Oscillation Phenomena in Nambu Quantum Mechanics

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Talk Based on:

- D. Minic and C. H. Tze, "Nambu Quantum Mechanics: A nonlinear generalization of Geometric Quantum Mechanics," Phys. Lett. B 536 (2002) 305-314 [hep-th/0202173 [hep-th]],
- D. Minic, T. Takeuchi, and C. H. Tze, "Interference and Oscillation in Nambu Quantum Mechanics," Phys. Rev. D 104 (2021), L051301 [hep-ph/2012.06583 [hep-ph]],
- D. Minic, T. Takeuchi, and NB, "A Generalization of Geometric Quantum Mechanics" (In preparation).

Motivation for the Name

- Minic and Tze's paper was inspired by:

  Y. Nambu,
  "Generalized Hamiltonian Dynamics,"
  Phys. Rev. D 7 (1973) 2405-2412.

- Nambu QM is a particular generalization of canonical QM and can be viewed as its two-parameter deformation.

Study of Quantum Foundations

- Contrasting quantum mechanics (QM) with classical theories (ex. Bell's theorem(s), Legget-Garg inequality).
- Correspondence of mathematical structure to reality (interpretations of QM).
- Looking for new effects in QM (ex. triple interference).
- Putting the axioms to test $\rightarrow$ generalizing QM (ex. quaternionic QM, Nambu QM).

Classical Mechanics: Poisson Bracket

- For two physical observables f and g, their Poisson bracket is:

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} = \varepsilon_{ij}\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_j},\ \varepsilon_{12} = 1.$$

- The time evolution of a quantity f is given by:

$$\frac{df}{dt} = \{f,H\}.$$

- For a harmonic oscillator, $H = \omega\left(\frac{p^2}{2} + \frac{q^2}{2}\right),\ \frac{dp}{dt} = -\omega q.$

Nambu Classical Mechanics: Nambu Bracket

- Using Louiville theorem as a guiding principle, a structure more general than the Poisson bracket was postulated by Nambu.

- A *ternary* Nambu bracket is given by:

$$\{f, g, h\} = \varepsilon_{ijk} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_j} \frac{\partial h}{\partial q_k}, \ \varepsilon_{123} = 1.$$

- For two conserved quantities H_1 and H_2, an observable f evolves as:

$$\frac{df}{dt} = \{f, H_1, H_2\}$$

Dynamics of an Asymmetric Top

- For a free asymmetric top:

$$H_1 = E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \quad H_2 = \frac{L^2}{2} = \frac{1}{2}\left(L_1^2 + L_2^2 + L_3^2\right)$$

- The time evolution of L_i is given by the ternary Nambu bracket:

$$\frac{dL_i}{dt} = \{L_i, H_1, H_2\} = \varepsilon_{ijk}\left(\frac{1}{I_j} - \frac{1}{I_k}\right) L_j L_k \ .$$

- The solution is given by Jacobi elliptic functions.

$$L_1(t) = N_1\,\mathrm{cn}(\Omega t), \ L_2(t) = -N_2\,\mathrm{sn}(\Omega t), \ L_3(t) = -N_3\,\mathrm{dn}(\Omega t).$$

Geometric Quantum Mechanics

- The dynamics of $\psi \in \mathbb{C}$ can be thought as the dynamics of a set of classical harmonic oscillators.

- More concretely, expand ψ in terms of eigenstates of the Hamiltonian as

$$|\psi(t)\rangle = \sum_n |n\rangle \underbrace{\langle n|\psi(t)\rangle}_{\psi_n(t)} = \sum_n \psi_n(t)\,|n\rangle\,.$$

- Then the Schrodinger equation implies:

$$\psi_n(t) = Ne^{-i\omega_n(t-t_n)}.$$

Quantum Dynamics as Classical Harmonic Oscillators

- Making the change of notation

$$q_n \;=\; \mathrm{Re}\,\psi_n,\;\; p_n \;=\; \mathrm{Im}\,\psi_n\;,$$

we get $\dot{q}_n = \omega_n p_n$ and $\dot{p}_n = -\omega_n q_n$.

