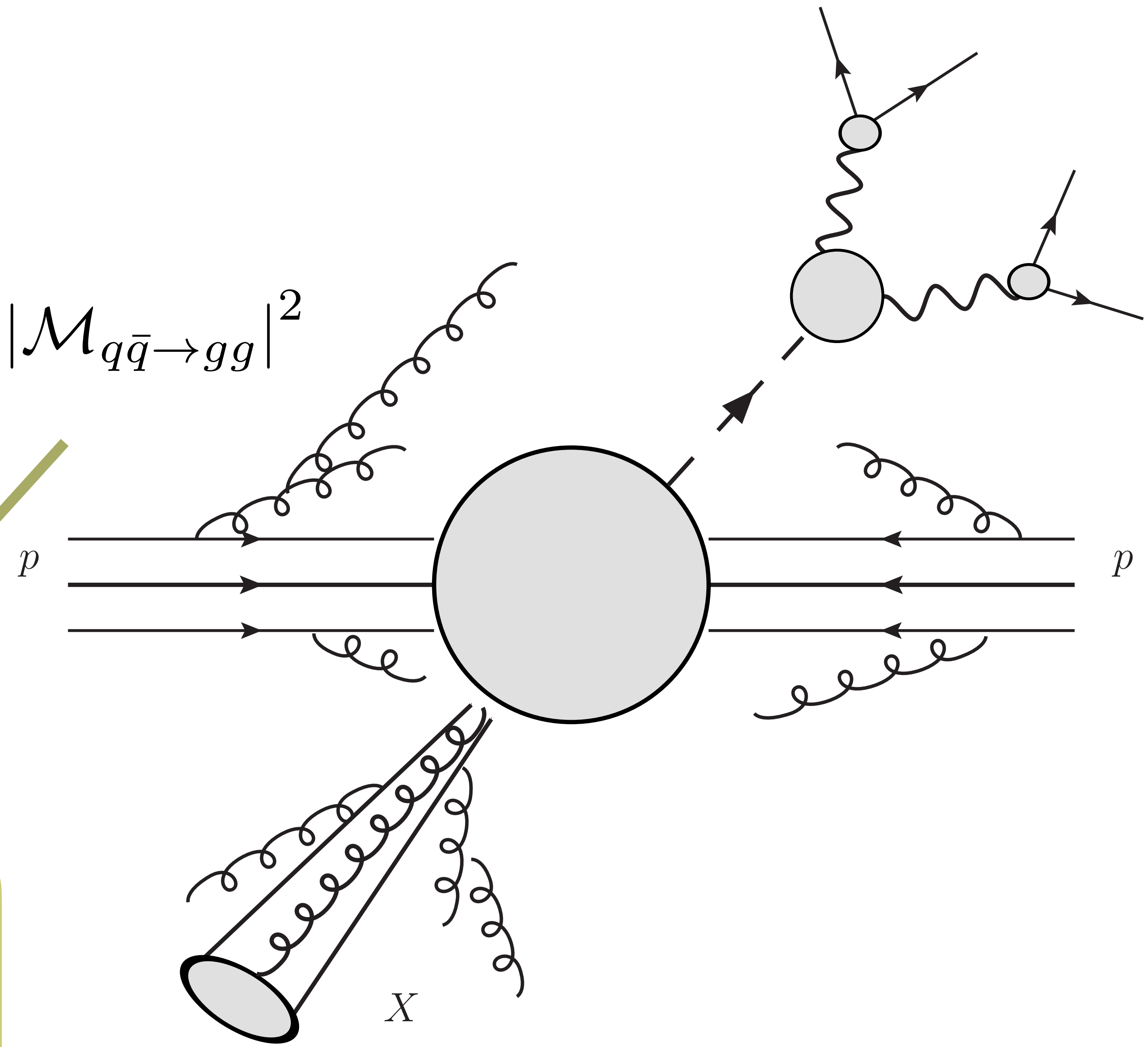
$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}] \, |\mathcal{M}_{q\bar{q}\to gg}|^2$$



AMPLITUDES FOR PHENOMENOLOGY AT THE LHC

QCD makes modelling of collisions very complicated

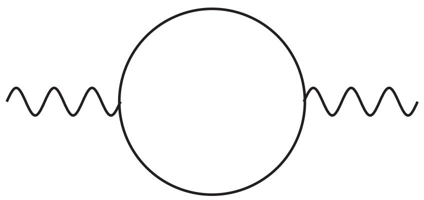
$$\sigma_{q\bar{q}\to gg} = \int [\mathrm{dPS}]\, |\mathcal{M}_{q\bar{q}\to gg}|^2$$



Building blocks are Scattering Amplitudes

WHY ANALYTIC CALCULATIONS?

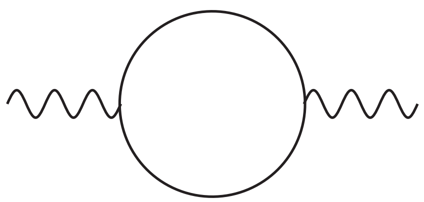
ANALYTIC ↔ NUMERICAL CONTROL!



$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

In which sense do we call this an **analytic result**?

ANALYTIC ↔ NUMERICAL CONTROL!

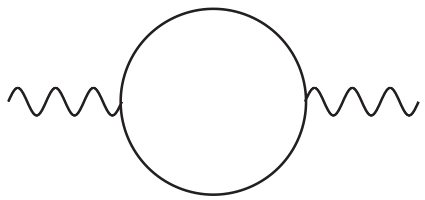


$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

In which sense do we call this an **analytic result**? →

Written in terms of known functions!

ANALYTIC ↔ NUMERICAL CONTROL!



$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

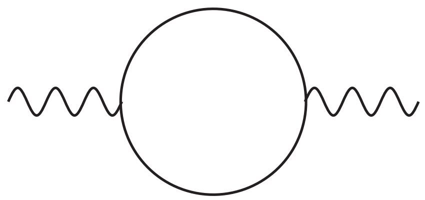
In which sense do we call this an **analytic result**? →

Written in terms of known functions!

Functional relations under control:
No hidden zeros!

$$\log 1/x + \log x = 0$$

ANALYTIC ↔ NUMERICAL CONTROL!



$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

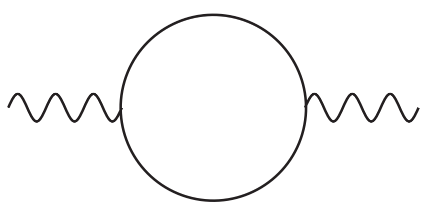
In which sense do we call this an **analytic result**? → Written in terms of known functions!

Functional relations under control:
No hidden zeros!

$$\log 1/x + \log x = 0$$

Branch cuts under control,
$$\log(x \pm i\epsilon) = \log x \pm i\pi$$

ANALYTIC ↔ NUMERICAL CONTROL!



$$\sim \frac{1}{\sqrt{s(s-4m^2)}} \ln\left(\frac{\sqrt{s-4m^2}+\sqrt{s}}{\sqrt{s-4m^2}-\sqrt{s}}\right)$$

In which sense do we call this an **analytic result**? → **Written in terms of known functions!**

Functional relations under control: No hidden zeros!

$$\log 1/x + \log x = 0$$

Branch cuts under control,

$$\log(x \pm i\epsilon) = \log x \pm i\pi$$

Argument transformation and Series expansion for <u>numerical evaluation</u>

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O\left(x^5\right)$$

ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

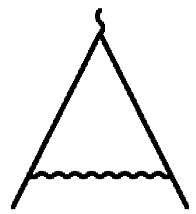
The "most famous calculation" in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the **g-2 of the electron**

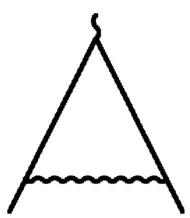
$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

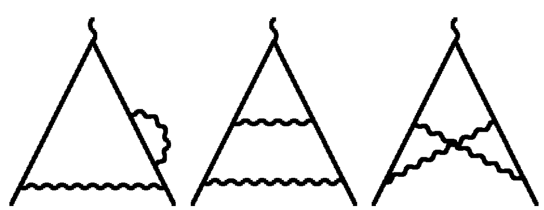
$C_1 =$  $= +0.50000000...$

ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

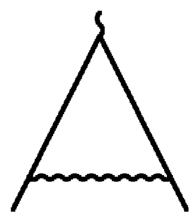
$C_1 =$  $= +0.50000000...$

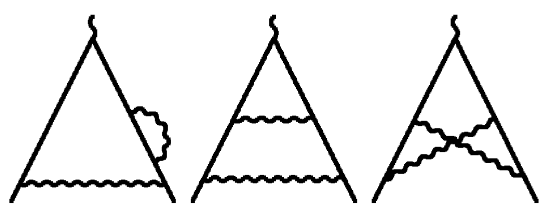
$C_2 =$  $= -0.328478965...$

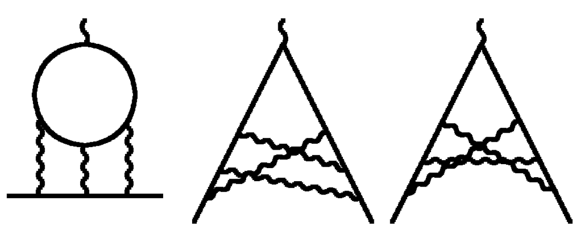
ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$C_1 =$  $= +0.50000000\ldots$

