Deformational Vorticity in Constitutive Equations: The Co-Rigid Rotational Maxwell Model

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ABSTRACT

The presented work incorporates an objective vorticity (kinematic variable) into a novel constitutive equation - the co-*rigid rotational Maxwell model*. This model is applied to large amplitude oscillatory shear test conditions and compared to experimental data where it showed to have an improvement over the corotational Maxwell model. Finally, this research establishes a foundation for further evaluation, analysis, and development of constitutive equations that use this objective vorticity for better characterization of non-Newtonian fluids.

INTRODUCTION

The vorticity tensor is widely known to capture a fluid's rotational properties and is thought to hold valuable kinematic information about the fluid. However, this information has largely gone unused in constitutive equations due to the vorticity tensor's failure to adhere to the material objectivity requirements. Attempts have been made to incorporate vorticity into constitutive equations. [1–3, 5–8, 11, 12] In prior work proposed by Wedgewood [9, 10], a vorticity decomposition was introduced that was able to separate the vorticity tensor $\boldsymbol{\omega}$ into a rigid-body rotational, non-objective part $\boldsymbol{\omega}_R$ and a deformational, objective part $\boldsymbol{\omega}_D$. However, their decomposition was not utilized within constitutive equations. In this paper, we build upon that work and present a mathematical formalism for incorporating the objective, deformational vorticity into a proposed constitutive equation model for large amplitude oscillatory shear (LAOS) flow.

METHODOLOGY

In this work, the fluid is assumed to be incompressible and homogeneous with unsteady, homogeneous shear flow fluid kinematics characterized as $v_x = \dot{\gamma}(t)y$, $v_y = 0$, and $v_z = 0$. The focus of this research is specifically on the *LAOS* experimental design and conditions.

Taking inspiration from the corotational Maxwell model (Eq. 1), which includes a Jaumann derivative that removes all the vorticity through a corotating reference frame, the co-rigid rotational Maxwell model is proposed as seen in Eq. 2.

$$\boldsymbol{\tau} + \lambda \frac{\mathscr{D}\boldsymbol{\tau}}{\mathscr{D}t} = -\eta_0 \dot{\boldsymbol{\gamma}} \tag{1}$$

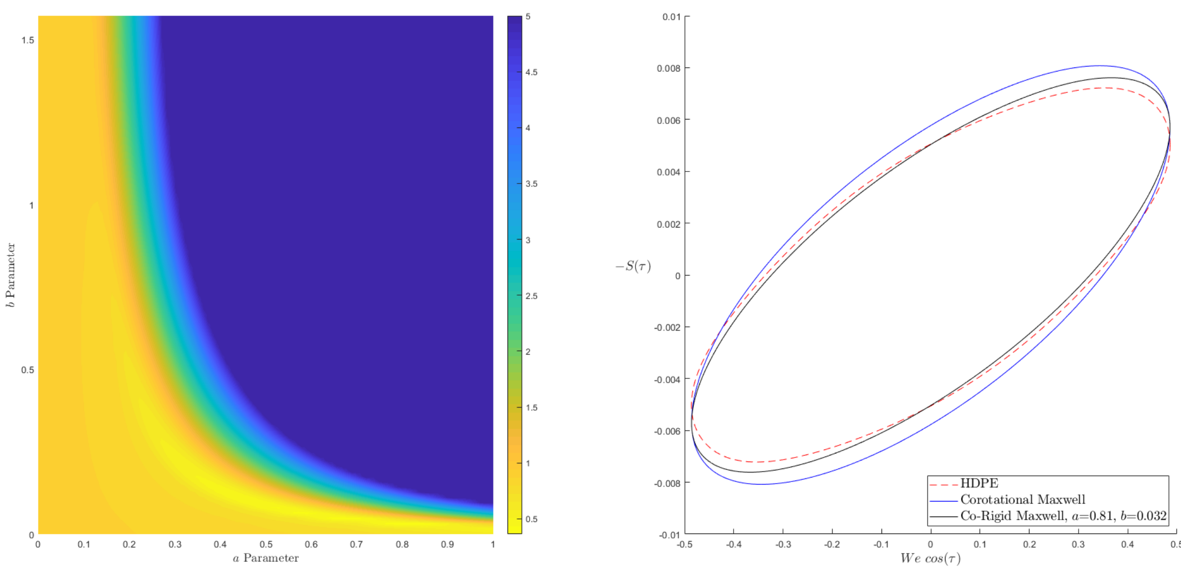
$$\boldsymbol{\tau} + \lambda \left[\frac{D}{Dt}\boldsymbol{\tau} + \frac{1}{2}\{\boldsymbol{\omega}_R \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}_R\} + \frac{1}{2}\{\hat{\boldsymbol{\omega}}_D \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \hat{\boldsymbol{\omega}}_D\} \right] = -\eta_0 \, \dot{\boldsymbol{\gamma}} \tag{2}$$

This model, however, removes only the *rigid-body vorticity* and allows the fluid particle to rotate with a portion of the deformational vorticity—the fluid's *observed deformational vorticity*—described as shown in Eq. 3, where a is the observed amplitude (magnitude) and b is the "lag" time (phase shift) response of the fluid particle. The range for a is from zero to one, where the particle rotates with none or all of the deformational vorticity, respectively. Parameter b ranges from zero to $\frac{\pi}{2}$, where it is in-phase or out-of-phase, respectively, of the shearing rate of strain. The corotational Maxwell model can be recovered when $a = 1$ and $b = 0$.

$$\hat{\omega}_D(t) = a \, \dot{\gamma}^0 \cos(t \, \omega - b) \tag{3}$$

An analytical solution was obtained using a perturbation method where the shear stress, the first normal stress, and the second normal stress are expanded as a power series in terms of strain-rate amplitude. Additionally, a numerical simulation of the LAOS co-rigid rotational Maxwell model was conducted using MATLAB R2022a. The dimensionless numbers used for the simulation were $De = 1.0$ and $We = 0.486$.

RESULTS AND DISCUSSION



(a) **(b)**

Figure 1: (a) Average error between co-rigid rotational Maxwell model and HDPE data. (b) LAOS result comparison for co-rigid rotational Maxwell model (black), corotational Maxwell model (blue), and HDPE data (red).

The numerical MATLAB results for the co-rigid rotational Maxwell model were compared to experimental data for HDPE at 160 °C obtained by Giacomin [4]. **Fig. 1a** is a heatmap of the error between simulated and experimental results that span the entire range of a and b. The yellow color depicts the areas of lowest error, to which a global minimum is found to be located at $a = 0.81$ and

$b = 0.032$. A Lissajous curve, shown in **Fig. 1b**, compares the HDPE data, corotational Maxwell model, and the proposed co-rigid rotational Maxwell model at the optimum a and b values.

CONCLUSION

The co-rigid rotational Maxwell model is a simple quasi-linear differential model. As expected, it was not able to capture the HDPE data perfectly; however, it did improve the accuracy of the predicted shear stress. More notably, this research acts as a stepping stone and illustrates a working methodology to incorporate an objective vorticity into constitutive equations, as well as lays the foundation to analyze and evaluate future models against experimental data.

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