Imprints of Axion's Evolution in CMB

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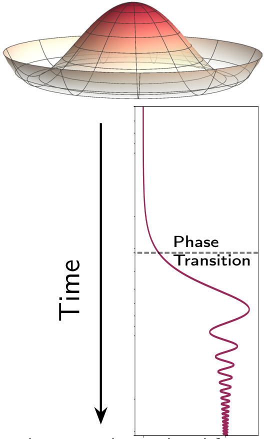

lower panel reproduced from Chadha-Day, Ellis, and Marsh (2022)

Axion-like particle (ALP) is a psudo-scalar field

- possible solution to strong CP problem
- candidate of dark matter
- appear after breaking a global $U(1)$ symmetry
  - called Peccei–Quinn symmtry if in QCD axion
- Depending on the time of symmtry breaking, ALP evolution can be
  - topological defects and non-linear dynamics
  - **non-zero initial amplitude and damped oscillations (the misaligned initial condition)**

1. How do axion-like particles evolve in a (thermal) medium?

2. What observables does such an evolution leave in the medium?

1. **How do axion-like particles evolve in a (thermal) medium?**

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Prelude

Snowmass2021 TF08 Whitepaper
Some open questions in axion theory

VI. AXIONS AND THERMAL FRICTION

field and the light degrees of freedom sources dark radiation, such that a steady-state-temperature ($T > H$) can be maintained even in an inflating universe. The equations that govern the time evolution of the scalar field and the radiation are given by:

?

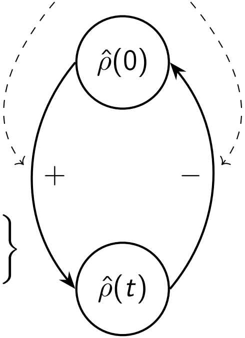
$$\ddot{a} + (3H + \Upsilon)\,\dot{a} + V'(a) = 0\,,$$
$$\dot{\rho}_{\rm dr} + 4H\rho_{\rm dr} = \Upsilon\dot{a}^2\,, \tag{15}$$

where $\rho_{\rm dr}$ is the energy density of dark radiation and $V(a)$ is the potential of a. This warm inflation scenario [219–225], has both interesting predictions for observations as well as theoretical upsides

Keldysh Formalism

Framework

$$\hat{\rho}(t) = \underline{e^{-i\hat{H}t}}\;\hat{\rho}(0)\;\underline{e^{i\hat{H}t}}$$



Close Time Path

1. Couple ALP with a bath χ via $H_I = ga\mathcal{O}_\chi$, e.g.,

$$ga\vec{E}\cdot\vec{B}, \quad g_s aG^{\mu\nu,b}\tilde{G}_{\mu\nu,b}, \quad g_\psi a\bar{\Psi}\gamma^5\Psi$$

2. Assume decoupled initial state $\hat{\rho}(0) = \hat{\rho}_a(0)\otimes\hat{\rho}_\chi(0)$
3. Trace over bath's degrees of freedom χ.

$$\rho^r(a_f^\pm;t) = \int_{a_i^\pm,a^\pm} \rho_a(a_i^\pm;0)\exp\Big\{\underbrace{i\int d^4x\left[\mathcal{L}_a^+ - \mathcal{L}_a^-\right] + \overbrace{i\mathcal{I}[a^+;a^-]}^{\text{influence function}}}_{iS_{eff}}\Big\}$$

All bath information is encoded in $\mathcal{I}[a^\pm]$

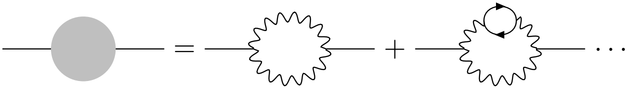
Keldysh Formalism

Influence function

4. Expand influence function in terms of g with $\langle \cdots \rangle = \mathrm{Tr}_\chi[\cdots \rho_\chi(0)]$.

$$\begin{aligned}\mathcal{I}[a^+, a^-] = &- g \int d^4x\, a^\pm(x) \langle \mathcal{O}_\chi \rangle \longleftarrow\!\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\; \text{vanish if parity of } \rho_\chi(0) \text{ is even} \\ &+ \frac{ig^2}{2} \int d^4x_1 d^4x_2\, a^\pm(x_1) \langle \mathcal{O}_\chi(x_1) \mathcal{O}_\chi(x_2) \rangle^{\pm\pm} a^\pm(x_2) \\ &+ \cdots\end{aligned}$$

- Four terms at $\mathcal{O}(g^2)$ with different labels
- Each term in expansion is <u>exact</u> in terms of couplings among degrees of freedom inside bath, e.g.



Equation of Motion

Langevin equation

Take the variation of the effective action. Up to $\mathcal{O}(g^2)$, equation of motion is a Langevin equation.

$$\ddot{\mathcal{A}}_{\vec{k}}(t) + \omega_{\vec{k}}^2 \mathcal{A}_{\vec{k}}(t) + \int_0^t \overbrace{\Sigma_{\vec{k}}(t-t')}^{\text{self energy}} \mathcal{A}_{\vec{k}}(t') d't = \boxed{\xi_{\vec{k}}(t)}$$

- $\mathcal{A} = \frac{1}{2}(a^+ + a^-)$ is the average of two branches.
- Initial conditions $\mathcal{A}_i, \dot{\mathcal{A}}_i$ are subject to initial density matrix $\rho_a(0)$.
- ξ is stochastic noise from bath and subject to Gaussian distribution $P[\xi]$

$$\langle\langle \xi \rangle\rangle = 0, \qquad \langle\langle \xi_{\vec{k}}(t) \xi_{\vec{k}'}(t') \rangle\rangle = \mathcal{N}_{\vec{k}}(t-t') \delta_{\vec{k},-\vec{k}'}$$

- Expectation values of observables $\overline{\langle\langle \cdots \rangle\rangle}$ are obtained after averaging over both initial condition $\overline{(\cdots)}$ and noise $\langle\langle \cdots \rangle\rangle$.