$C_2 =$  $= -0.328478965\ldots$

$C_3 =$  $= +1.181241456\ldots$

$C_4 =$ $= -1.912245764\ldots$

lots of Feynman diagrams

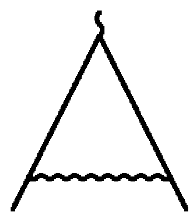
$C_5 =$ $= +6.737(159)$

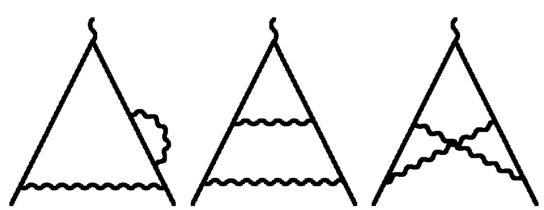
} Impressive numerical calculations by **Kinoshita et al**

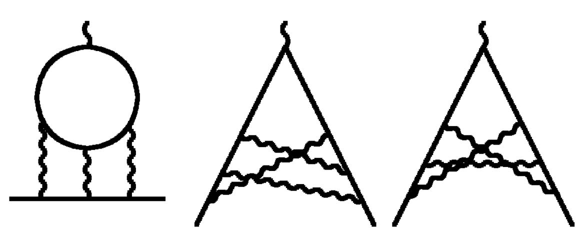
ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$C_1 =$  $= +0.50000000\dots$

$C_2 =$  $= -0.328478965\dots$

$C_3 =$  $= +1.181241456\dots$

$C_4 =$ $= -1.912245764\dots$

lots of Feynman diagrams

$C_5 =$ $= +6.737(159)$

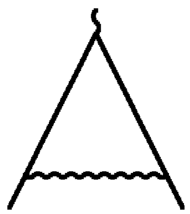
As numbers, they don't say much

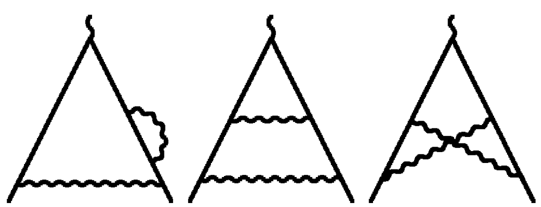
(except that the perturbative series seems to converge nicely once multiplied by 1/137 :-))

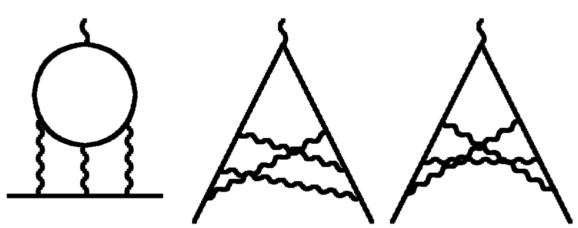
ANALYTIC "STRUCTURES": THE "DISCOVERY OF SPECIAL FUNCTIONS IN PARTICLE PHYSICS"

The "most famous calculation" in pQFT: the **g-2 of the electron**

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \dots$$

$C_1 =$  $= \frac{1}{2}$ [Schwinger '48]

$C_2 =$  $= \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$ [Petermann, Sommerfield '57]

$C_3 =$ 

$$= \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2 \ln^2 2}{24}\right]$$
$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2 \ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

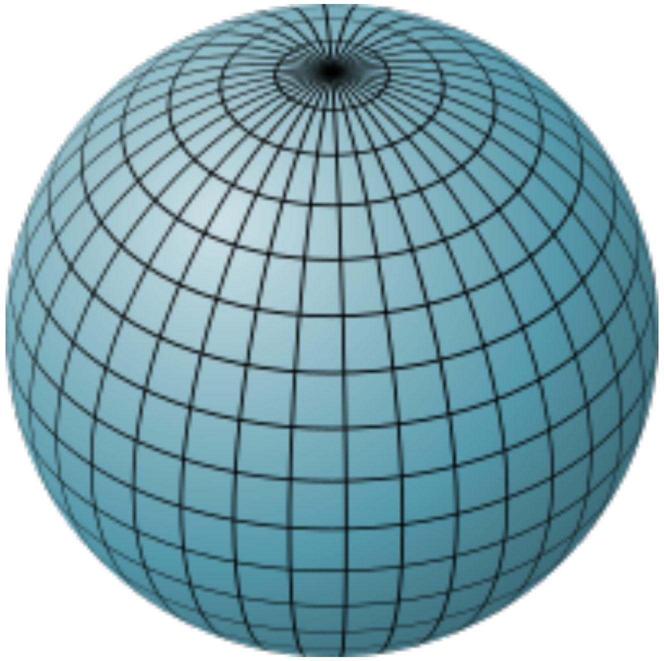
[Laporta, Remiddi '97]

But if we look at *analytic results*, some pattern starts to emerge:

rational numbers, Riemann zeta values, …, in general ***multiple polylogarithms*** *evaluated at special (rational) points*

SUCCESS OF THE PAST 20 YEARS: MULTIPLE POLYLOGARITHMS

Iterated integrals of **rational functions on the Riemann Sphere**

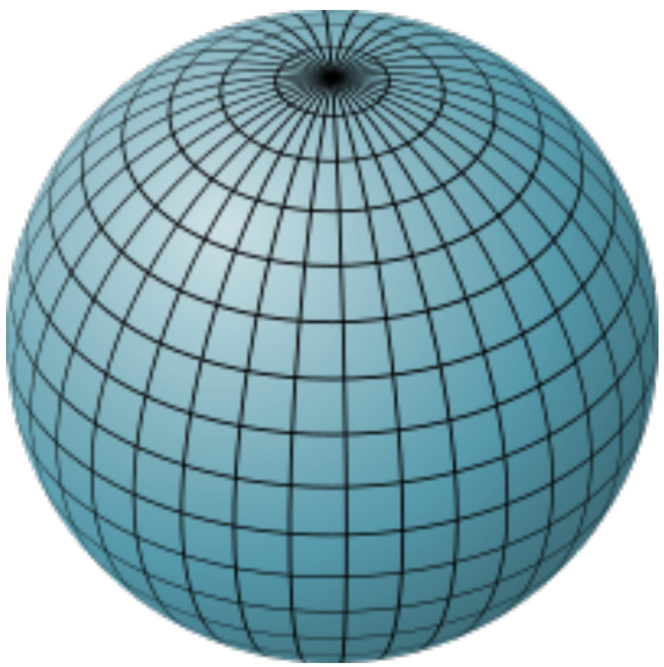


Multiple PolyLogarithms (MPLs)

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$
$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

SUCCESS OF THE PAST 20 YEARS: MULTIPLE POLYLOGARITHMS

Iterated integrals of **rational functions on the Riemann Sphere**



Multiple PolyLogarithms (MPLs)

$$G(c_1, c_2, ..., c_n, x) = \int_0^x \frac{dt_1}{t_1 - c_1} G(c_2, ..., c_n, t_1)$$

$$= \int_0^x \frac{dt_1}{t_1 - c_1} \int_0^{t_1} \frac{dt_2}{t_2 - c_2} ... \int_0^{t_{n-1}} \frac{dt_n}{t_n - c_n}$$

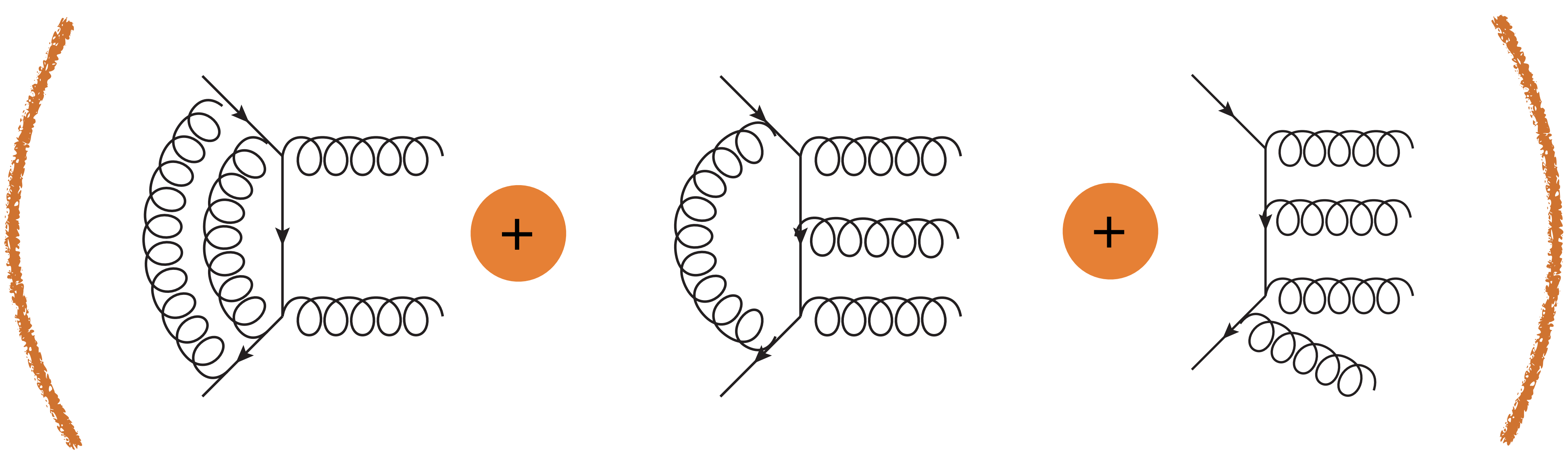
Provided us with the right language to make sense of a lot of the structure in scattering amplitudes

