Operator Basis of Chiral Perturbation Theory from On-Shell Amplitudes

— supported by package ABC4EFT[1]

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May 9, 2023 @ **Phenomenology 2023 Symposium, Pittsburgh**

[1] https://abc4eft.hepforge.org/

The Problem of Amplitude/Operator Basis

$$\mathcal{L}_{\rm EFT} = \sum_{d,i} \frac{C_i^{(d)}}{\Lambda^{d-4}} \mathcal{O}_i^{(d)}$$

- Enumerating operator basis has been an important and subtle problem for effective field theories.
- Difficulties: EOM, IBP, various group identities . . .
- Efforts are made for various important theories:
  - SMEFT: dim-5 (Weinberg 79'), dim-6 (Warsaw 10'), dim-7 (Liao&Ma 16'), dim-8 (Murphy 20') . . .
  - ChPT: $O(p^6)$ (Bijnens, *et al.* 99'), $O(p^8)$? (Bijnens, *et al.* 18')
  - Hilbert Series (Lehman&Martin; Henning, *et al.* 15').

Amplitude Basis and Young Tensor Method

- Amplitude/Operator basis correspondence (Shu, Ma, **MLX** 19')
  - Schematic definition: $\mathcal{B}^{\Psi}(\lambda,\tilde{\lambda}) \simeq \int \mathrm{d}^4x \, \langle\Psi|\mathcal{O}(x)|0\rangle$.
  - Intuition: On-shell amplitude bootstrap.
  - Isomorphism under redundancy relations:
    1. EOM $\simeq$ on-shell condition $\langle ii\rangle = [ii] = 0$.
    2. IBP $\simeq$ momentum conservation $\sum_i p_i = \sum_i |i\rangle[i| = 0$.
    3. Lorentz symmetry $\simeq \mathcal{B}(\langle ij\rangle,[ij])$ + Schouten identities.
- Amplitudes as Harmonics (Henning&Melia 19'):
  - Amplitude basis form irreps of $SU(N)$
- Young Tensor Method (Li, Ren, Shu, **MLX**, Yu, Zheng 20'):
  - Explicit monomial operators $\mathcal{O} \simeq \mathcal{B} = \mathcal{M} \times \mathcal{T}$.
  - Y-basis $\mathcal{M}^{\mathrm{y}}$ for a given type $\Psi \equiv \{h_i\}$.
  - Algorithm of reducing to y-basis $\mathcal{M} = \sum_i c_i \mathcal{M}^{\mathrm{y}}_i$.
  - Flavor relations (operators as flavor irrep.).
  - Code implementation (ABC4EFT.m), generality and efficiency.

Nambu-Goldstone Bosons

What makes Numbu-Goldstone bosons different from generic scalars?

- In terms of operators: non-linearly realized symmetry on G/H.

$$\pi_i^a \to \pi_i^a + \epsilon_i^a + O(\pi^2)$$

G-invariant operators: $\mathrm{tr}(u_\mu u^\mu) \supset \pi^2\partial^2, \pi^4\partial^2, \pi^6\partial^2, \ldots$

- In terms of amplitudes: Adler's zero condition.

$$\lim_{p_i \to 0} \mathcal{B}(\ldots \pi_i^a \ldots) = 0$$

Only the leading interaction term satisfies Adler's zero condition.

  - Soft recursion relation (Cheung *et al.* 15'): The rest can be reconstructed by Adler's zero for tree-level amplitudes.

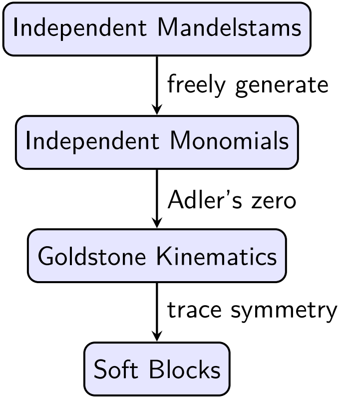
- Our target:
  - $\mathcal{B}(\pi_1, \ldots, \pi_N) \sim \mathcal{Y}\left(\mathcal{M}^{\text{soft}}(p_1, \ldots, p_N) \times \mathcal{T}^{a_1,\ldots a_N}\right)$
  - Amplitude/Operator Correspondence: $\nabla^{\{\mu_1} \cdots \nabla^{\mu_{n-1}} u_{a_i}^{\mu_n\}} \Leftrightarrow p_i^{\mu_1} \cdots p_i^{\mu_n}$