The self energy $i\Sigma(\vec{x}, t)$ and noise kernel $\mathcal{N}(\vec{x}, t)$

- are given by the *influence function* $\mathcal{I}[a^+, a^-]$
- depend on bath property $\rho_\chi(0)$ and the coupling operator $\mathcal{O}_\chi$

$$\mathcal{N}(x_1 - x_2) = \frac{g^2}{2}\mathrm{Tr}\Big(\{\mathcal{O}_\chi(t_1), \mathcal{O}_\chi(t_2)\}\hat{\rho}_\chi(0)\Big)$$
$$i\Sigma(x_1 - x_2) = g^2\mathrm{Tr}\Big([\mathcal{O}_\chi(t_1), \mathcal{O}_\chi(t_2)]\hat{\rho}_\chi(0)\Big)$$

Theorem (fluctuation-dissipation)

Assume a bath is in thermal equilibrium initially and couples with the system via bosonic operators.

$$i\Sigma(\vec{k}, \omega)\coth\left[\frac{\beta\omega}{2}\right] = 2\mathcal{N}(\vec{k}, \omega)$$

Langevin Equation

Decoherence and thermalization

For misaligned initial conditions,

- Amplitude damps.

$$\langle \mathcal{A}_{\vec{k}}(t)\rangle = e^{-\frac{\Gamma_{\vec{k}}}{2}t}\left[\overline{\mathcal{A}}_{i,\vec{k}}\cos(\Omega_{\vec{k}}t) + \overline{\dot{\mathcal{A}}}_{i,\vec{k}}\frac{\sin(\Omega_{\vec{k}}t)}{\Omega_{\vec{k}}}\right] + \mathcal{O}(g^2)$$

- Energy distribution approaches to thermal equilibrium, indicating <u>thermalization</u>.

$$\frac{E}{V} = \underbrace{\frac{e^{-\Gamma_0 t}}{2}\left[\overline{\dot{\mathcal{A}}_i^2 + m_a^2\,\mathcal{A}_i^2}\right]}_{\text{initial cold component decays}} + \underbrace{\int \frac{d^3k}{(2\pi)^3}\,\Omega_k\, n(\Omega_k)\left(1 - e^{-\Gamma_k t}\right)}_{\text{thermalized component grows}} + \mathcal{O}(g^2)$$

where $n(\Omega_{\vec{k}})$ is Bose-Einstein distribution.

A warming-up scenario for cold ALPs is exhibited.

A Classical Example: Ink drop in water

Brownian motion

Red ink drop in water [Vecteezy.com] (image is cropped)

- Described by a Langevin equation

$$m\boldsymbol{a}(t) + \lambda \boldsymbol{v}(t) = \boldsymbol{\eta}(t)$$

- Ink drop in water experiences two effects.

Drag dissipates the initial stream

Brownian Motion causes ink to diffuse in and thermalize with water

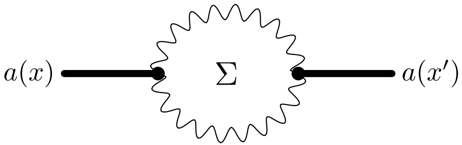
- Both effects originate from *random collisions with water molecules*, consequently are connected by

$$\langle \underbrace{\eta_i(t)\eta_j(t')}_{\text{fluctuation}} \rangle = 2\underbrace{\lambda}_{\text{dissipation}} k_B T \delta_{ij}\delta(t - t')$$

Consider photon-ALP coupling.

$$\mathcal{L}_I = -ga(x)\vec{E}(x) \cdot \vec{B}(x)$$

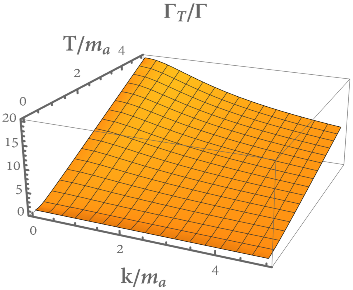
Up to the second order of g



For cosmic interest, this calculation is valid from recombination onward. Otherwise we need to consider plasma instead of a pure photon bath.

Application to Photon Bath

Enhanced relaxation rate



- Relaxation rate is substantially enhanced at finte temperature.
- In long wavelength $k \ll m_a$ and high temperature limit $T \gg m_a$,

$$\frac{\Gamma_T}{\Gamma_0} = 4\frac{T}{m_a} \qquad \Gamma_0 = \frac{g^2 m_a^4}{64\pi\Omega_k}$$

- As an estimation,

$$T_{\text{recombination}} \approx 0.26\,\mathrm{eV}$$
$$T_{\text{CMB}} \approx 2.3 \times 10^{-4}\,\mathrm{eV}$$
$$m_a \sim \mu\mathrm{eV}$$