- leading singularities and dlog forms (local integrals) **[Arkani-Hamed et al '10]**
- differential equations in canonical form **[Henn '13]**
- hint towards generalisations (elliptic multiple polylogs, more general diff forms, Calabi-Yau geometries etc)

TOWARDS A NNLO REVOLUTION (?)

$$\left|\mathcal{M}_{q\bar{q}\to gg}\right|^2 = \left|\mathcal{M}^{LO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)\left|\mathcal{M}^{NLO}_{q\bar{q}\to gg}\right|^2 + \left(\frac{\alpha_s}{2\pi}\right)^2 \left|\mathcal{M}^{NNLO}_{q\bar{q}\to gg}\right|^2 + \ldots$$

By understanding analytic structure of amplitudes + how to handle and subtract IR divergences, past 2 decades have seen the beginning of a NNLO revolution…



Double Virtual | Real Virtual | Double Real

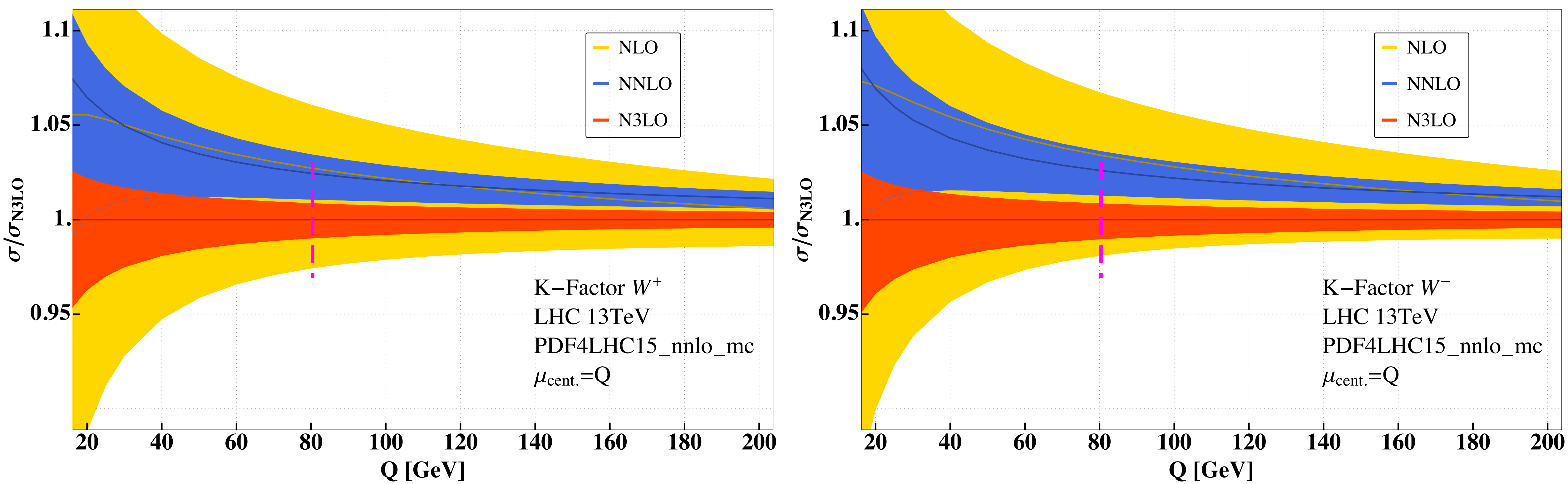
- complex integrals
- involved **IR structure**

BEYOND NNLO FOR 2→2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface...!

BEYOND NNLO FOR 2 → 2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!



[Duhr, Dulat, Mistlberger '20]

Non trivial uncertainty patterns observed going from NNLO to N3LO for W,γ Drell-Yan

We are far from being able to do N^3LO pheno for generic processes…

BEYOND NNLO FOR 2→2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!

We are far from being able to do N^{3}LO pheno for generic processes…

IR singularities and new sources for possible **factorisation breaking** (di-jet / $t\bar{t}$ @ N^{3}LO…)

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities (bootstrap?)…

BEYOND NNLO FOR 2 → 2 THERE IS STILL A LOT TO LEARN

We are just scratching the surface…!

We are far from being able to do N^{3}LO pheno for generic processes…

New challenges from pushing methods to compute **scattering amplitudes** from two to **three loops**:

Higher combinatorial complexity, *new special functions and new geometries*, discontinuities (bootstrap?)…

IR singularities and new sources for possible **factorisation breaking** (di-jet / $t\bar{t}$ @ N^{3}LO…)

Particularly interesting

di-jet production @ N3LO!

TOWARDS DI-JET AT N3LO

First step is **3 loop scattering amplitudes**:

- Informs on **complexity of functions involved**
- Informs on **IR structure** in three-loop QCD: **quadrupole correlations!**
- Informs on **all-order structure of QCD**: High Energy limit, **Regge factorisation etc**

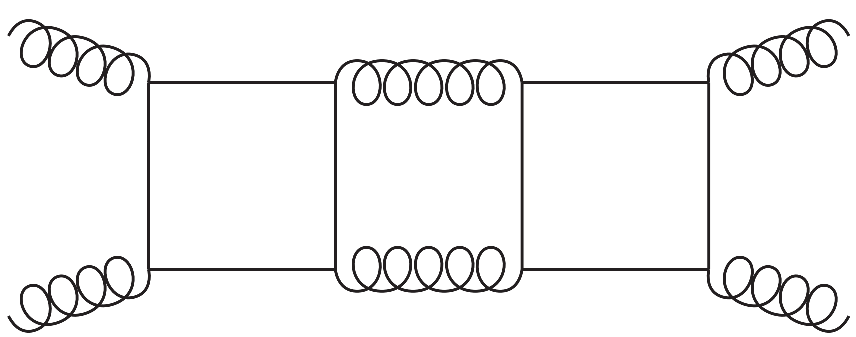
TOWARDS DI-JET AT N3LO

First step is **3 loop scattering amplitudes**:

- Informs on **complexity of functions involved**
- Informs on **IR structure** in three-loop QCD: **quadrupole correlations!**
- Informs on **all-order structure of QCD**: High Energy limit, **Regge factorisation etc**

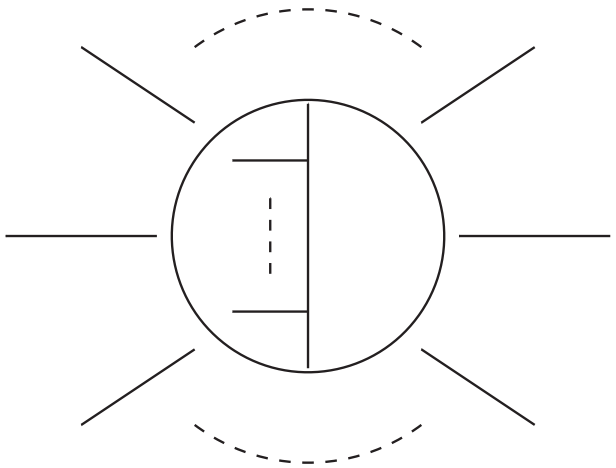
3 main channels: $gg \to gg\,,\; q\bar{q} \to gg\,,\; q\bar{q} \to Q\bar{Q}$

We will focus mainly on the most complicated one: $gg \to gg$



FROM AMPLITUDES TO INTEGRALS

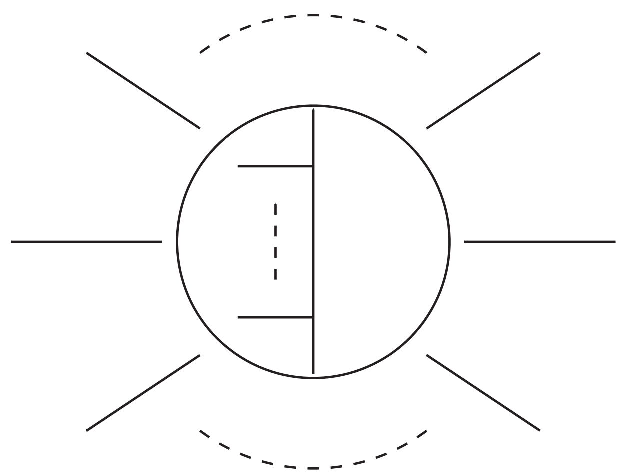
Scattering Amplitudes: flashing through standard approach [See Harald Ita's talk]



$$\sim \qquad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1,\ldots,\mu_n} \, u(p)$$

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes: flashing through standard approach [See Harald Ita's talk]



$$\sim \qquad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1,\ldots,\mu_n} \, u(p)$$

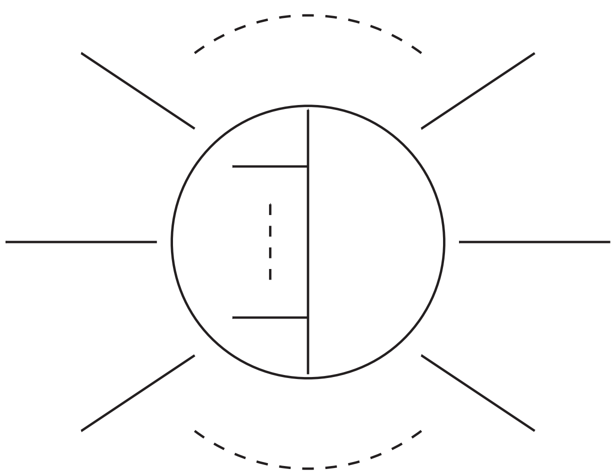
(Scalar) Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes: flashing through standard approach [See Harald Ita's talk]



$$\sim \quad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1,\ldots,\mu_n} \, u(p)$$