Chiral Perturbation Theory

On-shell approach (Dai *et al.* 20'):

- Only P even Building block s_{ij} (no $\epsilon^{\mu\nu\rho\sigma}$).
- Mandelstams suffer from Gram det.
  - $D = 4$ at $O(p^{10})$: $\det s_{ij} = 0$
- The resulting soft blocks are polynomials.
  - Do not correspond to monomial operator
- Assuming independent traces (large N_f).
  - Cayley-Hamilton when $N_f < N$.

$$\langle A^4 \rangle = \frac{1}{2} \langle A^2 \rangle^2 , \quad \forall A \in \mathfrak{su}(3)$$



Chiral Perturbation Theory

Young Tensor approach (Low, Shu, **MLX**, Zheng 20'):

- Both parities are included (not separated).

$$\langle ij\rangle[jk]\langle kl\rangle[li] \pm [ij]\langle jk\rangle[kl]\langle li\rangle$$
$$= p_{i\mu}p_{j\nu}p_{k\rho}p_{l\sigma}\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma(\gamma^5)]$$

- Free from Gram det.

	ϕ^6p^8	ϕ^6p^{10}	ϕ^6p^{12}	ϕ^7p^8	ϕ^7p^{10}	ϕ^7p^{12}	ϕ^8p^{10}	ϕ^8p^{12}
$f^{\text{even}}(s_{ij})$	–	1287	3003	–	8568	27132	42504	177100
y-basis	–	1286	2994	–	8547	26873	42308	173915
$f^{\text{odd}}(s_{ij},\epsilon)$	225	825	2475	1575	8400	35700	53900	309925
y-basis	180	600	1650	1106	5019	18305	28196	132335

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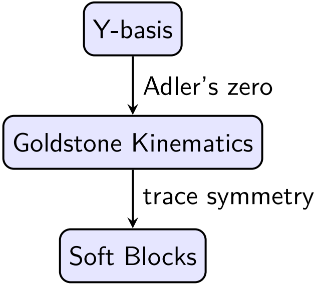
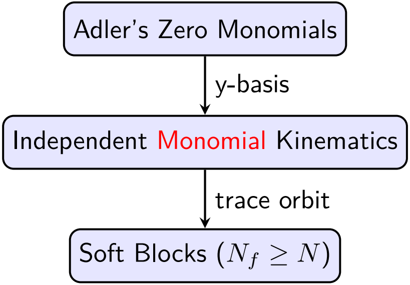


Chiral Perturbation Theory

Young Tensor approach (Low, Shu, **MLX**, Zheng 20'):

- Both parities are included separately.
  $s_{12}s_{34}s_{56} = \sum_{i=1}^{205} c_i \mathcal{M}_i^{\mathrm{y}}, \ \epsilon_{1234}s_{56} = \sum_{i=1}^{205} c'_i \mathcal{M}_i^{\mathrm{y}}$



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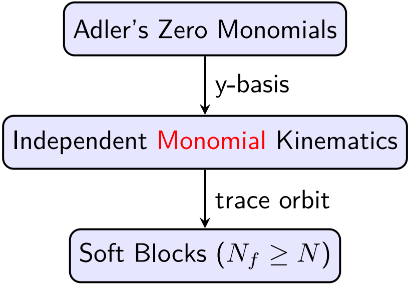

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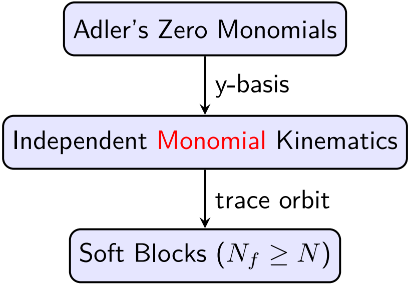
- Trace orbit of residual group $H^{(\mathcal{T})}$:

  $\mathcal{Y}(\mathcal{T}\mathcal{M}) = \mathcal{Y}(\mathcal{T}\mathcal{M}')$ if $\mathcal{M}' \in H^{(\mathcal{T})}(\mathcal{M})$
  $\Leftrightarrow\ \mathcal{O} = \mathcal{O}'$