Application to Photon Bath

Reduced finite temperature effective mass

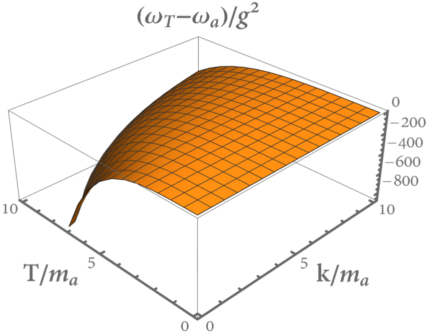


Figure: $\Delta\omega/g^2$ is in units of m_a^3

- Finite temperature self-energy correction is negative
- In high temperature limit $T \gg m_a$,

$$m_a^2(T) \approx m_a^2\left[1 - \left(\frac{T}{T_c}\right)^4\right]$$

$$T_c = \left(\frac{15}{\pi^2}\right)^{\frac{1}{4}} \sqrt{\frac{m_a}{g}}$$

- At $T > T_c$, $m_a^2(T)$ is negative, leading to instability, a signal for an inverted phase transition.

Divergences in zero-temperature part of self-energy require higher order derivatives.

$$\Sigma_R^{(0)} = -\frac{g^2}{64\pi^2}\left[\underbrace{\frac{1}{2}\Lambda^4 + 2K^2\Lambda^2}_{\text{regularized away}} + \frac{3}{2}(K^2)^2 + \underbrace{(K^2)^2 \ln\left(\frac{\Lambda^2}{|K^2|}\right)}_{\text{require } (\partial^2 a)^2 \text{ term}} \right]$$

Ginzburg-Landau description

$$F = \underbrace{\underbrace{\frac{1}{2}(\partial a)^2 + C(\partial^2 a)^2 + \cdots}_{\substack{\text{possible density wave} \\ \text{if } C<0}} + \underbrace{\frac{1}{2}m_a^2(T)a^2 + D\,a^4}_{\substack{\text{possible condensate} \\ \text{when } m_a^2(T)<0}} + \cdots}_{\text{possible new exotic phase}}$$

- Solve the quantum master equation

$$\dot{\hat{\rho}}_I(t) = -i[H_I, \hat{\rho}_I(0)] - \int_0^t [H_I(t), [H_I(t'), \hat{\rho}_I(t')]]dt'$$

- Use *Markove approximation*, *rotating wave approximation*.

- Recover decay rate, self energy, decoherence, thermalization

1. How do axion-like particles evolve in a (thermal) medium?
2. **What observables does such an evolution leave in the medium?**

Condensate Induced by Coherent ALP Field

1. Begin with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial a)^2 - \frac{1}{2}m_a^2 a^2 + \mathcal{L}_\chi - g a \mathcal{O}_\chi$$

2. Find the equation of motion for operators in Heisenberg picture.

$$\frac{\partial^2}{\partial t^2}a(\vec{x},t) - \nabla^2 a(\vec{x},t) + m_a^2 a(\vec{x},t) = -g\mathcal{O}_\chi(\vec{x},t)$$

3. Expectation values are found by tracing over the intial density matrix.

$$\langle \mathcal{O}_\chi(\vec{x},t)\rangle = -\frac{1}{g}\left[\frac{\partial^2}{\partial t^2}\,\overline{a}(\vec{x},t) - \nabla^2\overline{a}(\vec{x},t) + m_{0a}^2\,\overline{a}(\vec{x},t)\right]$$

   - $\langle \mathcal{O}_\chi\rangle$ and $\overline{a} \triangleq \langle a\rangle$ are macroscopic condensates.

4. **NOT the end of the story**.

1. Decompose a coherent ALP as its amplitude expectation value $\overline{a}$ and quantum fluctuations $\widetilde{a}$ around the amplitude.

$$a(\vec{x},t) = \overline{a}(\vec{x},t) + \widetilde{a}(\vec{x},t)$$

2. Neglect the fluctuations $\widetilde{a}(x)$ (**mean field approximation**).

$$\mathcal{L}_I = -g\,\overline{a}\,\mathcal{O}_\chi \qquad \text{or} \qquad H_I(t) = g\int d^3x\;\overline{a}(\vec{x},t)\,\mathcal{O}_\chi(\vec{x})$$

3. Result in a system driven by a classical source. Up to the linear order,

$$\langle\mathcal{O}_\chi(\vec{x})\rangle(t) \triangleq \mathrm{Tr}\big(\mathcal{O}_\chi(\vec{x})\,\rho_\chi(t)\big) = \int d^3x' \int_{t_0}^{t} \Xi(\vec{x}-\vec{x}', t-t')\,\overline{a}(\vec{x}', t')\,dt' + \cdots$$

- The **linear response kernel** $\Xi(\vec{x}-\vec{x}', t-t')$ is also called **dynamical susceptibility**.

$$\Xi(\vec{x}-\vec{x}', t-t') = -ig\,\mathrm{Tr}\Big(\Big[\mathcal{O}_\chi^{(H_\chi)}(\vec{x},t), \mathcal{O}_\chi^{(H_\chi)}(\vec{x}',t')\Big]\,\rho_\chi(t_0)\Big) \quad ; \quad t > t'$$

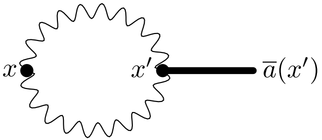
  - The superscript (H_χ) means Heisenberg picture in absence of the source $\bar{a}$.

