

Dark photon conversions in the presence of multiple resonances

Phenomenology Symposium 2023

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Dark photons and ordinary photons

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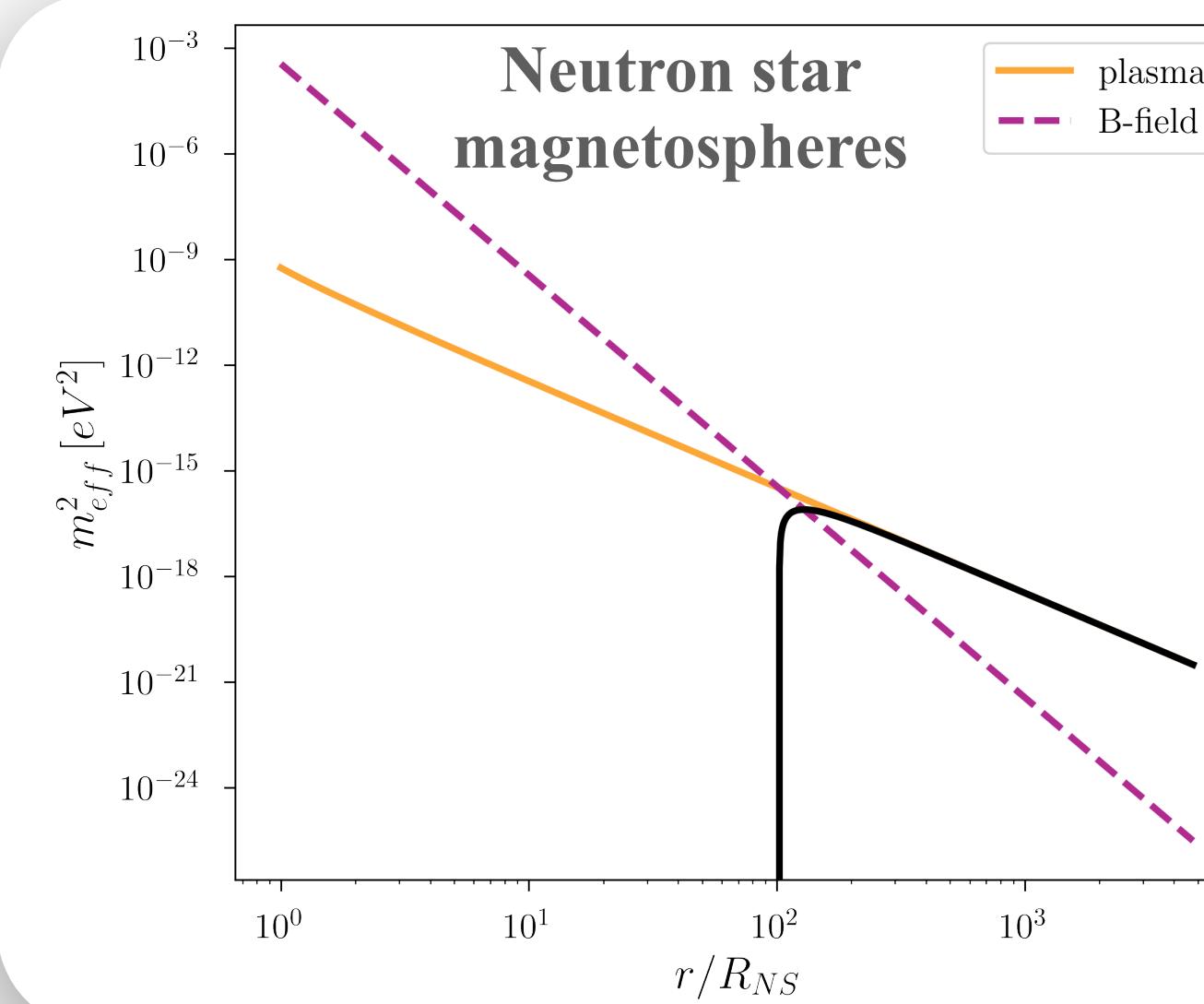