(Scalar) Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

IBPs (Finite fields etc)

differential equations

Feynman parameters

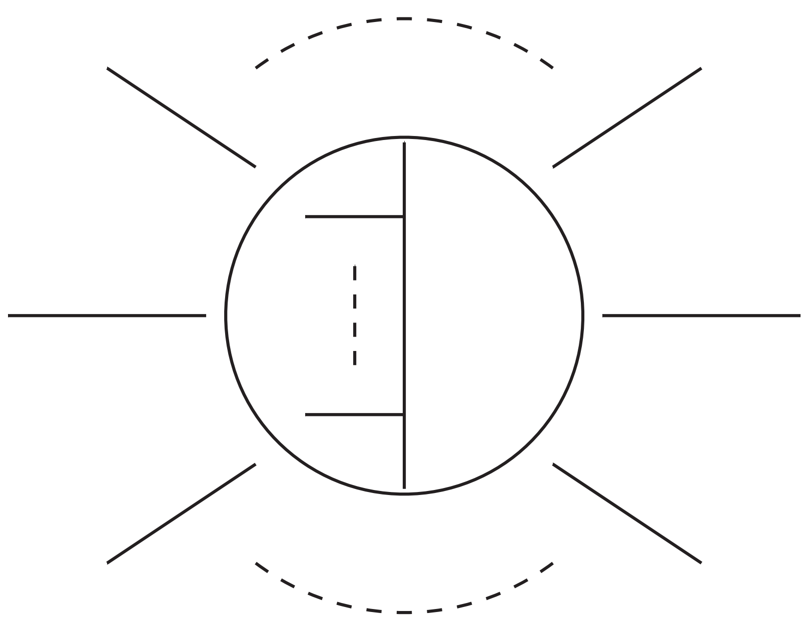
Numerical methods …

Some analytic or numerical result for the amplitudes

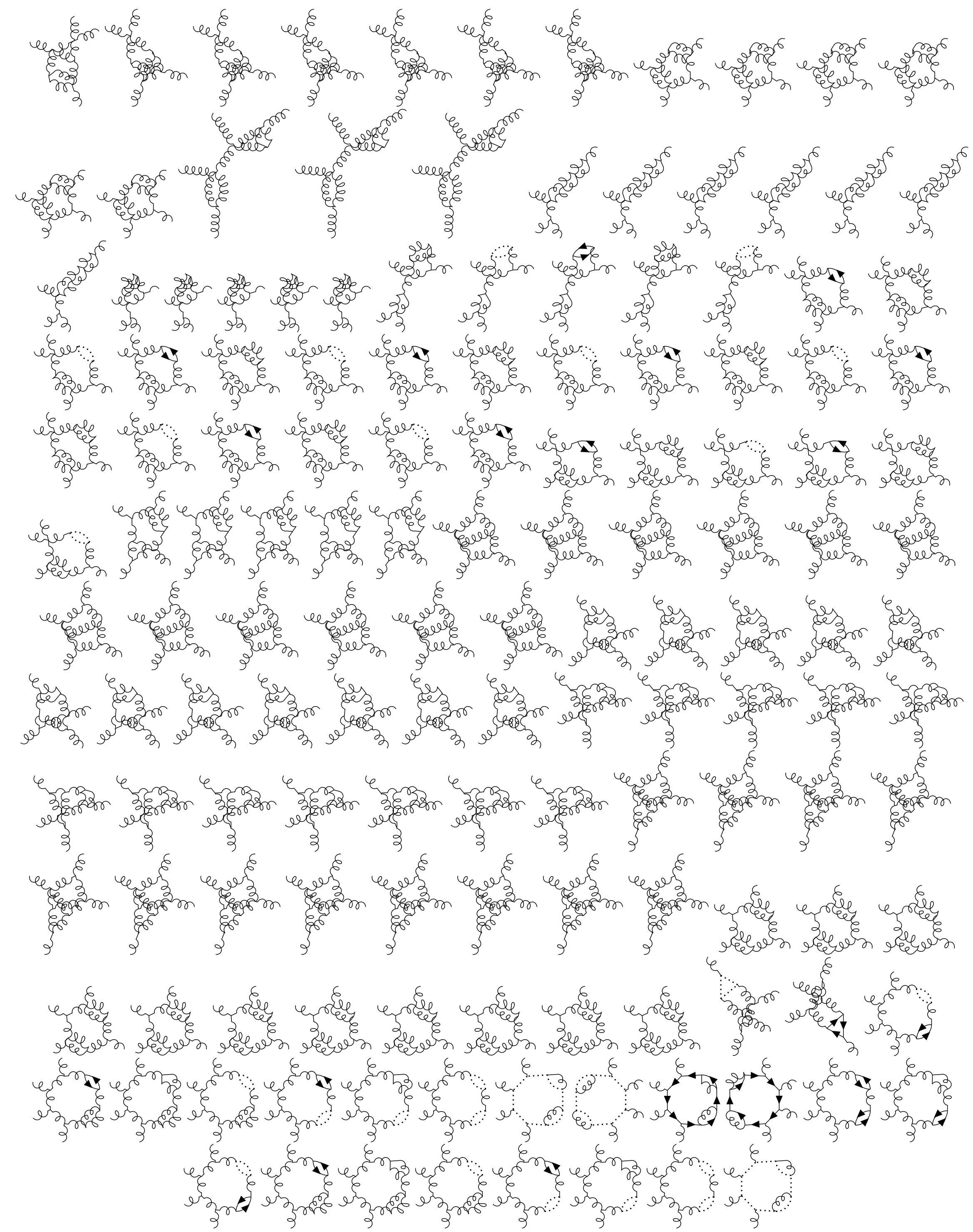
[See Gudrun's and Mao's talks]

FROM AMPLITUDES TO INTEGRALS

In reality, for $gg \to gg$ @ 3 loops



~



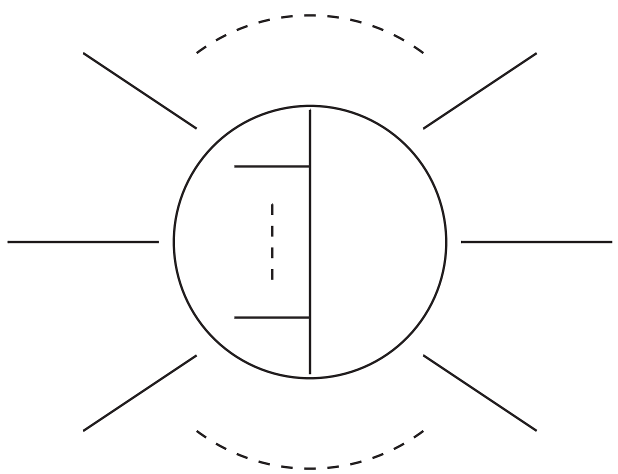
+500 more pages

(~50000 Feynman diagrams — 10^7 **integrals**!!)

So we need a way to organise this mess…

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes: flashing through standard approach [See Harald Ita's talk]



$$\sim \qquad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1,\ldots,\mu_n} \, u(p)$$

(Scalar) Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

→

IBPs (Finite fields etc)

differential equations

Feynman parameters

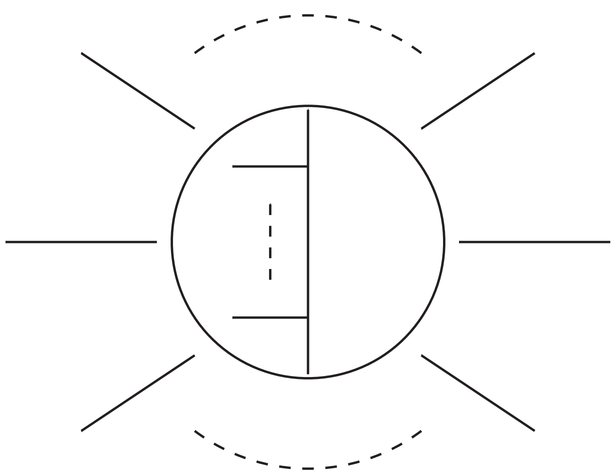
Numerical methods …

Some analytic or numerical result for the amplitudes

[Harald's talk (not only)]

FROM AMPLITUDES TO INTEGRALS

Scattering Amplitudes: flashing through standard approach [See Harald Ita's talk]



$$\sim \quad \mathscr{A} = \epsilon_1^{\mu_1} \cdots \epsilon_n^{\mu_n} \, \bar{v}(q) \, \Gamma_{\mu_1, \ldots, \mu_n} \, u(p)$$

[This talk starts here…]

Tensor reduction: projectors - form factors

(Scalar) Feynman Integrals

$$\mathscr{I} = \int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \frac{S_1^{b_1} \ldots S_m^{b_m}}{D_1^{a_1} \ldots D_n^{a_n}}$$

with $S_i \in \{k_i \cdot k_j, \ldots, k_i \cdot p_j\}$

IBPs (Finite fields etc)