- Dynamical susceptibility $\Xi(x-x')$ and self-energy $\Sigma(x-x')$ are simply related by the coupling strength g.

$$\Sigma(x-x') = g\,\Xi(x-x')$$

Chern-Simons Condensate

- Suppose the medium states are photons, i.e., $\mathcal{O}_\chi = \vec{E} \cdot \vec{B}$,



- This pseudoscalar density $\vec{E} \cdot \vec{B}$ is a total surface term, hence the name, **Chern-Simons condensate** $\langle \vec{E} \cdot \vec{B} \rangle$.

$$\vec{E} \cdot \vec{B} \propto F_{\mu\nu} \widetilde{F}^{\mu\nu} \propto \partial_\mu \left(\varepsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta \right)$$

Chern-Simons Condensate Induced by ALP

- For simiplicity in this talk, assume a homogeneous ALP field.

$$\bar{a}(t) = e^{-\frac{\Gamma}{2}\,t}\left(a_0\, e^{-i m_a t} + a_0^*\, e^{i m_a t}\right)$$

- The induced Chern-Simons condensate is

$$\langle \vec{E} \cdot \vec{B} \rangle (t) = \frac{1}{g}\left[\Sigma(\vec{0}, m_a)\, \bar{a}(t) + \Gamma\, \dot{\bar{a}}(t)\right]$$

Note that $\Sigma(\vec{0}, m_a), \Gamma \propto g^2$. Therefore,

$$\langle \vec{E} \cdot \vec{B} \rangle \propto g$$

- At hight temperature,

$$\langle \vec{E} \cdot \vec{B} \rangle (t) = -\frac{g\, \pi^2\, T^4}{15}\, \bar{a}(t) + \frac{g\, m_a^2\, T}{16\, \pi}\, \dot{\bar{a}}(t) + \mathcal{O}(m_a^2/T^2)$$

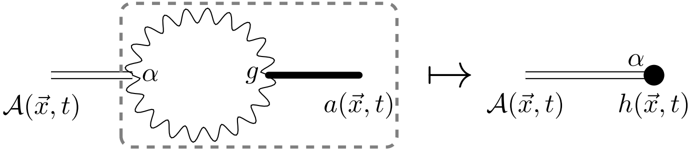
Probe the Chern-Simons Condensdate

Mixing with emergent axion quasiparticles

- Axion-Like quasi-particle $\mathcal{A}$ can be created in some novel materials, e.g., topological insulator.

$$g_{\mathcal{A}\gamma\gamma}\mathcal{A}\vec{E}\cdot\vec{B}, \qquad g_{\mathcal{A}\gamma\gamma} \propto \alpha_{EM}$$

- It can couple/mix with cosmic ALP via photons, be driven by the condensate.



Possibly Improved Detection Efficiency

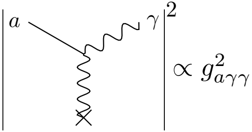
Detect ALP at the linear order of the coupling

- The mixing effect provides detection schemes with efficiency **linearly proportional** to ALP-photon coupling.

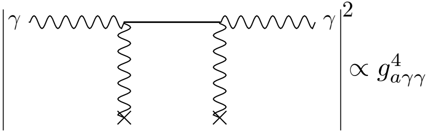
$$\overline{\mathcal{A}} \propto g_{a\gamma\gamma}\alpha_{EM}\bar{a}$$

- This is achieved by exploiting the **coherence** of cosmic ALP
- Many search schemes rely on higher order processes.

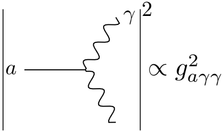
Solar Axion Helioscopes



Light Shinning Through Wall



(Stimulated) Emission Line



Conclusions

- The evolution of axion-like particles in a thermal medium
  - Noise terms are important, leading to warming-up ALPs.
- Chern-Simions condensate induced by a coherent ALP field
- Possible search schemes at the linear order of ALP-photon coupling

Future Directions

- Add cosmological expansion
- Explore entangled initial condition instead of $\rho_a(0) \otimes \rho_\chi(0)$
- Give analytical measurable signals for the proposed search scheme.
- ......

Thank You For Your Attention

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Map between canonical quantization and path-integral formalism

$$\langle a_f;\chi_f|U(t)|a_i;\chi_i\rangle = \int \mathcal{D}a^+\mathcal{D}\chi^+\, e^{i\int_0^t d\tau \int d^3x \mathcal{L}[a^+,\chi^+]}$$

$$\langle a_i';\chi_i'|U^{-1}(t)|a_f';\chi_f'\rangle = \int \mathcal{D}a^-\mathcal{D}\chi^-\, e^{-i\int_0^t d\tau \int d^3x \mathcal{L}[a^-,\chi^-]}$$

Map between CTP varaibles and field operators

$$A^+B^+ \to \mathrm{Tr}[T(AB)\rho] \qquad A^-B^- \to \mathrm{Tr}[\rho\widetilde{T}(AB)] \qquad A^+B^- \to \mathrm{Tr}A\rho B$$

Trace over bath's degrees of freedom

$$e^{i\mathcal{I}[a^+;a^-]} = \mathrm{Tr}_\chi\Big[\mathcal{U}(t;a^+)\,\rho_\chi(0)\,\mathcal{U}^{-1}(t;a^-)\Big]$$

1. Introduce Keldysh variables.

$$\mathcal{A} = \frac{1}{2}(a^+ + a^-), \quad \mathcal{R} = a^+ - a^-$$

They induce a Wigner transform $\rho_a(0) \longrightarrow W[\mathcal{A}_i, \pi_i]$