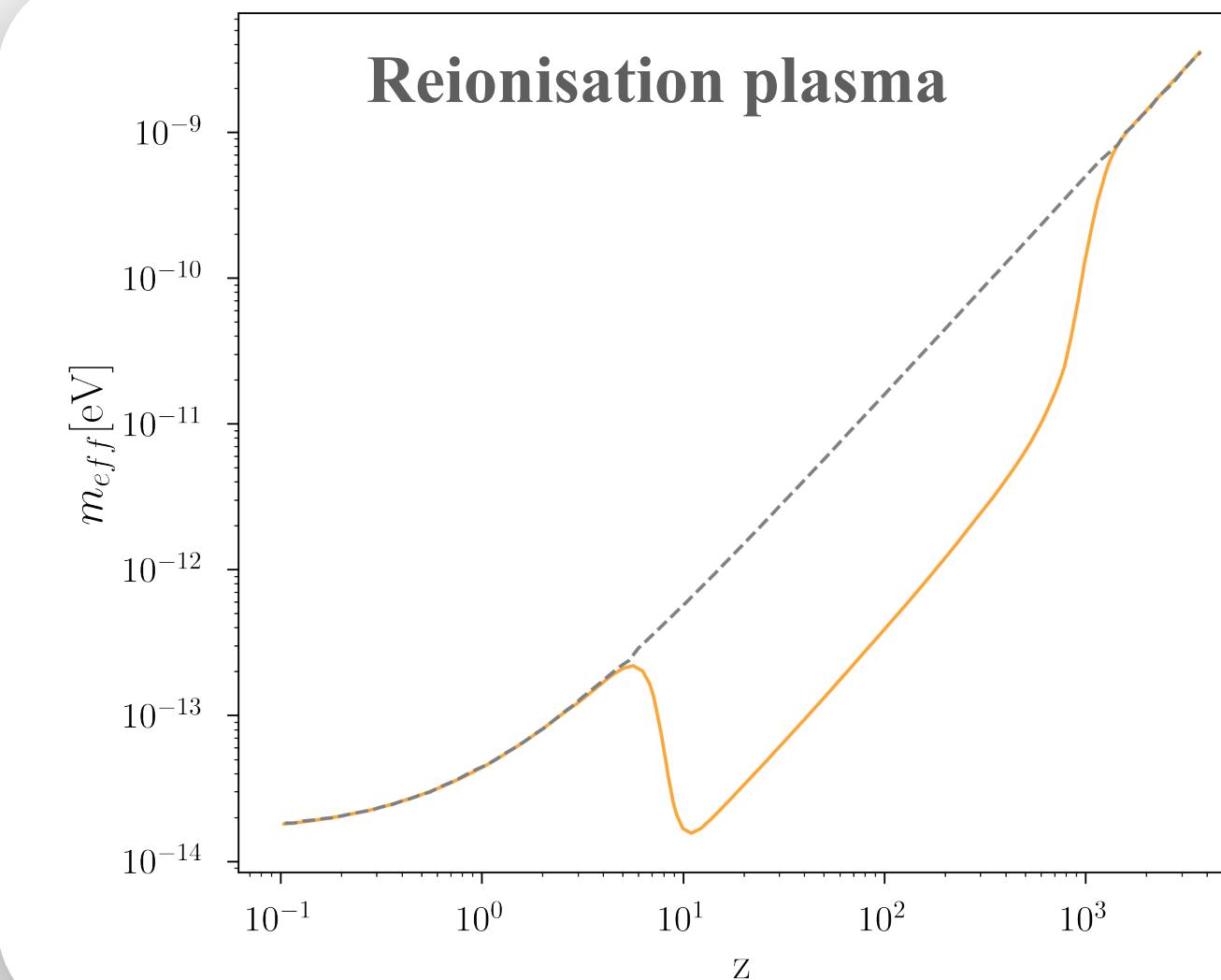
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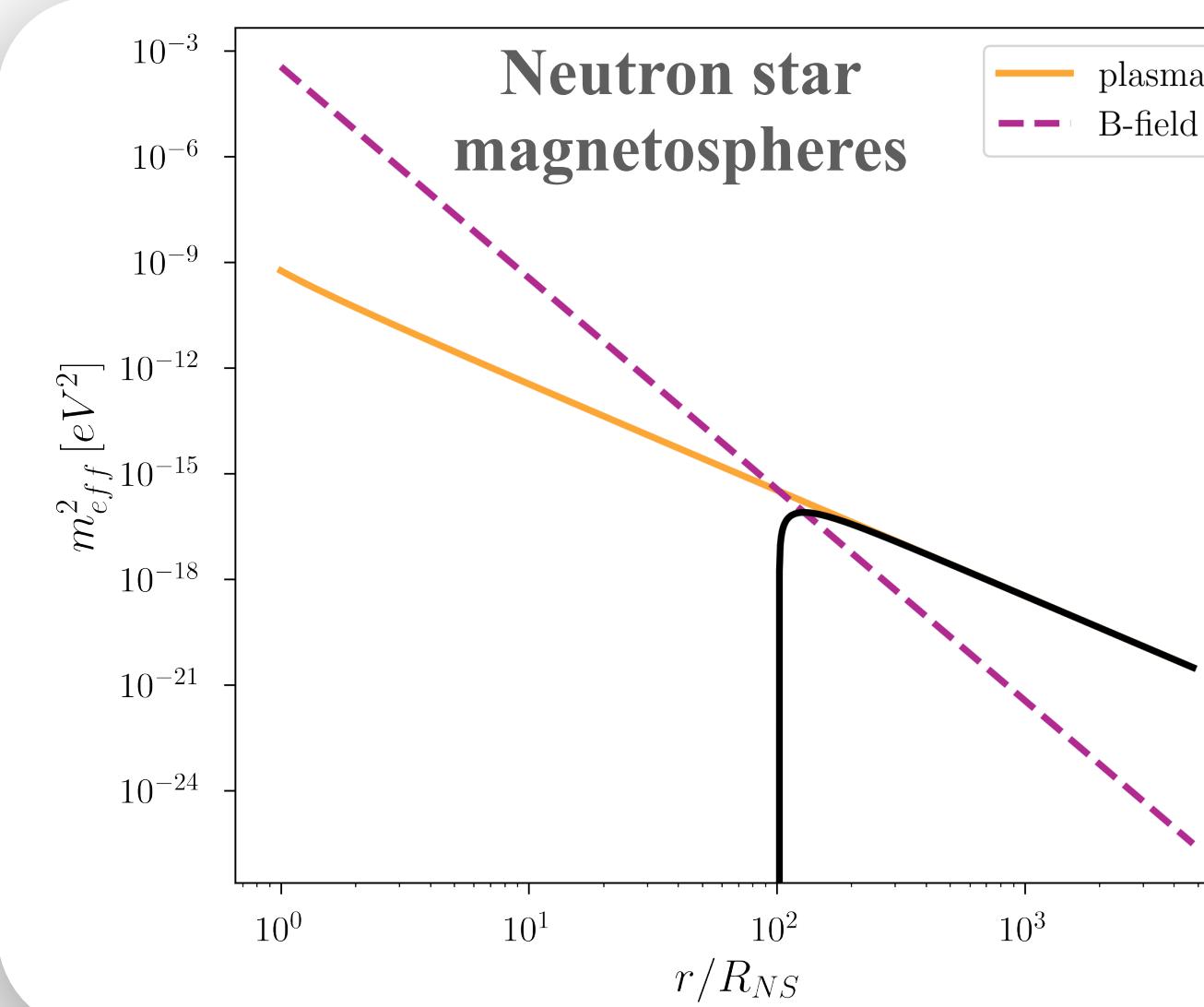
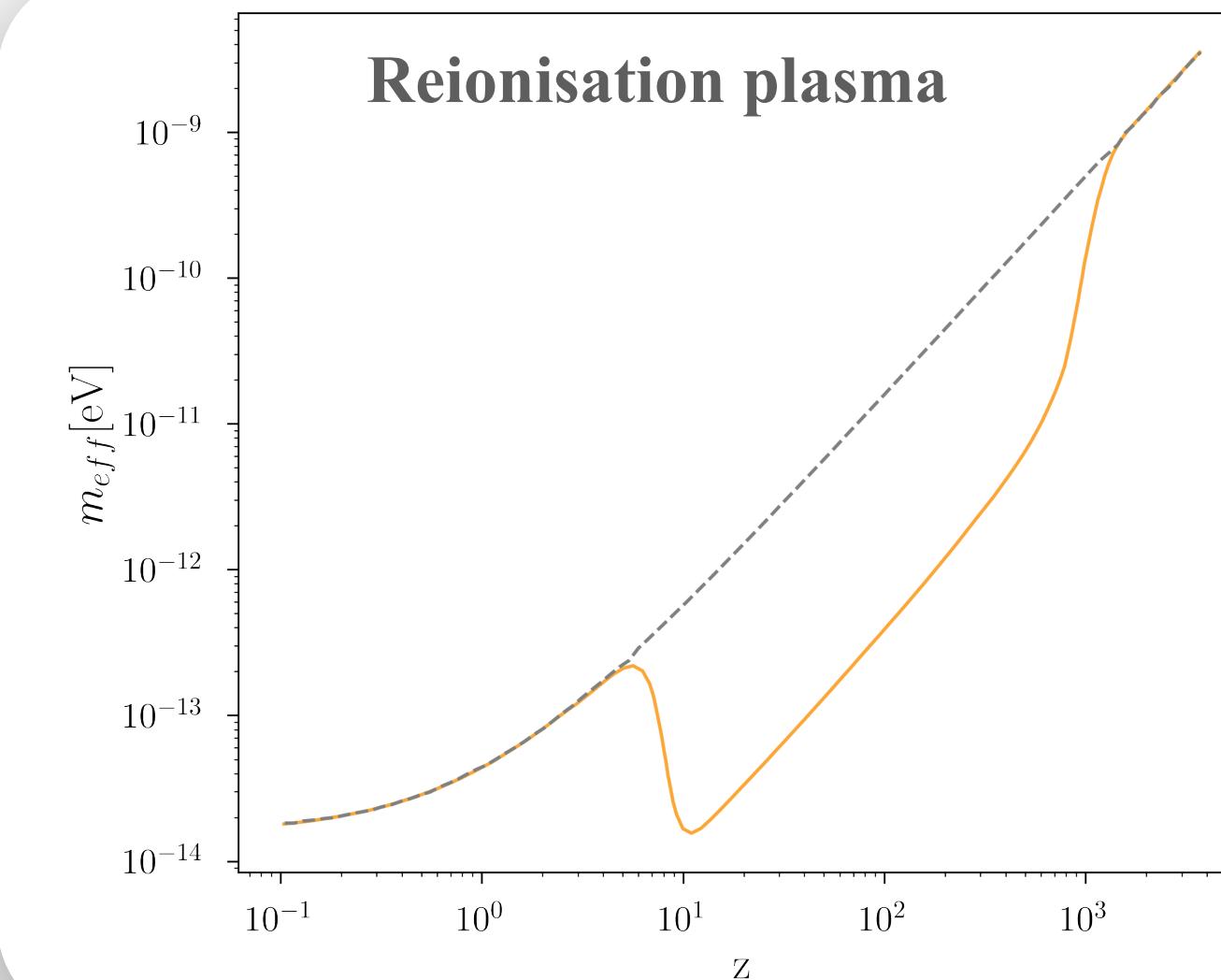
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- Photons can acquire an effective (non-zero) mass in the presence of a medium. This can heavily modify the mixing properties.
- Moreover, this induced effective mass may not be constant and can vary with space and time.
- Hence, a careful treatment of dark photon-photon oscillations in such potential profiles is important.