- The generic solution is

$$\vec{\psi}(t) \;\equiv\; \begin{bmatrix} q_n(t) \\ p_n(t) \end{bmatrix} \;=\; N_n \begin{bmatrix} \cos\omega_n(t-t_n) \\ -\sin\omega_n(t-t_n) \end{bmatrix}\;.$$

- The dynamics is given by

$$\dot{p}_n = \{p_n, H\},\;\; \dot{q}_n = \{q_n, H\},\;\; H = \sum_n \omega_n \left(\frac{q_n^2}{2} + \frac{p_n^2}{2} \right).$$

Generalization to Nambu QM

- Phase space: Harmonic oscillator $\rightarrow$ Asymmetric top:

$$\vec{\psi}_n = \begin{bmatrix} q_n \\ p_n \end{bmatrix} \quad \rightarrow \quad \vec{\Psi}_n = \begin{bmatrix} q_n \\ p_n \\ s_n \end{bmatrix}$$

- Dynamics governed by: $H_1 = \sum_n \left(\frac{p_n^2}{2I_1} + \frac{q_n^2}{2I_2} + \frac{s_n^2}{2I_3} \right)$ and $H_2 = \sum_n \left(\frac{q_n^2}{2} + \frac{p_n^2}{2} + \frac{s_n^2}{2} \right)$.
- New time evolution:

$$\vec{\Psi}_n(t) = N_n \begin{bmatrix} c_\xi \,\mathrm{cn}(\Omega_n(t - t_n), k) \\ -\kappa_\xi \,\mathrm{sn}(\Omega_n(t - t_n), k) \\ -s_\xi \,\mathrm{dn}(\Omega_n(t - t_n), k) \end{bmatrix},$$

where $0 \leq k < 1$ and $-\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}$ are the deformation parameters, and $c_\xi = \cos\xi,\ s_\xi = \sin\xi,\ \kappa_\xi = \sqrt{c_\xi^2 + k^2 s_\xi^2}$.

Asymptotics of Elliptic Functions

Expansion of the elliptic functions in k for $\Omega = \left(\frac{2K}{\pi}\right)\omega$:

$$\begin{aligned}
\mathrm{sn}(\Omega t, k) &= \left(1 + \frac{k^2}{16} + \frac{7k^2}{256} + \cdots\right)\sin(\omega t) \\
&\quad + \left(\frac{k^2}{16} + \frac{k^4}{32} + \cdots\right)\sin(3\omega t) + \left(\frac{k^4}{256} + \cdots\right)\sin(5\omega t) + \cdots \;, \\
\mathrm{cn}(\Omega t, k) &= \left(1 - \frac{k^2}{16} - \frac{9k^4}{256}\right)\cos(\omega t) \\
&\quad + \left(\frac{k^2}{16} + \frac{k^4}{32}\right)\cos(3\omega t) + \left(\frac{k^4}{256} + \cdots\right)\cos(5\omega t) + \cdots \;, \\
\mathrm{dn}(\Omega t, k) &= \left(1 - \frac{k^2}{4} - \frac{5k^4}{64}\right) \\
&\quad + \left(\frac{k^2}{4} + \frac{k^4}{16}\right)\cos(2\omega t) + \left(\frac{k^4}{64} + \cdots\right)\cos(4\omega t) + \cdots
\end{aligned}$$

Phase Space of Nambu QM

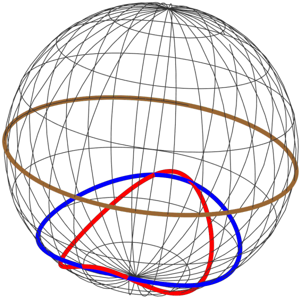


Figure: Phase portraits of an asymmetric top for various values of the deformation parameters. Blue: $k = 0.8,\ \xi = \frac{\pi}{4}$, red: $k = 0.9,\ \xi = \frac{\pi}{3}$. It is clear that for $k = \xi = 0$ (brown), we get the phase portrait of a harmonic oscillator.