2. Introduce external source $\mathcal{J}$ in iS_{eff} and define the generating functional $Z[\mathcal{J}]$ by setting $\mathcal{R}_f = 0$ and tracing over $\mathcal{A}_f$ in $\rho_f(a_f^{\pm}; t)$.

$$Z[\mathcal{J}] \propto \int_{\mathcal{A}_i \cdots} \underbrace{W[\mathcal{A}_i, \pi_i]}_{\text{initial condition}} \times \underbrace{P[\xi]}_{\substack{\text{probability distribution} \\ \text{from bath}}} \times \exp\left\{ i \int dt \sum_{\vec{k}} \mathcal{A}_{\vec{k}}(t) \mathcal{J}_{-\vec{k}}(t) \right\}$$

$$\times \prod_{\vec{k}} \delta \underbrace{\Big[\ddot{\mathcal{A}}_{\vec{k}}(t) + \omega_{\vec{k}}^2 \mathcal{A}_{\vec{k}}(t) + \int_0^t \overbrace{\Sigma_{\vec{k}}(t-t')}^{\text{self energy}} \mathcal{A}_{\vec{k}}(t') d't - \xi_{\vec{k}}(t) \Big]}_{\text{Langevin Equation}} \times \prod_{\vec{k}} \delta\left[\pi_{i,\vec{k}} - \dot{\mathcal{A}}_{i,\vec{k}}\right]$$

Use functional Gaussian integral to convert the quadratic term in R to a quadratic term in ξ.

$$\exp\left\{-\frac{1}{2}\int d^4x_1 d^4x_2 R(x_1)\mathcal{N}(x_1-x_2)R(x_2)\right\} = \frac{\int D\xi \exp\left\{-\frac{1}{2}\int d^4x_1 d^4x_1 \xi(x_1)\mathcal{N}^{-1}(x_1-x_2)\xi(x_2) + i\int d^4x R(x)\xi(x)\right\}}{\int D\xi \underbrace{\exp\left\{-\frac{1}{2}\int d^4x_1 d^4x_2 \xi(x_1)\mathcal{N}^{-1}(x_1-x_2)\xi(x_2)\right\}}_{= P[\xi] \text{ up to normalization}}}$$

In momentum space, $\langle\langle\xi\rangle\rangle = 0$ and $\langle\langle\xi_{\vec{k}}(t)\xi_{\vec{k}'}(t')\rangle\rangle = \mathcal{N}_{\vec{k}}(t-t')\delta_{\vec{k},-\vec{k}'}$.

$$P[\xi] \propto \prod_{\vec{k}} \exp\left\{-\frac{1}{2}\int dt_1 \int dt_2\, \xi_{-\vec{k}}(t_1)\, \mathcal{N}_{\vec{k}}^{-1}(t_1-t_2)\, \xi_{\vec{k}}(t_2)\right\}$$

Classical Limit of Fluctuation-Dissipation Theorem

In the literature it is usually assumed that the noise kernal $\mathcal{N}_{\vec{k}}(t-t')$ has very short time correlation, i.e.

$$\mathcal{N}_{\vec{k}}(t-t') \propto \delta(t-t')$$

which entails that

$$i\Sigma_{\vec{k}}(\vec{k},\omega)\coth\left[\frac{\beta\omega}{2}\right] \propto \text{constant} \quad \xrightarrow[\coth\omega/2T\simeq 2T/\omega]{\text{classical limit}} \quad i\Sigma(\vec{k},\omega) \propto \omega$$

The classical limit is an ohmic spectral density, which in general is NOT compatible with a relativistic bath.

Formal solution is

$$\mathcal{A}_{\vec{k}}(t) = \mathcal{A}_{i,\vec{k}}\dot{\mathcal{G}}_{\vec{k}}(t) + \dot{\mathcal{A}}_{i,\vec{k}}\mathcal{G}_{\vec{k}}(t) + \int_0^t \mathcal{G}_{\vec{k}}(t-t')\xi_{\vec{k}}(t')dt'$$

The Green's function is

$$\mathcal{G}_{\vec{k}}(t) = -\int_{-\infty}^{\infty} \frac{1}{(\nu - i\epsilon)^2 - \omega_{\vec{k}}^2 - \Sigma(\nu, \vec{k})} \frac{d\nu}{2\pi} \approx e^{-\frac{\Gamma_{\vec{k}}}{2}t} \frac{\sin(\Omega_{\vec{k}}t)}{\Omega_{\vec{k}}} + \mathcal{O}(g^2)$$

$\Sigma_{\vec{k}}(\nu, \vec{k})$ is complex in general, inducing decay and correction to dispersion relation.

Amplitude & Energy

$$\mathcal{A}_{\vec{k}}(t) = \mathcal{A}_{i,\vec{k}}\dot{\mathcal{G}}_{\vec{k}}(t) + \dot{\mathcal{A}}_{i,\vec{k}}\mathcal{G}_{\vec{k}}(t) + \int_0^t \mathcal{G}_{\vec{k}}(t-t')\,\xi_{\vec{k}}(t')\,dt'$$

$$\overline{\langle\langle \mathcal{A}_{\vec{k}}\rangle\rangle} = \overline{\mathcal{A}_{i,\vec{k}}\dot{\mathcal{G}}_{\vec{k}}(t) + \dot{\mathcal{A}}_{i,\vec{k}}\mathcal{G}_{\vec{k}}(t)} + \int_0^t \mathcal{G}_{\vec{k}}(t-t')\langle\langle\,\xi_{\vec{k}}(t')\,\rangle\rangle dt'$$