Photon-dark photon Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} + \frac{1}{2}\epsilon F_{\mu\nu}X^{\mu\nu} + \frac{1}{2}m_{\gamma'}^2 A'_{\mu}A'^{\mu} + eJ^{\mu}A_{\mu}$$

A^{μ} : photon field

A'^{μ} : dark photon field

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“Kinetic mixing term”

Dark Photon oscillation

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} + \frac{1}{2} \begin{pmatrix} A_1^\mu & A_2^\mu \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & m_{\gamma'}^2 \end{pmatrix} \begin{pmatrix} A_{1\mu} \\ A_{2\mu} \end{pmatrix} + eJ^\mu (A_{1\mu} + \epsilon A_{2\mu})$$

$$A_1^\mu = A^\mu - \epsilon A'^\mu$$

“Mass eigenbasis”

$$A_2^\mu = A'^\mu$$

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$$A_a^\mu = A_1^\mu + \epsilon A_2^\mu : \text{active state}$$

“Interaction eigenbasis”

$$A_s^\mu = A_1^\mu - \epsilon A_2^\mu : \text{sterile state}$$

Schrodinger equation

$$i\partial_z \begin{pmatrix} A_a \\ A_s \end{pmatrix} = H \begin{pmatrix} A_a \\ A_s \end{pmatrix}$$

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Diagonal

$$H_1 = \frac{1}{2\omega} \begin{pmatrix} 0 & \epsilon m_{\gamma'}^2 \\ \epsilon m_{\gamma'}^2 & 0 \end{pmatrix}$$