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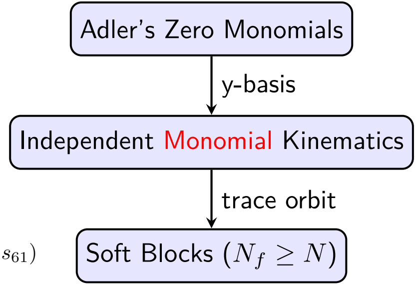
$$\begin{aligned} \mathcal{Y}(\mathrm{tr}[123456]s_{12}s_{34}s_{56}) &= \mathcal{Y}(\mathrm{tr}[123456]s_{23}s_{45}s_{61}) \\ \Rightarrow \ \langle u_\mu u^\mu u_\nu u^\nu u_\rho u^\rho \rangle &= \langle u_\rho u^\mu u_\mu u^\nu u_\nu u^\rho \rangle \end{aligned}$$



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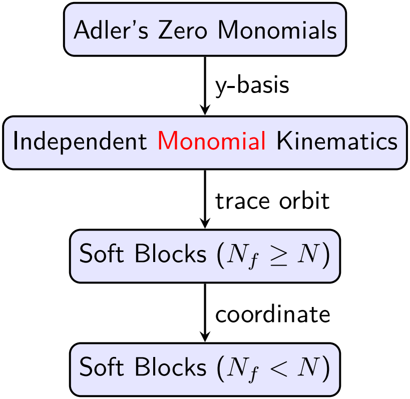
- $\mathcal{Y}(\mathrm{tr}[123456]s_{12}s_{34}s_{56}) = \sum_i c_i(\mathcal{T}\mathcal{M})_i$



Independent Trace Structures

Group $SU(N_f)$	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$	$SU(7)$	Trace
$\mathcal{T}^{a_1a_2a_3}$	1	2	2	2	2	2	2
$\mathcal{T}^{a_1a_2a_3a_4}$	3	8	9	9	9	9	9
$\mathcal{T}^{a_1a_2a_3a_4a_5}$	6	32	43	44	44	44	44
$\mathcal{T}^{a_1a_2a_3a_4a_5a_6}$	15	145	245	264	265	265	265

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$\mathcal{T}^{a_1a_2a_3}$	1	2	2	2	2	2	2
$\mathcal{T}^{a_1a_2a_3a_4}$	3	8	9	9	9	9	9
$\mathcal{T}^{a_1a_2a_3a_4a_5}$	6	32	43	44	44	44	44
$\mathcal{T}^{a_1a_2a_3a_4a_5a_6}$	15	145	245	264	265	265	265

Inner Product Treatment

Define $g_{ij} \equiv \langle \mathcal{T}_i, \mathcal{T}_j \rangle = (\mathcal{T}_i^\dagger)_{a_1\ldots a_N} (\mathcal{T}_j)^{a_1\cdots a_N}$, we have

$$\det g(N_f) \neq 0 \quad \Leftrightarrow \quad \{\mathcal{T}_i\} \text{ is independent}$$

$$\text{where} \qquad T^a_{ij} T^a_{kl} = \delta_{il}\delta_{kj} - \delta_{ij}\delta_{kl}/N_f$$

Tensor reduction for $\mathcal{T} = \sum_i c^i \mathcal{T}_i$: $\quad c^i = \left\langle (g^{-1})^{ij} \mathcal{T}_j, \mathcal{T} \right\rangle$.

The Operator Basis

1. Obtain inequivalent orbits $\mathcal{Y}(\mathcal{T}\mathcal{M})_i$;
2. Evaluate $\mathcal{Y}$ and find the unique coordinate on the $\mathcal{T}_j\mathcal{M}_k$;
3. Select independent monomials via linear algebra;
4. Translate to operators

$$p_{i\mu_1} p_{i\mu_2} \cdots p_{i\mu_n} \quad \Leftrightarrow \quad \nabla_{(\mu_1} \nabla_{\mu_2} \cdots \nabla_{\mu_{n-1}} u_{\mu_n)}$$

$O(p^6)$ and $O(p^8)$ bases for 6-point soft blocks (agree with Graf *et al.* 20'):

6-pt $O(p^6)$	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
P-even	3	8	13	14	15
P-odd	0	3	4	4	4

6-pt $O(p^8)$	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
P-even	9	40	68	74	76
P-odd	2	20	33	35	35