$$\begin{aligned}
\overline{\langle\langle \mathcal{A}_{\vec{k}}(t)\mathcal{A}_{-\vec{k}}(t)\rangle\rangle} = {} & \overline{\left(\mathcal{A}_{i,\vec{k}}\dot{\mathcal{G}}_{\vec{k}}(t) + \dot{\mathcal{A}}_{i,\vec{k}}\mathcal{G}_{\vec{k}}(t)\right)\left(\mathcal{A}_{i,-\vec{k}}\dot{\mathcal{G}}_{-\vec{k}}(t) + \dot{\mathcal{A}}_{i,-\vec{k}}\mathcal{G}_{-\vec{k}}(t)\right)} \\
& + \overline{\left(\mathcal{A}_{i,\vec{k}}\dot{\mathcal{G}}_{\vec{k}}(t) + \dot{\mathcal{A}}_{i,\vec{k}}\mathcal{G}_{\vec{k}}(t)\right)} \int_0^t \mathcal{G}_{-\vec{k}}(t-t')\langle\langle\,\xi_{-\vec{k}}(t')\,\rangle\rangle dt' \\
& + \overline{\left(\mathcal{A}_{i,-\vec{k}}\dot{\mathcal{G}}_{-\vec{k}}(t) + \dot{\mathcal{A}}_{i,-\vec{k}}\mathcal{G}_{-\vec{k}}(t)\right)} \int_0^t \mathcal{G}_{\vec{k}}(t-t')\langle\langle\,\xi_{\vec{k}}(t')\,\rangle\rangle dt' \\
& + \int_0^t\int_0^t \mathcal{G}_{\vec{k}}(t-t')\mathcal{G}_{-\vec{k}}(t-t'')\langle\langle\,\xi_{\vec{k}}(t')\xi_{-\vec{k}}(t'')\,\rangle\rangle dt'\,dt''
\end{aligned}$$

Description of Misaligned Initial Condition

A initially non-zero amplitude state is described by a coherent state of the form

$$|\Delta\rangle = \prod_{\vec{k}} e^{\Delta_{\vec{k}} b^\dagger_{\vec{k}} - \Delta^*_{\vec{k}} b_{\vec{k}}} |0\rangle$$

In the Schroedinger representation,

$$\Psi[a] = e^{i\int d^3x \overline{\pi}_i(x) a(x)} \Psi_0[a - \overline{\mathcal{A}}_i]$$

The density matrix for a pure, misaligned initial state is

$$\rho_a[a, a'; 0] = \Psi^*[a']\Psi[a]$$

Experimental Constraints

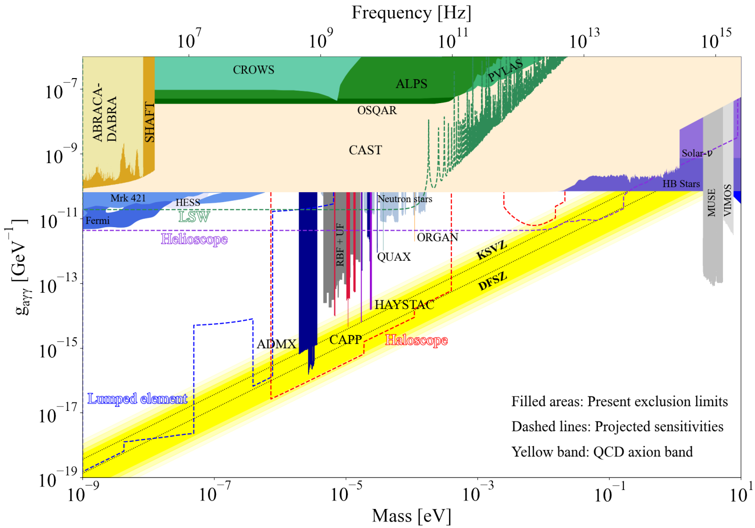

Image Credit: Semertzidis and Youn (2021)

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Quantum Master Equation

- Let the Hamiltonian be $H = H_0 + H_I$. In the interaction picture, the density matrix is
$$\hat{\rho}_I(t) = e^{iH_0 t}\hat{\rho}(t)e^{-iH_0 t}$$
- Taking time derivative gives the equation of motion.
$$\dot{\hat{\rho}}(t) = -i[H_I(t), \hat{\rho}_I(t)]$$
- Formal solution is found by integrating, inserting the solution back, and iterating. After one iteration,
$$\dot{\hat{\rho}}(t) = -i[H_I, \hat{\rho}_I(0)] - \int_0^t [H_I(t), [H_I(t'), \hat{\rho}_I(t')]]dt'$$

- Trace over χ to find the reduced density matrix of $\hat{\rho}_{Ia} = \mathrm{Tr}_\chi \hat{\rho}_I(t)$

$$\dot{\hat{\rho}}_{Ia}(t) = -g^2 \int_0^t dt' \int d^3x \int d^3x' \Big\{ \hat{a}_I(x)\, \hat{a}_I(x')\, \hat{\rho}_{Ia}(t')\; G^>(x-x')$$

$$+\hat{\rho}_{Ia}(t')\, \hat{a}_I(x')\, \hat{a}_I(x)\, G^<(x-x') - \hat{a}_I(x)\, \hat{\rho}_{Ia}(t')\, \hat{a}_I(x')\, G^<(x-x') - \hat{a}_I(x')\, \hat{\rho}_{Ia}(t')\, \hat{a}_I(x)\, G^>(x-x') \Big\}$$

where

$$G^>(x-x') = \mathrm{Tr}_\chi \hat{\rho}_\chi(0)\mathcal{O}_\chi(x)\mathcal{O}_\chi(x') \qquad G^<(x-x') = \mathrm{Tr}_\chi \hat{\rho}_\chi(0)\mathcal{O}_\chi(x')\mathcal{O}_\chi(x)$$