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Conversion probability

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$$\Phi(z) = \int_{z_i}^z dz' \left(\frac{m_{\gamma'}^2}{2\omega} - \frac{m_{eff}^2}{2\omega} \right)$$

“Accumulated relative phase”

Dark photon phase

Photon phase

Conversion probability

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$$\Phi(z) = \int_{z_i}^z dz' \frac{m_{\gamma'}^2 - m_{eff}^2}{2\omega}$$

“Accumulated relative phase”

- In vacuum, the photon state is massless and we have $m_{eff}^2 = 0$

$$\langle P_{\gamma \leftrightarrow \gamma'}^{vac} \rangle = 2\epsilon^2$$

Resonance and stationary phase approximation

$$P_{\gamma \leftrightarrow \gamma'} = \epsilon^2 \left| \int_{z_i}^z dz' \frac{m_{\gamma'}^2}{2\omega} e^{-i\Phi(z')} \right|^2$$

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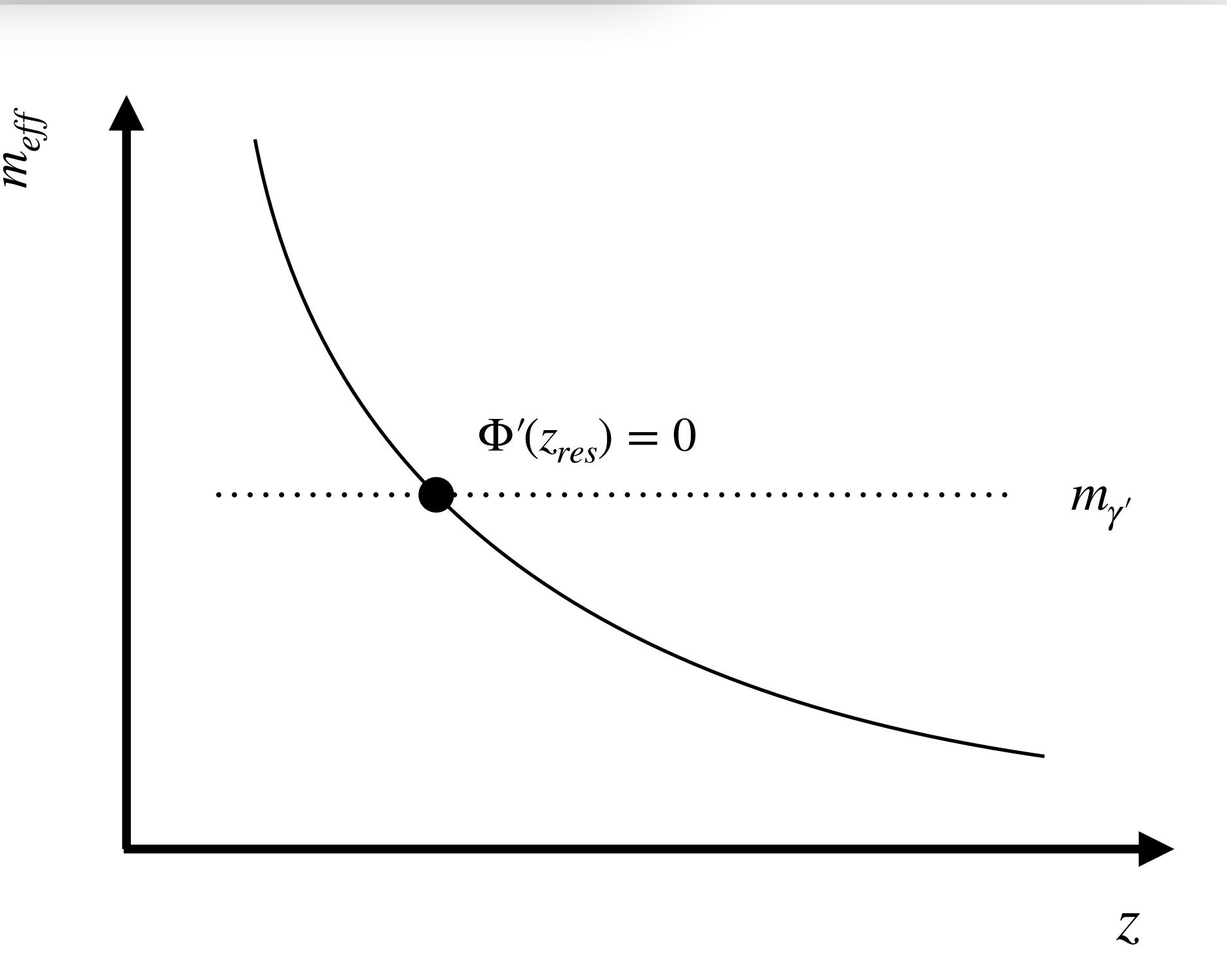
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- Except at stationary points, $\Phi' = 0 \longrightarrow m_{eff} = m_{\gamma'}$ “MSW effect”
- Integral gets most of it's contribution from stationary points