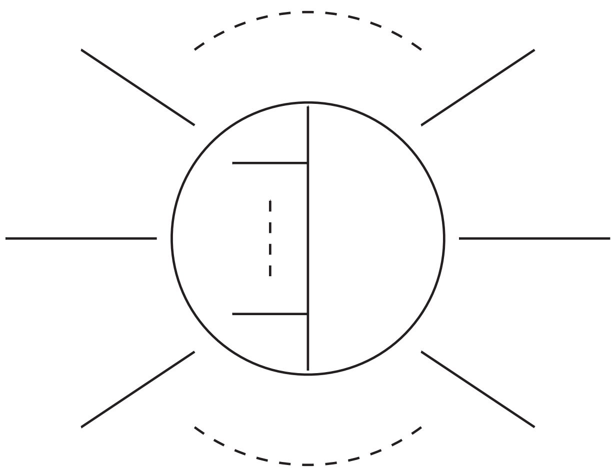
differential equations

Feynman parameters

Numerical methods …

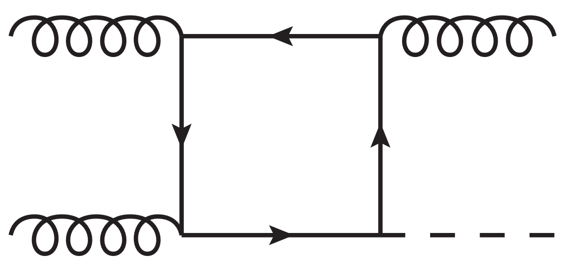
Some analytic or numerical result for the amplitudes

TENSOR DECOMPOSITION



$$= \int \prod_{i=1}^{L} d^D k_i \; R_i(k_1, \ldots, k_L, p_1, \ldots, p_E, m_j)$$

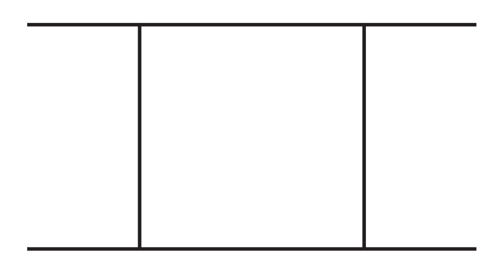
$\mathcal{M}_{gg \to Hg} \sim$



First step:

Strip it of Lorentz and Dirac structures

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p_2)^2 (k-p_2-p_3)^2 (k-p_1-p_2-p_3)^2} =$$



Scalar Feynman Integrals are what we know how to compute

PROJECTOR - FORM FACTOR METHOD IN A NUTSHELL

- Pick your favourite process *(for example* $q\bar{q} \to Zg$ *or* $q\bar{q} \to Q\bar{Q}$*)*

PROJECTOR - FORM FACTOR METHOD IN A NUTSHELL

- Pick your favourite process *(for example $q\bar{q} \to Zg$ or $q\bar{q} \to Q\bar{Q}$)*

- Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$\mathscr{A} = \sum_{j=1}^{n} F_j T_j$$

PROJECTOR - FORM FACTOR METHOD IN A NUTSHELL

- Pick your favourite process *(for example $q\bar{q} \to Zg$ or $q\bar{q} \to Q\bar{Q}$)*

- Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$\mathscr{A} = \sum_{j=1}^{n} F_j T_j$$

- Derive **projectors operators** *(dual vectors)* to single out corresponding form factors:

$$M_{ij} = \sum_{pol} T_i^\dagger T_j \qquad\qquad \mathscr{P}_j = \sum_k \left(M^{-1}\right)_{jk} T_k^\dagger$$

$$\boxed{\mathscr{P}_j \mathscr{A} = F_j}$$

PROJECTOR - FORM FACTOR METHOD IN A NUTSHELL

- Pick your favourite process (*for example* $q\bar{q} \to Zg$ *or* $q\bar{q} \to Q\bar{Q}$)

- Use **Lorentz + gauge + any symmetry** (parity, Bose etc…) to find minimal set of "tensor structures" in d space-time dimensions *(vectors in a vector space)*:

$$\mathscr{A} = \sum_{j=1}^{n} F_j T_j$$

- Derive **projectors operators** *(dual vectors)* to single out corresponding form factors:

$$M_{ij} = \sum_{pol} T_i^\dagger T_j \qquad \mathscr{P}_j = \sum_k \left(M^{-1}\right)_{jk} T_k^\dagger$$

$$\boxed{\mathscr{P}_j \mathscr{A} = F_j}$$

- Apply these projectors on **Feynman diagrams** (or any other representation of the scattering amplitude) —> obtain combination of scalar integrals

FROM "TENSORS" TO HELICITY AMPLITUDES

Ultimately, we are interested in **helicity amplitudes**

(minimal, physical objects which retain full physical information on final states)

FROM "TENSORS" TO HELICITY AMPLITUDES

Ultimately, we are interested in **helicity amplitudes**

(minimal, physical objects which retain full physical information on final states)

Fix helicities (assuming that **external states** are in $d = 4$ **dimensions**)

$$\mathscr{A} = \sum_{i=1}^{n} F_j T_j \quad \longrightarrow \quad \mathscr{A}(\lambda_1, \ldots, \lambda_E) = \sum_{i=1}^{n} F_j T_j(\lambda_1, \ldots, \lambda_E) = \sum_{j=1}^{m<n} \bar{F}_j S_j(\lambda_1, \ldots, \lambda_E)$$

All "internal" indices are in d dimensions

Combinations of original form factors

Helicity amplitudes, spinor products, momentum twistors…

This allows us to have the full structure of the amplitude under control

True at every number of loops!

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

Typical case 4 quark scattering $q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$

$$D_i \sim \bar{u}(p_1)\Gamma^{\mu_1,\ldots,\mu_n} u(p_2)\ \bar{u}(p_3)\Gamma_{\mu_1,\ldots,\mu_n} u(p_4)$$

Infinite number of tensor structures in d dimensions

TENSOR DECOMPOSITION: PROS AND CONS

Problems in d-dimensions: it is a powerful and very general method but:

When applied in standard dimensional regularisation **(CDR)**, it can become intractable for complicated problems due to **evanescent structures in d=4**

Typical case 4 quark scattering $q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$

$$D_i \sim \bar{u}(p_1)\Gamma^{\mu_1,\ldots,\mu_n} u(p_2)\, \bar{u}(p_3)\Gamma_{\mu_1,\ldots,\mu_n} u(p_4)$$

Infinite number of tensor structures in *d* dimensions

$$\begin{aligned}
\mathcal{D}_1 &= \bar{u}(p_1)\gamma_{\mu_1}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}u(p_4),\\
\mathcal{D}_2 &= \bar{u}(p_1)\not{p}_3u(p_2)\ \bar{u}(p_3)\not{p}_1u(p_4),\\
\mathcal{D}_3 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_4),\\
\mathcal{D}_4 &= \bar{u}(p_1)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\not{p}_1\gamma_{\mu_3}u(p_4),\\
\mathcal{D}_5 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_4),\\
\mathcal{D}_6 &= \bar{u}(p_1)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_1\gamma_{\mu_4}\gamma_{\mu_5}u(p_4).
\end{aligned}$$

→ up to 2 loops!

TENSOR DECOMPOSITION: UPGRADE IN THV

Improvements in **'t Hooft - Veltman (tHV)** scheme [Peraro, Tancredi '19,'20]

2 independent helicity configurations: $q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$

—> only **two "tensors" are linearly independent** if external states are in $d = 4$

$$\mathcal{D}_1 = \bar{u}(p_1)\gamma_{\mu_1}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}u(p_4),$$
$$\mathcal{D}_2 = \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4),$$

TENSOR DECOMPOSITION: UPGRADE IN THV

Improvements in **'t Hooft - Veltman (tHV)** scheme [Peraro, Tancredi '19,'20]

2 independent helicity configurations: $q(p_2) + \bar{q}(p_1) \to Q(p_3) + \bar{Q}(p_4)$

—> only **two "tensors" are linearly independent** if external states are in $d = 4$

$$\mathcal{D}_1 = \bar{u}(p_1)\gamma_{\mu_1}u(p_2)\ \bar{u}(p_3)\gamma_{\mu_1}u(p_4),$$
$$\mathcal{D}_2 = \bar{u}(p_1)\not{p}_3 u(p_2)\ \bar{u}(p_3)\not{p}_1 u(p_4),$$

4 dimensional tensors alone are enough to obtain **full result in 't Hooft-Veltman scheme**

and also **the finite remainder in CDR!**

Used successfully for $pp \to pp$ @ 3 loops [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21,'22]

And first **full colour** calculation for a $2 \to 3$ amplitude: $q\bar{q} \to \gamma\gamma j$ at 2 loops in QCD

[Agarwal, Buccioni, Manteuffel, Tancredi '21]

TENSOR DECOMPOSITION: CHIRAL THEORIES

Let's see how this works for chiral theories

Consider the decay of a Z-boson and to three jet

$$Z(p_4) \to q(p_1) + \bar{q}(p_2) + g(p_3)$$

TENSOR DECOMPOSITION: CHIRAL THEORIES

Let's see how this works for chiral theories

Consider the decay of a Z-boson and to three jet

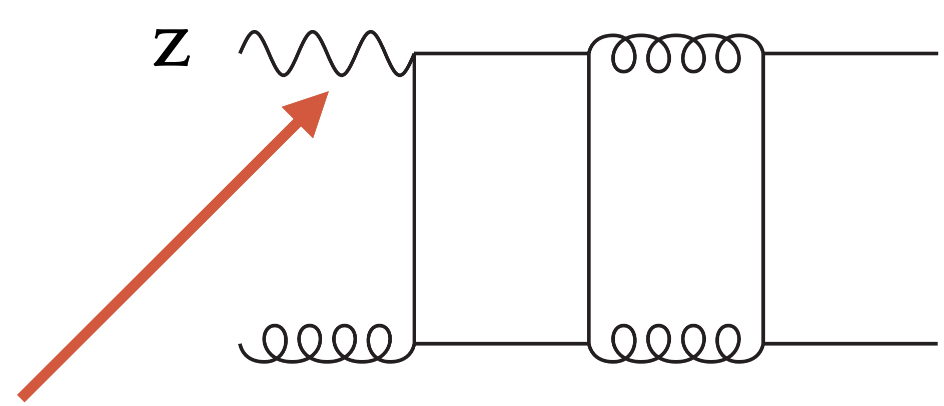
$$Z(p_4) \to q(p_1) + \bar{q}(p_2) + g(p_3)$$