- Upon taking the trace over the χ degrees of freedom, the first term $-i[H_I, \hat{\rho}_I(0)]$ vanishes under the assumption that the thermal density matrix of the environmental fields is even under parity, hence $\mathrm{Tr}_\chi(\mathcal{O}_\chi \hat{\rho}(0)) = 0$.

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This approximation entails replacing $\rho_{Ia}(t') \to \rho_{Ia}(t)$ in the time integral.

- Take the first term in last page as an exmaple.

$$-g^2 a(\vec{x},t)\int_0^t \frac{d\mathcal{K}(t')}{dt'}\,\hat{\rho}_{Ia}(t')\,dt' \quad ; \quad \mathcal{K}(t') \equiv \int_0^{t'} a(\vec{x}',t'')\,G^{>}(\vec{x}-\vec{x}',t-t'')dt''$$

- Integrate by parts.

$$-g^2 a(\vec{x},t)\mathcal{K}(t)\hat{\rho}_{Ia}(t) + g^2 a(\vec{x},t)\int_0^t \mathcal{K}(t')\,\frac{d\hat{\rho}_{Ia}(t')}{dt'}dt'$$

- In the second term $d\hat{\rho}_{Ia}(t')/dt' \propto g^2$ so this term yields a contribution that is formally of order g^4 and can be neglected to second order.

$$\dot{\hat{\rho}}(t) = -i[H_I,\hat{\rho}_I(0)] - \int_0^t [H_I(t),[H_I(t'),\hat{\rho}_I(\,t'\,)]]dt'$$

Quantum Master Equation

Rotating wave approximation

- The (ALP) field in the interaction picture $a_I(\vec{x},t)$ is
$$a_I(\vec{x},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left[b_{\vec{k}}\, e^{-i\omega_k t}\, e^{i\vec{k}\cdot\vec{x}} + b^{\dagger}_{\vec{k}}\, e^{i\omega_k t}\, e^{-i\vec{k}\cdot\vec{x}} \right]$$
where the operators $b_{\vec{k}}, b^{\dagger}_{\vec{k}}$ do not depend on time, and $\omega_k = \sqrt{k^2 + m_a^2}$.
- In writing the products $a_I(\vec{x},t)$, $a_I(\vec{x}',t')$, there two types of terms.
  - Slow terms, $b^{\dagger}_{\vec{q}}\, b_{\vec{q}}\, e^{i\omega_q(t-t')}$
  - Fast terms, $b^{\dagger}_{\vec{q}}\, b^{\dagger}_{-\vec{q}}\, e^{2i\omega_q t}\, e^{i\omega_q(t-t')}\,;\; b_{\vec{q}}\, b_{-\vec{q}}\, e^{-2i\omega_q t}\, e^{-i\omega_q(t-t')}$
- The extra rapidly varying phases $e^{\pm 2i\omega_q t}$ lead to rapid dephasing on time scales $\simeq 1/\omega_q$ and do not yield resonant (nearly energy conserving) contributions.
- Keeping only the slow terms defines the **rotating wave approximation**.

Trace over the time-dependent reduced density matrix.

- Amplitude

$$\frac{d}{dt}\langle b_{\vec{k}}\rangle(t) = \left[-i\,\Delta_k(t) - \frac{\Gamma_k(t)}{2}\right]\langle b_{\vec{k}}\rangle(t) \qquad \frac{d}{dt}\langle b^\dagger_{\vec{k}}\rangle(t) = \left[i\,\Delta_k(t) - \frac{\Gamma_k(t)}{2}\right]\langle b^\dagger_{\vec{k}}\rangle(t)$$

- Variance

$$\frac{dN_q(t)}{dt} = \mathrm{Tr}_a\left\{b^\dagger_{\vec{q}}\, b_{\vec{q}}\,\dot{\hat{\rho}}_{Ia}(t)\right\} = -\Gamma_q(t)N_q(t) + \Gamma^<_q(t)$$

$$\frac{d}{dt}\langle b_{\vec{k}}\ b_{-\vec{k}}\rangle(t) = \left[-2i\,\Delta_k(t) - \Gamma_k(t)\right]\langle b_{\vec{k}}\ b_{-\vec{k}}\rangle(t) \quad \xrightarrow{h.c.} \quad \langle b^\dagger_{\vec{k}}\ b^\dagger_{-\vec{k}}\rangle(t)$$

- where $\Gamma_q = \Gamma_q^> - \Gamma_q^<$.

$$\Delta_q(t) = \frac{g^2}{2\omega_q} \int \frac{dq_0}{2\pi}\, \varrho(q_0, q)\, \frac{\Big[1 - \cos[(\omega_q - q_0)t]\Big]}{(\omega_q - q_0)}$$

$$\Gamma_q^>(t) = \frac{g^2}{\omega_q} \int \frac{dq_0}{2\pi}\, \varrho(q_0, q)\, \big[1 + n(q_0)\big] \frac{\sin[(\omega_q - q_0)t]}{(\omega_q - q_0)}$$

$$\Gamma_q^<(t) = \frac{g^2}{\omega_q} \int \frac{dq_0}{2\pi}\, \varrho(q_0, q)\, n(q_0) \frac{\sin[(\omega_q - q_0)t]}{(\omega_q - q_0)}$$

- Results are recovered in long time limit.