The Operator Basis

$SU(N_f)$	Operator Basis	Amplitude Basis
$SU(2)$	$\mathcal{O}_1 = \langle u^\mu u^\nu u^\rho u_\mu u_\nu u_\rho \rangle$	$\mathcal{B}_1 = \mathcal{Y} \circ \mathrm{tr}[123456] s_{14} s_{25} s_{36}$
	$\mathcal{O}_2 = \langle u^\mu u^\nu u^\rho u_\mu u_\rho u_\nu \rangle$	$\mathcal{B}_2 = \mathcal{Y} \circ \mathrm{tr}[123456] s_{14} s_{26} s_{35}$
	$\mathcal{O}_3 = \langle u^\mu u^\nu u^\rho u_\rho u_\mu u_\nu \rangle$	$\mathcal{B}_3 = \mathcal{Y} \circ \mathrm{tr}[123456] s_{15} s_{26} s_{34}$
$SU(3)$	$\mathcal{O}_4 = \langle u^\mu u^\nu u^\rho u_\rho u_\nu u_\mu \rangle$	$\mathcal{B}_4 = \mathcal{Y} \circ \mathrm{tr}[123456] s_{16} s_{25} s_{34}$
	$\mathcal{O}_5 = \langle u^\mu u^\nu u_\nu u^\rho u_\rho u_\mu \rangle$	$\mathcal{B}_5 = \mathcal{Y} \circ \mathrm{tr}[123456] s_{16} s_{23} s_{45}$
	$\mathcal{O}_6 = \langle u^\mu u^\nu u^\rho u_\mu \rangle \langle u_\nu u_\rho \rangle$	$\mathcal{B}_6 = \mathcal{Y} \circ \mathrm{tr}[1234\|56] s_{14} s_{25} s_{36}$
	$\mathcal{O}_7 = \langle u^\mu u^\nu u^\rho u_\nu \rangle \langle u_\mu u_\rho \rangle$	$\mathcal{B}_7 = \mathcal{Y} \circ \mathrm{tr}[1234\|56] s_{15} s_{24} s_{36}$
	$\mathcal{O}_8 = \langle u^\mu u^\nu u_\mu u^\nu \rangle \langle u_\rho u_\rho \rangle$	$\mathcal{B}_8 = \mathcal{Y} \circ \mathrm{tr}[1234\|56] s_{13} s_{24} s_{56}$
$SU(4)$	$\mathcal{O}_9 = \langle u^\mu u^\nu u_\nu u_\mu \rangle \langle u^\rho u_\rho \rangle$	$\mathcal{B}_9 = \mathcal{Y} \circ \mathrm{tr}[1234\|56] s_{14} s_{23} s_{56}$
	$\mathcal{O}_{10} = \langle u^\mu u^\nu u^\rho \rangle \langle u_\mu u_\nu u_\rho \rangle$	$\mathcal{B}_{10} = \mathcal{Y} \circ \mathrm{tr}[123\|456] s_{14} s_{25} s_{36}$
	$\mathcal{O}_{11} = \langle u^\mu u^\nu u^\rho \rangle \langle u_\mu u_\rho u_\nu \rangle$	$\mathcal{B}_{11} = \mathcal{Y} \circ \mathrm{tr}[123\|456] s_{14} s_{26} s_{35}$
	$\mathcal{O}_{12} = \langle u^\mu u^\nu u_\mu \rangle \langle u^\rho u_\nu u_\rho \rangle$	$\mathcal{B}_{12} = \mathcal{Y} \circ \mathrm{tr}[123\|456] s_{13} s_{25} s_{46}$
	$\mathcal{O}_{13} = \langle u^\mu u^\rho \rangle \langle u^\nu u_\mu \rangle \langle u_\rho u_\nu \rangle$	$\mathcal{B}_{13} = \mathcal{Y} \circ \mathrm{tr}[12\|34\|56] s_{14} s_{25} s_{36}$
$SU(5)$	$\mathcal{O}_{14} = \langle u^\mu u^\nu \rangle \langle u^\rho u_\rho \rangle \langle u_\mu u_\nu \rangle$	$\mathcal{B}_{14} = \mathcal{Y} \circ \mathrm{tr}[12\|34\|56] s_{15} s_{26} s_{34}$
$SU(N_f \geq 6)$	$\mathcal{O}_{15} = \langle u^\mu u_\mu \rangle \langle u^\nu u_\nu \rangle \langle u^\rho u_\rho \rangle$	$\mathcal{B}_{15} = \mathcal{Y} \circ \mathrm{tr}[12\|34\|56] s_{12} s_{34} s_{56}$