Resonance and stationary phase approximation

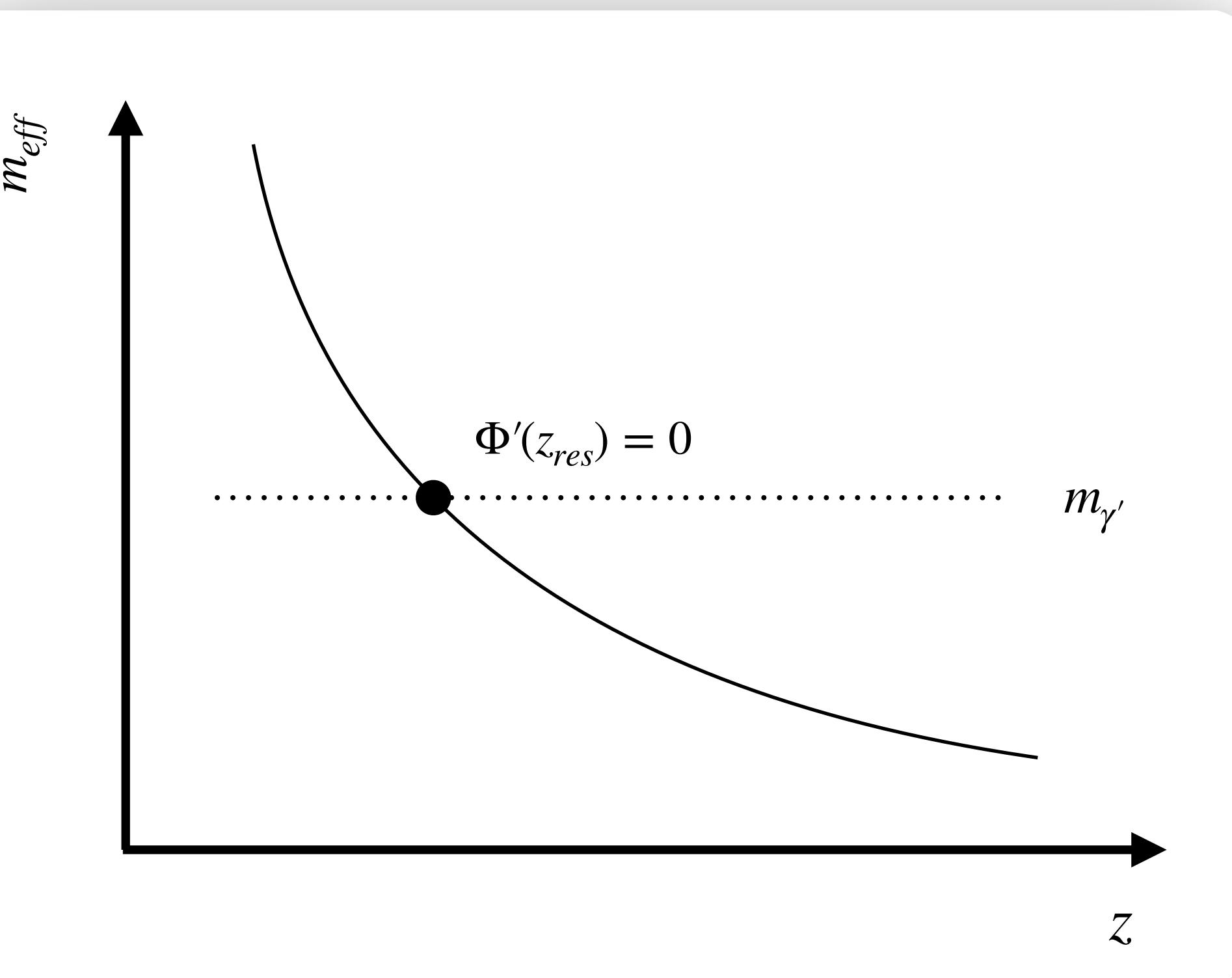
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Resonance and stationary phase approximation