Inner Product in Canonical QM

- In canonical QM, for two states ψ and ϕ,

$$\begin{aligned} |\psi\rangle \;&=\; \sum_n \psi_n |n\rangle \;\; and\, |\phi\rangle \;=\; \sum_m \phi_m |m\rangle \\ \implies \langle\psi|\phi\rangle &= \left(\sum_n \psi_n^* \langle n|\right)\left(\sum_m \phi_m |m\rangle\right) \\ &= \underbrace{\sum_n (\vec{\psi}_n \cdot \vec{\phi}_n)}_{g(\psi,\phi)} + i \underbrace{\sum_n (\vec{\psi}_n \times \vec{\phi}_n)}_{\varepsilon(\psi,\phi)}, \\ |\langle\psi|\phi\rangle|^2 &= g(\psi,\phi)^2 + \varepsilon(\psi,\phi)^2. \end{aligned}$$

Inner product in Nambu QM

- In Nambu QM, the wavefunctions are generalized to vectors in 3D:

$$g(\Psi,\Phi) \;=\; \sum_n (\vec{\Psi}_n \cdot \vec{\Phi}_n) \;\text{ and }\; \vec{\varepsilon}(\Psi,\Phi) \;=\; \sum_n (\vec{\Psi}_n \times \vec{\Phi}_n),$$

$$\text{so that } \;|\langle\Psi|\Phi\rangle|^2 \;=\; g(\Psi,\Phi)^2 + \vec{\varepsilon}(\Psi,\Phi)\cdot\vec{\varepsilon}(\Psi,\Phi)\;.$$

- This generalization has physical consequences in particle oscillation phenomena.

Neutrino Flavor Oscillation

- Consider two flavor oscillation with flavor eigenstates $|\alpha\rangle$ and $|\beta\rangle$ and mass eigenstates $|1\rangle$ and $|2\rangle$,

$$\begin{bmatrix} |\alpha\rangle \\ |\beta\rangle \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} |1\rangle \\ |2\rangle \end{bmatrix} .$$

- Transition probability in canonical QM:

$$\begin{aligned} P(\alpha \to \beta) &= g(\beta, \psi(t))^2 + \varepsilon(\beta, \psi(t))^2 \\ &= \sin^2 2\theta \, \sin^2 \underbrace{\left[\frac{(\omega_1 - \omega_2)t}{2} \right]}_{\frac{\delta m_{12}^2}{4E} L} . \end{aligned}$$

Flavor Oscillation in Nambu QM

- In Nambu QM, transition probability up to order k^2:

$$P(\alpha \to \beta) = \left(c_\xi^2 + \frac{k^2}{2} s_\xi^2 \right) \sin^2 2\theta \, \sin^2 \left(\frac{\delta m_{12}^2 L}{4E} \right) ,$$

where $s_\xi = \sin\xi$ and $c_\xi = \cos\xi$.

- The bounds on $\sin^2 2\theta$ can be interpreted as bounds on $\left(c_\xi^2 + \frac{k^2}{2} s_\xi^2 \right) \sin^2 2\theta$.

- From atmospheric neutrinos (de Salas et al. arXiv:2006.1123), $s_\xi^2 \left(1 - \frac{k^2}{2} \right) < 0.027(1\sigma),\ 0.037(2\sigma),\ 0.048(3\sigma)$.

Time-dependent CP Violation

- Consider the oscillation of B^0 mesons into $\bar{B}^0$ and vice-versa.
- The weak eigenstates B_L and B_H are given by linear combinations of the above states as:

$$|B_{L,H}\rangle = p\left|B^0\right\rangle \pm q\left|\bar{B}\right\rangle .$$

- Let f be a common final state into which B^0 and $\bar{B}^0$ decay.
- The time-dependent asymmetry in the processes $B^0 \to f$ and $\bar{B}^0 \to f$ is given by

$$A(t) = \frac{|\left\langle f \middle| B^0(t) \right\rangle|^2 - |\left\langle f \middle| \bar{B}^0(t) \right\rangle|^2}{|\left\langle f \middle| B^0(t) \right\rangle|^2 + |\left\langle f \middle| \bar{B}^0(t) \right\rangle|^2}.$$

CP-Asymmetry in Nambu QM I

- For canonical QM, $A(t)$ is given by:

$$\underbrace{2S\sin\left(\frac{\Delta mt}{2}\right)\cos\left(\frac{\Delta mt}{2}\right)}_{S\ \sin(\Delta mt)}-\underbrace{C\left(\cos^2\left(\frac{\Delta mt}{2}\right)-\sin^2\left(\frac{\Delta mt}{2}\right)\right)}_{C\ \cos(\Delta mt)},$$

where $\Delta m = (m_H - m_L)$.