Quantum Master Equation

A complementary check

- Solve the quantum master equation with *Markove approximation*, *rotating wave approximation*.

$$\dot{\hat{\rho}}_I(t) = -i[H_I, \hat{\rho}_I(0)] - \int_0^t [H_I(t), [H_I(t'), \hat{\rho}_I(t')]]dt'$$

- Recover decay rate, self energy, decoherence, thermalization
  - More convenient to study coherence.

- Because of the **breakdown** of approximations, the QME approach misses
  - Inverted phase transition
  - Requirements of higher order derivative terms

1. Decompose a coherent ALP as it amplitude expectation value $\bar{a}$ and quantum fluctuations $\tilde{a}$ around the amplitude.

$$a(\vec{x},t) = \bar{a}(\vec{x},t) + \tilde{a}(\vec{x},t)$$

2. Neglect the fluctuations $\tilde{a}(x)$ (**mean field approximation**).

$$\mathcal{L}_I = -g\,\bar{a}\,\mathcal{O}_\chi \qquad \text{or} \qquad H_I(t) = g\int d^3x\;\bar{a}(\vec{x},t)\,\mathcal{O}_\chi(\vec{x})$$

3. Result in a system driven by a classical source

$$\rho_\chi(t) = U(t,t_0)\,\rho_\chi(t_0)\,U^{-1}(t,t_0)$$

with $U(t,t_0)$ being the evolution operator.

$$i\frac{d}{dt}U(t,t_0) = (H_\chi + H_I(t))\,U(t,t_0) \quad ; \quad U(t_0,t_0) = 1$$

4. Up to the linear order, the expectation value $\langle \mathcal{O}_\chi(\vec{x}) \rangle(t) = \mathrm{Tr}\big(\mathcal{O}_\chi(\vec{x})\, \rho_\chi(t)\big)$ is

$$\langle \mathcal{O}_\chi(\vec{x}) \rangle(t) = \langle \mathcal{O}(\vec{x}) \rangle(t_0) + \int d^3x' \int_{t_0}^{t} \Xi(\vec{x} - \vec{x}', t - t')\, \bar{a}(\vec{x}', t')\, dt' + \cdots$$

where the **linear response kernel**, namely the **dynamical susceptibility**, is

$$\Xi(\vec{x} - \vec{x}', t - t') = -ig\,\mathrm{Tr}\Big(\Big[\mathcal{O}_\chi^{(H_\chi)}(\vec{x}, t), \mathcal{O}_\chi^{(H_\chi)}(\vec{x}', t')\Big]\, \rho_\chi(t_0)\Big) \quad ; \quad t > t'$$

The superscript (H_χ) means Heisenberg picture in absence of the source $\bar{a}$.

5. Structure of $\Xi(x - x')$ and self-energy $\Sigma(x - x')$ are similar.

$$\Sigma(\vec{x} - \vec{x}', t - t') = g\, \Xi(\vec{x} - \vec{x}', t - t')$$

A Complementary Check

Exploit the exact relation

Keldysh Formalism

Linear Response Theory

$$
\begin{tikzcd}
& \mathcal{L} \arrow[dl] \arrow[dr] & \\
\overline{a}(\vec{x}, t) \arrow[dr] & & \langle \mathcal{O}_\chi \rangle = \Xi(\vec{x}, t) \circledast \overline{a}(\vec{x}, t) \arrow[dl] \\
& \langle \vec{E} \cdot \vec{B} \rangle &
\end{tikzcd}
$$

Exact Relation

Ansatz: Exponentially Decaying Oscillation

Complementary Check Using The Exact Relation

The exact relation in momentum space

$$\langle \mathcal{O}_\chi(\vec{x},t)\rangle = -\frac{1}{g}\left[\frac{\partial^2}{\partial t^2}\,\overline{a}(\vec{x},t) + k^2\overline{a}(\vec{x},t) + m_{0a}^2\,\overline{a}(\vec{x},t)\right]$$

Plug in

$$\overline{a}(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\,\overline{a}_k(t), \qquad \overline{a}_k(t) = \left[A_k e^{-i\omega_k(t-t_0)} + A_k^* e^{i\omega_k(t-t_0)}\right] e^{-\frac{\Gamma_k}{2}(t-t_0)}$$

Result in

$$\langle \mathcal{O}_\chi(\vec{x},t)\rangle_k = \frac{1}{g}[\Omega_k^2 - k^2 - m_{0a}^2]e^{-i\Omega_k(t-t_0)} + h.c. \qquad \Omega_k = \omega_k - i\frac{\Gamma_k}{2}$$

Ω_k satisfies $\Omega_k^2 - k^2 - m_{0a}^2 = \Sigma_k(\Omega_k)$. Thus, to leading order,

$$\langle \mathcal{O}_\chi(\vec{x},t)\rangle_k = \frac{1}{g}\Sigma_k(\Omega_k)e^{-i\Omega_k(t-t_0)} + h.c. \approx \frac{1}{g}[\Sigma_k(\omega_k)\overline{a}_k(t) + \Gamma_k\dot{\overline{a}}_k(t)]$$