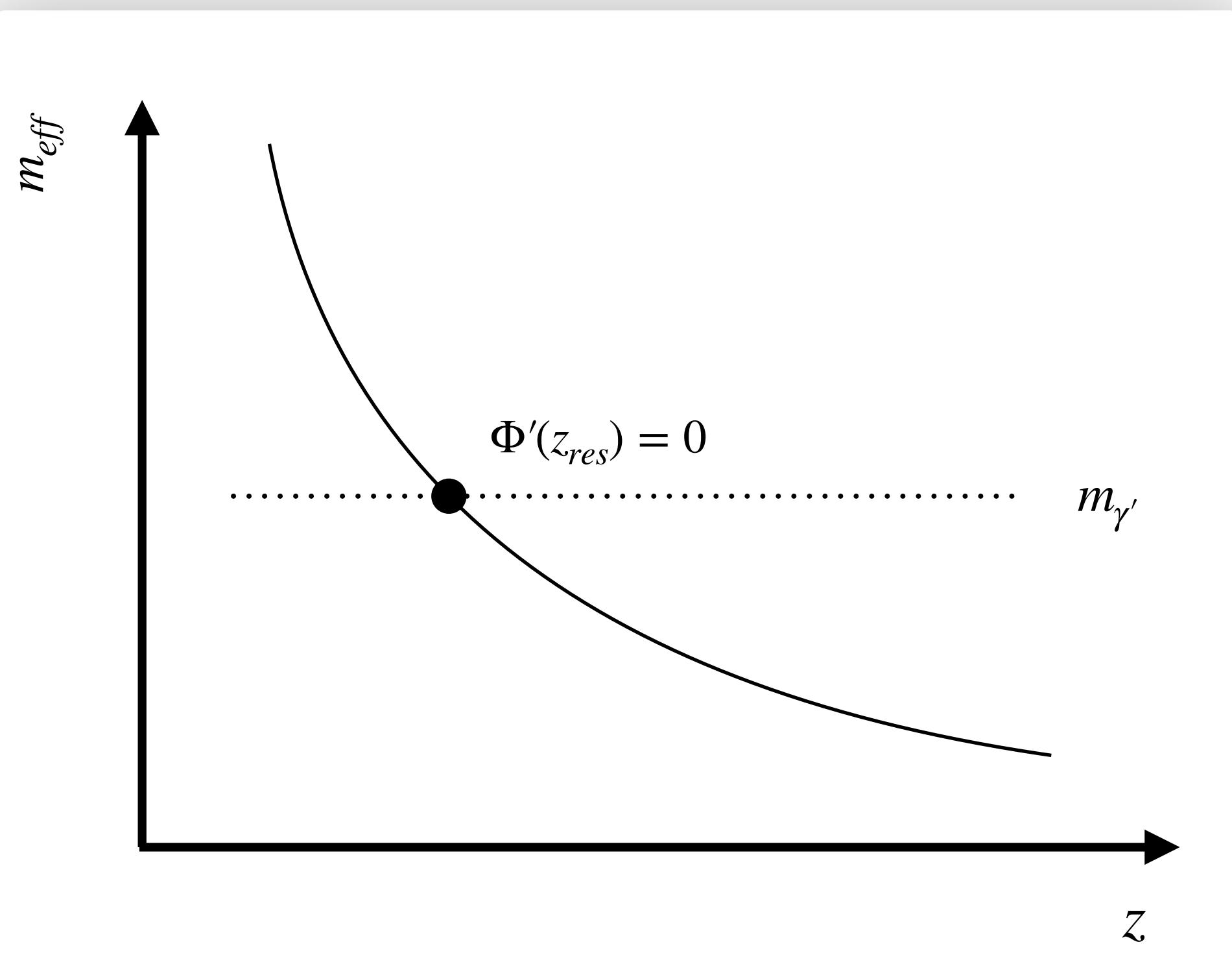
$$P_{\gamma \leftrightarrow \gamma'} \approx \epsilon^2 \left| \sqrt{\frac{2\pi}{|\Phi^{(2)}(z_{res})|}} \frac{m_{\gamma'}^2}{2\omega} e^{-i\Phi(z_{res})} \right|^2$$