Status:

Pheno @ NNLO including **only vector-like** couplings of singlet type

Amplitudes [Garland, Gerhmann et al '02]

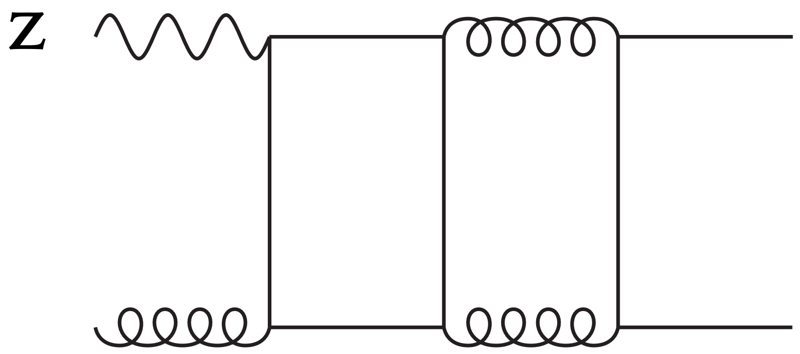
Pheno [Gehrmann-De Ridder et al '17, '18] etc etc



$\gamma^\mu \gamma^5$ — axial coupling neglected in *singlet* contributions —

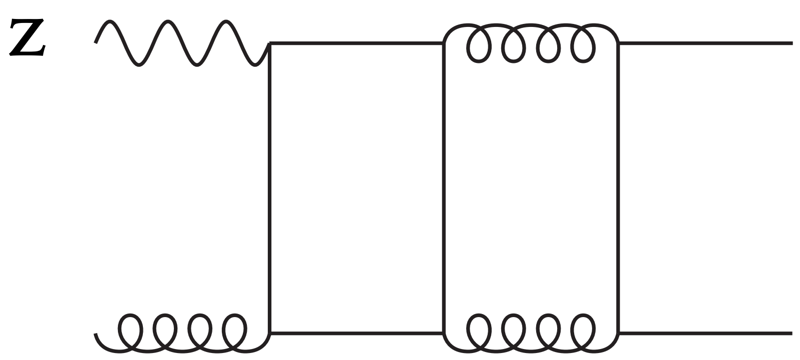
Need to include top+bottom to get consistent result (anomaly!)

TENSOR DECOMPOSITION: CHIRAL THEORIES



One issue for **axial couplings** is
evanescent structures in chiral tensor

TENSOR DECOMPOSITION: CHIRAL THEORIES



One issue for **axial couplings** is **evanescent structures in chiral tensor**

In our approach, only **tensors in $d = 4$ are relevant**, we can **span amplitude** with a basis of vectors in $d = 4$: $p_1^\mu, p_2^\mu, p_3^\mu$, plus the fourth **parity-odd one**

$$\epsilon_{\nu\rho\sigma\mu} p_1^\nu p_2^\rho p_3^\sigma = \epsilon^{p_1 p_2 p_3 \mu} = v_A^\mu$$

With these, a possible basis can be written as: (could be **further optimised** for singlet contributions)

$$\begin{aligned} A^{\mu\nu} = & \bar{u}(p_2)\not{p}_3 u(p_1)\left[F_1 p_1^\mu p_1^\nu + F_2 p_2^\mu p_1^\nu + F_3 g^{\mu\nu} + G_1 p_1^\mu v_A^\nu + G_2 p_2^\mu v_A^\nu + G_3 v_A^\mu p_1^\nu\right] \\ & + \bar{u}(p_2)\gamma^\nu u(p_1)\left[F_4 p_1^\mu + F_5 p_2^\mu\right] + \bar{u}(p_2)\gamma^\mu u(p_1) F_6 p_1^\nu \\ & + \bar{u}(p_2)\not{v}_A u(p_1)\left[G_4 p_1^\mu p_1^\nu + G_5 p_2^\mu p_1^\nu\right] + G_6\left[\bar{u}(p_2)\gamma^\mu u(p_1) v_A^\nu + \bar{u}(p_2)\gamma^\nu u(p_1) v_A^\mu\right] \end{aligned}$$

[Gehrmann, Peraro, Tancredi ’22]

TENSOR DECOMPOSITION: CHIRAL THEORIES

$$A^{\mu\nu} = \bar{u}(p_2)\not{p}_3 u(p_1)\Big[F_1 p_1^\mu p_1^\nu + F_2 p_2^\mu p_1^\nu + F_3 g^{\mu\nu} + G_1 p_1^\mu v_A^\nu + G_2 p_2^\mu v_A^\nu + G_3 v_A^\mu p_1^\nu\Big]$$
$$+ \bar{u}(p_2)\gamma^\nu u(p_1)\Big[F_4 p_1^\mu + F_5 p_2^\mu\Big] + \bar{u}(p_2)\gamma^\mu u(p_1) F_6 p_1^\nu$$
$$+ \bar{u}(p_2)\not{v}_A u(p_1)\Big[G_4 p_1^\mu p_1^\nu + G_5 p_2^\mu p_1^\nu\Big] + G_6\Big[\bar{u}(p_2)\gamma^\mu u(p_1) v_A^\nu + \bar{u}(p_2)\gamma^\nu u(p_1) v_A^\mu\Big]$$

The counting is straightforward:

- **2 helicities** for the $q\bar{q}$ line (massless)
- **2 helicities** for the (physical) gluon
- **3 helicities** for the (physical) Z boson

} Gives a total of = **12 helicity amplitudes**

matched by the number of tensors and form factors

Note that manipulations are done in **tHV / Larin scheme**

$$p_i \cdot v_A = 0\,, \qquad v_A \cdot v_A = \epsilon^{p_1 p_2 p_3 \mu}\epsilon^{p_1 p_2 p_3 \mu} = \frac{d-3}{4} s_{12} s_{13} s_{23}$$

[Gehrmann, Peraro, Tancredi '22]

THE CASE OF 4-GLUON SCATTERING

Applied all these ideas to $gg \to gg$ [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi ’21]

8 helicity amplitudes ~ 8 form factors for each colour ordered amplitude

$$\mathcal{A}^{a_1 a_2 a_3 a_4} = 4\pi\alpha_{s,b} \sum_{i=1}^{6} \mathcal{A}^{[i]} \mathcal{C}_i \quad \longrightarrow \quad \mathscr{A} = \sum_{j=1}^{8} \mathscr{F}_i T_i$$

$\mathcal{C}_1 = \mathrm{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \mathrm{Tr}[T^{a_1} T^{a_4} T^{a_3} T^{a_2}]$ etc…

THE CASE OF 4-GLUON SCATTERING

Applied all these ideas to $gg \to gg$ [Caola, Chakraborty, Gambuti, Manteuffel, Tancredi '21]

8 helicity amplitudes $\sim$ 8 form factors for each colour ordered amplitude

$$\mathcal{A}^{a_1 a_2 a_3 a_4} = 4\pi\alpha_{s,b} \sum_{i=1}^{6} \mathcal{A}^{[i]} \mathcal{C}_i \quad \longrightarrow \quad \mathscr{A} = \sum_{j=1}^{8} \mathscr{F}_i T_i$$

$$\mathcal{C}_1 = \mathrm{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}] + \mathrm{Tr}[T^{a_1} T^{a_4} T^{a_3} T^{a_2}] \quad \text{etc...}$$

Helicity amplitudes $\mathscr{A}_\lambda = s_\lambda H_\lambda$ where $\lambda = \{++++,-+++,+-++,etc\}$

$$s_{++++} = \frac{\langle 12\rangle\langle 34\rangle}{[12][34]}, \quad s_{-+++} = \frac{\langle 12\rangle\langle 14\rangle[24]}{\langle 34\rangle\langle 23\rangle\langle 24\rangle}$$

$$s_{+-++} = \frac{\langle 21\rangle\langle 24\rangle[14]}{\langle 34\rangle\langle 13\rangle\langle 14\rangle}, \quad s_{++-+} = \frac{\langle 32\rangle\langle 34\rangle[24]}{\langle 14\rangle\langle 21\rangle\langle 24\rangle}$$