- In Nambu QM, the same quantity becomes:

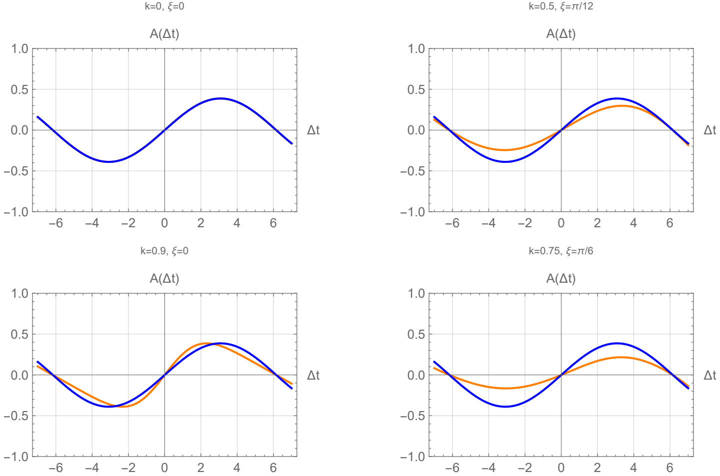
$$\frac{2\ S\ \kappa\ c_\xi^2\ \mathrm{cn}\ \mathrm{sn} - C\left(c_\xi^2\ \mathrm{cn}^2 - \kappa^2\ \mathrm{sn}^2\right)}{1 - 2\left(S\kappa\ \mathrm{sn} - (1 + C\ cn)\right)\ s_\xi\ c_\xi\ \mathrm{dn}}$$

- For $k = \xi = 0$, we have

$$s_\xi = \sin\xi = 0,\ c_\xi = \cos\xi = 1,\ \kappa = \sqrt{1 + k^2\tan^2\xi} = 1,$$

and we get back the canonical QM result.

CP-Asymmetry in Nambu QM II



Blue: Canonical QM, Orange: Nambu QM.

Bounds from $B^0 - \bar{B}^0$ Oscillation at Belle

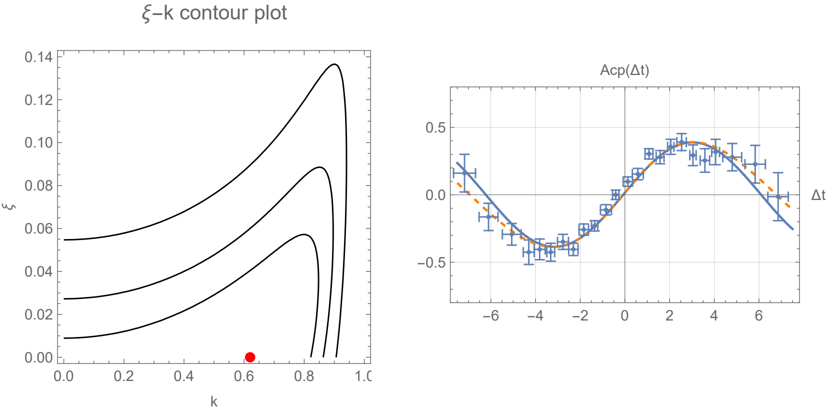


Figure: Data from Adachi et al., Phys. Rev. Lett. 108, 171802 (2012)

Bounds on k and ξ

- Comparing with the $B^0 - \bar{B}^0$ oscillation data from Belle, the allowable values of k and ξ are as follows:

$$0 < k < 0.848,\ 0 < \xi < 0.057\ (1\sigma)$$

$$0 < k < 0.896,\ 0 < \xi < 0.088\ (2\sigma)$$

$$0 < k < 0.941,\ 0 < \xi < 0.137\ (3\sigma).$$

- They easily satisfy the weaker bounds obtained from atmospheric neutrinos.

Summary

- An understanding of QM in terms of physical principles is yet to be achieved, so the study of its foundations is still active.
- Relaxation of the underlying assumptions of QM and confrontation with experiments could be one way to make progress.
- Nambu QM is a minimalistic yet experimentally testable generalization of QM.
- Particle oscillation data already provide bounds on the deformation parameters of Nambu QM.

Thank You!