Induced Linear Relations

For example $N_f = 2$, choose $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3\}$ as the basis:

$$\begin{pmatrix}\mathcal{B}_4\\\mathcal{B}_5\\\mathcal{B}_6\\\mathcal{B}_7\\\mathcal{B}_8\\\mathcal{B}_9\end{pmatrix} = \begin{pmatrix}1 & 1 & -1\\1 & 1 & -1\\1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 2\\2 & 2 & -2\end{pmatrix} \times \begin{pmatrix}\mathcal{B}_1\\\mathcal{B}_2\\\mathcal{B}_3\end{pmatrix}, \quad \begin{pmatrix}\mathcal{B}_{10}\\\mathcal{B}_{11}\\\mathcal{B}_{12}\\\mathcal{B}_{13}\\\mathcal{B}_{14}\\\mathcal{B}_{15}\end{pmatrix} = \begin{pmatrix}0 & -1 & 1\\0 & 1 & -1\\0 & 0 & 0\\1 & 2 & 1\\2 & 2 & 0\\4 & 4 & -4\end{pmatrix} \times \begin{pmatrix}\mathcal{B}_1\\\mathcal{B}_2\\\mathcal{B}_3\end{pmatrix}$$

Summary and Outlook

- Summary of new results:
  1. Monomial operators basis (consistent with Hilbert Series counting);
  2. Parity-odd amplitude/operator basis;
  3. Gram determinant considered;
  4. Cayley-Hamilton theorem considered (finite N_f);
  5. Capable of deriving linear relations among the operators.
- Easy to add external sources: EWChEFT (NLO, NNLO) (Sun, **MLX**, Yu 22').

Classes	$\mathcal{N}_{\text{type}}$	$\mathcal{N}_{\text{term}}$	$\mathcal{N}_{\text{operator}}$
UhD^4	$3+6+0+0$	15	15
X^2Uh	$6+4+0+0$	10	10
$XUhD^2$	$2+6+0+0$	8	8
X^3	$4+2+0+0$	6	6
ψ^2UhD	$4+8+0+0$	13(16)	$13{n_f}^2 \quad (16{n_f}^2)$
ψ^2UhD^2	$6+10+0+0$	60(80)	$60{n_f}^2 \quad (80{n_f}^2)$
ψ^2UhX	$7+7+0+0$	22(28)	$22{n_f}^2 \quad (28{n_f}^2)$
ψ^4	$12+24+4+8$	117(160)	$\frac{1}{4}{n_f}^2(31-6n_f+335{n_f}^2) \quad ({n_f}^2(9-2n_f+125{n_f}^2))$
Total	123	261(313)	$\frac{335{n_f}^4}{4}-\frac{3{n_f}^3}{2}+\frac{411{n_f}^2}{4}+39 \quad (39+133{n_f}^2-2{n_f}^2-2{n_f}^3+125{n_f}^4)$ $\mathcal{N}_{\text{operatrs}}(n_f=1)=224(295), \quad \mathcal{N}_{\text{operatrs}}(n_f=3)=7704(11307)$

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Thank you for your attention!

Gram Det Makes a Difference

ONLY at $O(p^{10})$ or beyond (# independent trace orbits):

Trace Class	(6)	$(4\|2)$	$(3\|3)$	$(2\|2\|2)$
General D	112	91	43	25
$D=4$	111	90	42	24

	(8)	$(6\|2)$	$(5\|3)$	$(4\|4)$	$(4\|2\|2)$	$(3\|3\|2)$	$(2\|2\|2\|2)$
General D	435	320	226	129	149	117	26
$D=4$	427	314	222	126	146	115	25

	(10)	$(8\|2)$	$(7\|3)$	$(6\|4)$	$(5\|5)$	$(6\|2\|2)$
General D	105	74	45	50	29	37
$D=4$	99	71	43	47	27	35

	$(5\|3\|2)$	$(4\|4\|2)$	$(4\|3\|3)$	$(4\|2\|2\|2)$	$(3\|3\|2\|2)$	$(2\|2\|2\|2\|2)$
General D	35	30	21	18	18	7
$D=4$	35	28	20	17	17	6