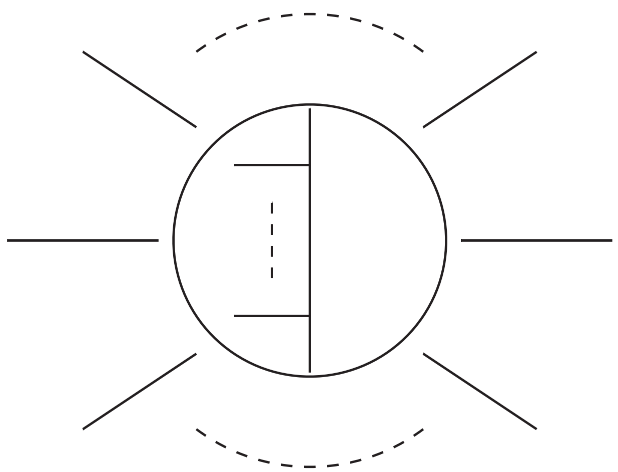
$$s_{+++-} = \frac{\langle 42\rangle\langle 14\rangle[12]}{\langle 13\rangle\langle 23\rangle\langle 12\rangle}, \quad s_{++--} = \frac{\langle 12\rangle[34]}{[12]\langle 34\rangle},$$

$$s_{+-+-} = \frac{\langle 13\rangle[24]}{[13]\langle 24\rangle}, \quad s_{+--+} = \frac{\langle 14\rangle[23]}{[14]\langle 23\rangle}.$$

For example:

$$\longrightarrow \quad H_{-+-+} = t^2 \left(\frac{\mathscr{F}_8}{su} - \frac{\mathscr{F}_3}{2s} + \frac{\mathscr{F}_6}{2u} - \frac{\mathscr{F}_1}{4} \right)$$

REDUCTION TO MASTER INTEGRALS



$$= \int \prod_{i=1}^{L} d^D k_i \; R_i(k_1, \ldots, k_L, p_1, \ldots, p_E, m_j)$$

$$= \sum_{i=1}^{N} R_i(x_1, \ldots, x_r) \, \mathcal{I}_i(x_1, \ldots, x_n)$$

Path to get there extremely complicated, became possible thanks to new mathematical tools

[See Harald's talk]

Numerical methods (Finite Fields), avoid complexity in intermediate steps, reconstruct final result

[Manteuffel, Schabinger '14]
[Peraro '16, '19]
[Klappert, Lange '19]

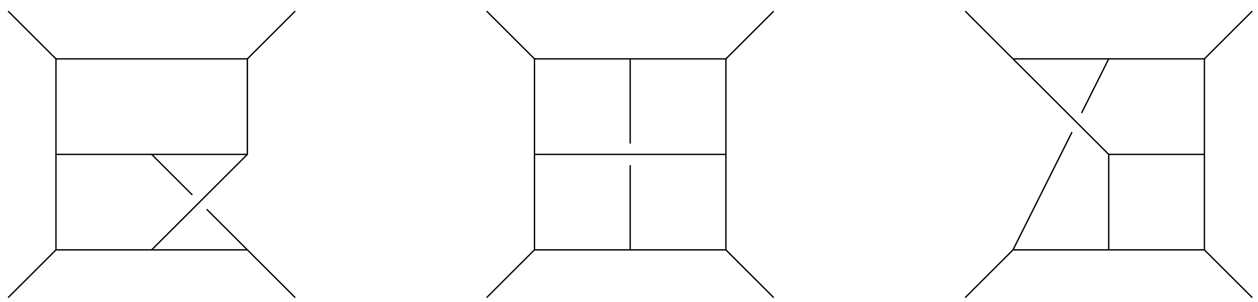
alternative representation for rational functions: multivariate partial-fractioning

[Remiddi,…, '99…] [Abreu et al '18] [Boehm, et al'20]
[Heller, Manteuffel '21]

THE CASE OF 4-GLUON SCATTERING

Three-loop calculation is very non-trivial, it took "20 more years"!

- **Many master integrals** ($\sim$ but only 500 vs 10^7 integrals before reduction!)



Approached by **differential equations method** [Kotikov '97; Remiddi '99; Gehrmann Remiddi '00]

$$d\vec{I} = \epsilon A(x)\vec{I}$$

[Arkani-Hamed '10; Kotikov '07 '10; Henn '13, Lee '15]

- Finding a so-called **"canonical basis"** is **very non-trivial** [Henn, Mistlberger, Smirnov, Wasser, 2020]

Result can be written in terms of **simple functions**: (harmonic) **multiple polylogarithms**

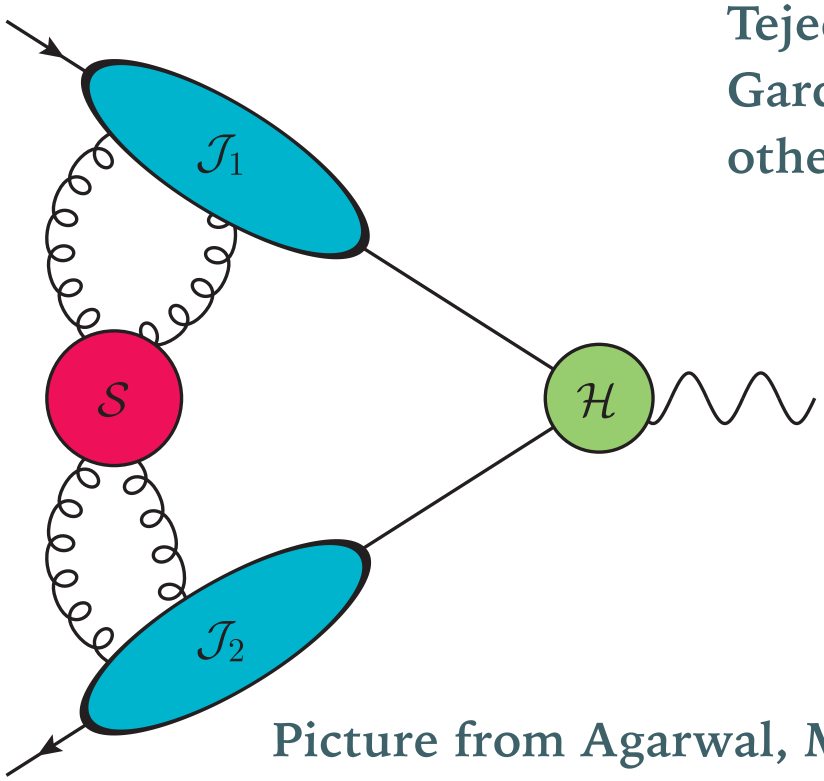
$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{dt_1}{t_1 - a_1} G(a_2,\ldots,a_n;t_1)\,, \quad a_j = \{0,1\}\,, \qquad G(0,\ldots,0,x) = \frac{1}{n!}\log^n x$$

[Remiddi, Vermaseren '99]

INFRA-RED STRUCTURE

IR singularities are known to **factorise in gauge theories**

[Becher, Neubert, Dixon, Magnea, Sterman, Tejeda-Yeomans, Mert Aybat, Almelid, Duhr, Gardi, Ferroglia, Czakon, Mitov, ... many others ...]

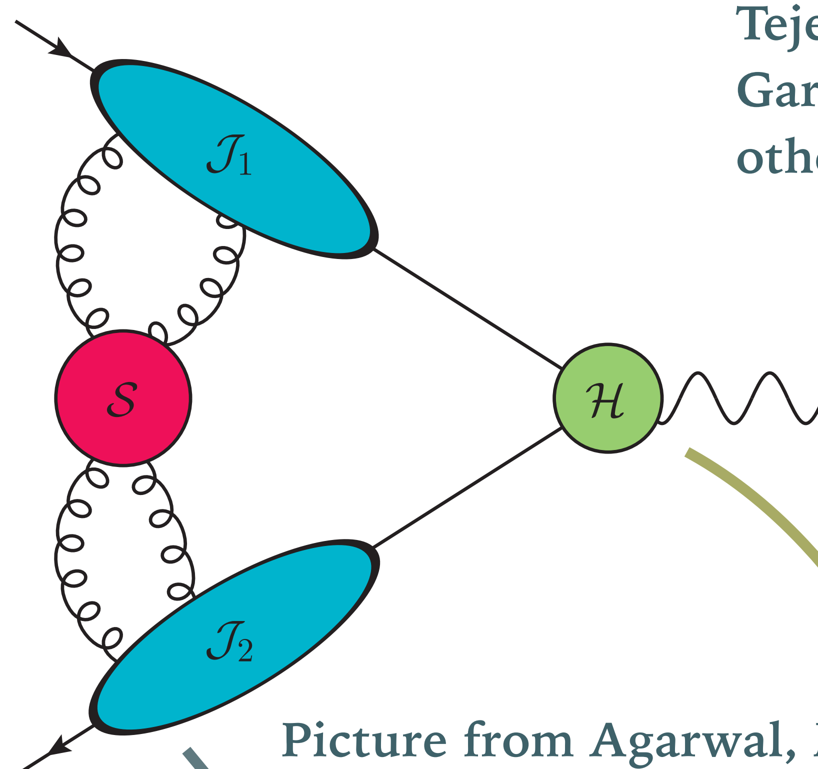


Picture from Agarwal, Magnea, Signorile-Signorile, Tripathi ’21

INFRA-RED STRUCTURE

IR singularities are known to **factorise in gauge theories**

[Becher, Neubert, Dixon, Magnea, Sterman, Tejeda-Yeomans, Mert Aybat, Almelid, Duhr, Gardi, Ferroglia, Czakon, Mitov, … many others …]



Picture from Agarwal, Magnea, Signorile-Signorile, Tripathi ’21

$$\mathcal{A}_n\left(\frac{p_i}{\mu},\alpha_s(\mu),\epsilon\right) = \prod_{i=1}^{n}\frac{\mathcal{J}_i\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)}{\mathcal{J}_{E,i}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right)} \times \mathcal{S}_n\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)$$

INFRA-RED STRUCTURE

The can be "multiplicatively renormalised away" similarly to UV divergences

$$\boldsymbol{\mathcal{H}}_{i,\,\mathrm{fin}}(\epsilon,\{p\}) = \lim_{\epsilon\to 0} \boldsymbol{\mathcal{Z}}^{-1}(\epsilon,\{p\},\mu)\, \boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}(\epsilon,\{p\})$$

$$\boldsymbol{\mathcal{Z}}(\epsilon,\{p\},\mu) = \mathbb{P}\exp\left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \mathbf{\Gamma}(\{p\},\mu')\right]$$

Z acts on the amplitude as a matrix in colour space, non-trivial correlations among partons

INFRA-RED STRUCTURE

The can be "multiplicatively renormalised away" similarly to UV divergences

$$\boldsymbol{\mathcal{H}}_{i,\,\mathrm{fin}}(\epsilon, \{p\}) = \lim_{\epsilon \to 0} \boldsymbol{\mathcal{Z}}^{-1}(\epsilon, \{p\}, \mu)\, \boldsymbol{\mathcal{H}}_{i,\,\mathrm{ren}}(\epsilon, \{p\})$$

$$\boldsymbol{\mathcal{Z}}(\epsilon, \{p\}, \mu) = \mathbb{P} \exp \left[\int_{\mu}^{\infty} \frac{\mathrm{d}\mu'}{\mu'} \boldsymbol{\Gamma}(\{p\}, \mu') \right]$$

Z acts on the amplitude as a matrix in colour space, non-trivial correlations among partons

The anomalous dimension Γ is fully known up to three loops

$$\boldsymbol{\Gamma}(\{p\}, \mu) = \boldsymbol{\Gamma}_{\mathrm{dipole}}(\{p\}, \mu) + \ldots$$

Dipole-like correlations among at most 2 partons are enough **up to two loops**

cusp anomalous dimension
soft-collinear double poles

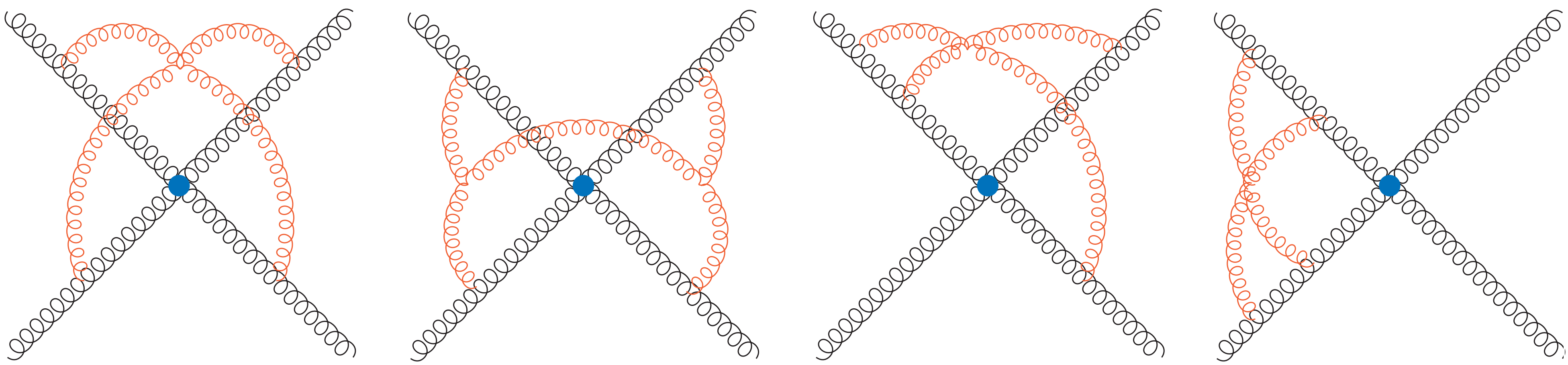
quark and gluon anomalous dimensions

$$\boldsymbol{\Gamma}_{\mathrm{dip}} = \sum_{1 \le i < j \le 4} \mathbf{T}_i^a\, \mathbf{T}_j^a\, \gamma^{\mathrm{K}} \ln\left(\frac{\mu^2}{-s_{ij} - i\delta} \right) + \sum_i \gamma^i$$

INFRA-RED STRUCTURE

At three loops we see for the first time **quadrupole correlations**

[Dixon, Gardi, Magnea '11]
[Becher, Neubert '13]
[Almelid, Duhr, Gardi '15]



$$\mathbf{\Gamma}(\{p\},\mu) = \mathbf{\Gamma}_{\text{dipole}}(\{p\},\mu) + \mathbf{\Delta}_4(\{p\})$$

$$\mathbf{\Delta}_4(\{p\}) = \sum_{L=3}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^L \mathbf{\Delta}_4^{(L)}(\{p\})$$

$$\mathbf{\Delta}_4^{(3)} = f_{abe} f_{cde} \Bigg[-16\, C \sum_{i=1}^{4} \sum_{\substack{1 \le j < k \le 4 \\ j,k \neq i}} \left\{\mathbf{T}_i^a, \mathbf{T}_i^d\right\} \mathbf{T}_j^b \mathbf{T}_k^c + 128 \left[\mathbf{T}_1^a \mathbf{T}_2^c \mathbf{T}_3^b \mathbf{T}_4^d D_1(x) - \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d D_2(x)\right] \Bigg],$$

HIGH ENERGY LIMIT (REGGE FACTORIZATION)

Calculation elucidates general structures in QCD

Verified **all-order structures in Regge kinematics** $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

Define even and odd amplitude under $(s \leftrightarrow u)$

$$\mathcal{H}_{\text{ren},\pm} = \frac{1}{2}\left[\mathcal{H}_{\text{ren}}(s,u) \pm \mathcal{H}_{\text{ren}}(u,s)\right]$$

And the even "large logarithm"

$$L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2}\left(\ln\left(\tfrac{-s-i\delta}{-t}\right) + \ln\left(\tfrac{-u-i\delta}{-t}\right)\right)$$

HIGH ENERGY LIMIT (REGGE FACTORIZATION)

Calculation elucidates general structures in QCD

Verified **all-order structures in Regge kinematics** $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

Define even and odd amplitude under $(s \leftrightarrow u)$

$$\mathcal{H}_{\text{ren},\pm} = \frac{1}{2}\left[\mathcal{H}_{\text{ren}}(s,u) \pm \mathcal{H}_{\text{ren}}(u,s)\right]$$

And the even "large logarithm"

$$L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2}\left(\ln\left(\tfrac{-s-i\delta}{-t}\right) + \ln\left(\tfrac{-u-i\delta}{-t}\right)\right)$$

At **leading power in x** and up to *next-to-leading logarithmic accuracy (NLL)* for even part and *NNLL for the odd part*, the amplitude factorises in a well understood way (LO BFKL) in terms of exchanges of "Reggeons" (multiple exchanges give rise to ***Regge cut contributions***)

$$\mathcal{H}_{\text{ren},\pm} = Z_g^2\, e^{L\mathbf{T}_t^2 \tau_g} \sum_{n=0}^{3} \bar{\alpha}_s^n \sum_{k=0}^{n} L^k \mathcal{O}_k^{\pm,(n)} \mathcal{H}_{\text{ren}}^{(0)}$$

↑

Regge trajectory

HIGH ENERGY LIMIT (REGGE FACTORIZATION)

Calculation elucidates general structures in QCD

Verified **all-order structures in Regge kinematics** $|s| \approx |u| \gg |t|$ (or $x = t/s \to 0$)

Define even and odd amplitude under ($s \leftrightarrow u$)

$$\mathcal{H}_{\mathrm{ren},\pm} = \frac{1}{2}\left[\mathcal{H}_{\mathrm{ren}}(s,u) \pm \mathcal{H}_{\mathrm{ren}}(u,s)\right]$$

And the even "large logarithm"

$$L = -\ln(x) - \frac{i\pi}{2} \approx \frac{1}{2}\left(\ln\left(\tfrac{-s-i\delta}{-t}\right) + \ln\left(\tfrac{-u-i\delta}{-t}\right)\right)$$

At **leading power in x** and up to *next-to-leading logarithmic accuracy (NLL)* for even part and *NNLL for the odd part*, the amplitude factorises in a well understood way (LO BFKL) in terms of exchanges of "Reggeons" (multiple exchanges give rise to ***Regge cut contributions***)

To test Regge factorisation at this order last needed ingredient was the gluon Regge trajectory at 3 loops, can be extracted from any three-loop process. We found agreement between $gg \to gg$ and $qq \to QQ$: this allows us to **predict $qq \to gg$ 3loop amplitude to NNLL accuracy**!

We verified this prediction to be **correct** by comparing to a successive explicit calculation

CONCLUSIONS

Multiloop amplitudes are **essential for pheno**, but they are also ***a lot of fun!***

Recent developments have allowed us to push investigations up to **3 loops** for **complete QCD $2 \to 2$ amplitudes** — *and beyond for simpler building blocks* —

Exploring QCD amplitudes at high loops we learn about **physics** and **mathematics**

1. All-order results in high-energy / Regge kinematics (beyond basic BFKL)
2. Structure of IR singularities in non-abelian QFTs
3. New ways to organise amplitudes in dim-regularisation
4. New geometries in pQFT (CY and higher genus) (didn't talk about this here)
5. … and much more …

Exciting times ahead!

THANK YOU VERY MUCH!