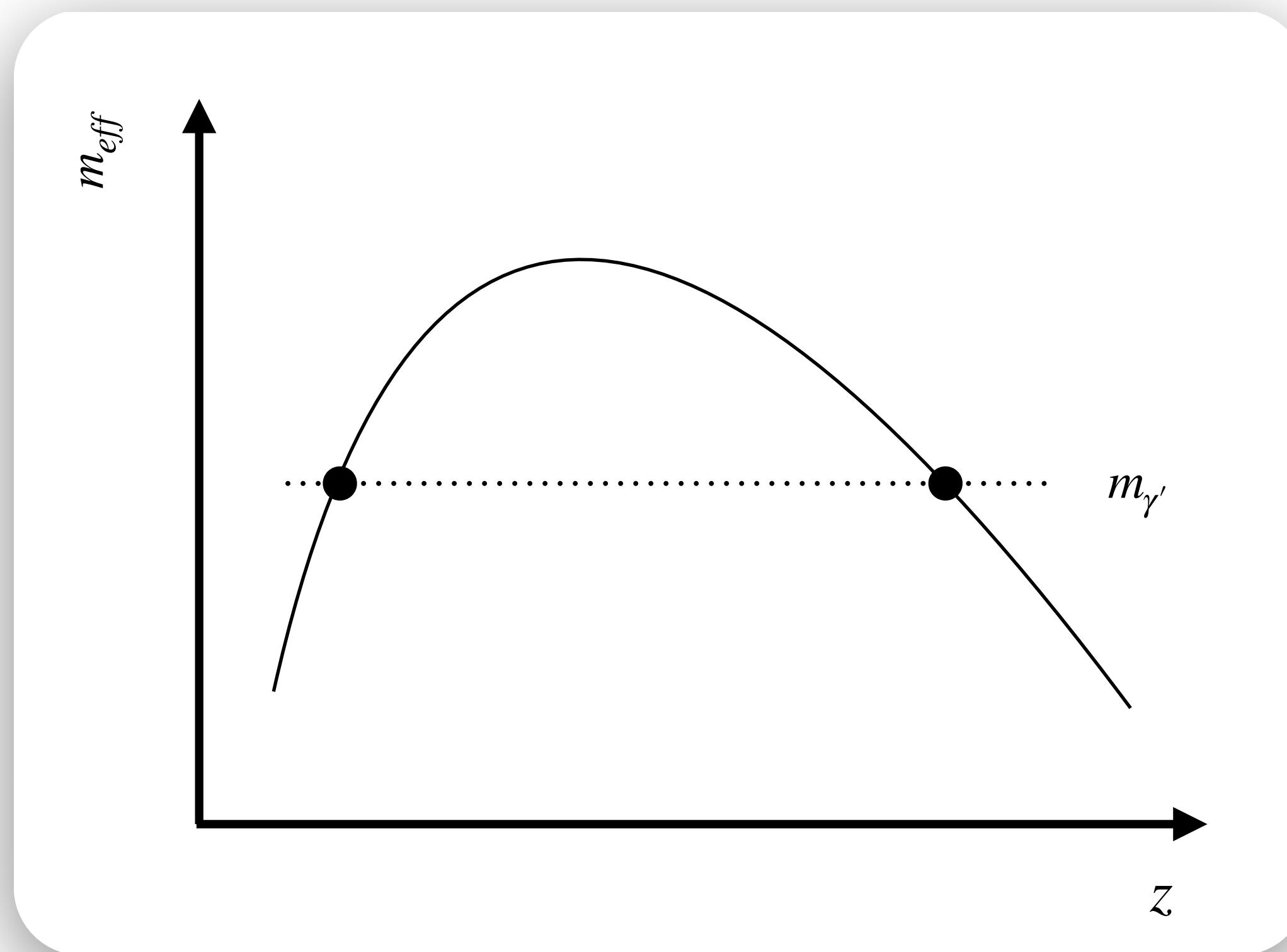
Resonance and stationary phase approximation

$$P_{\gamma \leftrightarrow \gamma'} \approx \epsilon^2 A^2 \quad \text{with} \quad A \equiv \sqrt{\frac{2\pi}{|\Phi^{(2)}(z_{res})|}} \left(\frac{m_{\gamma'}^2}{2\omega} \right)$$

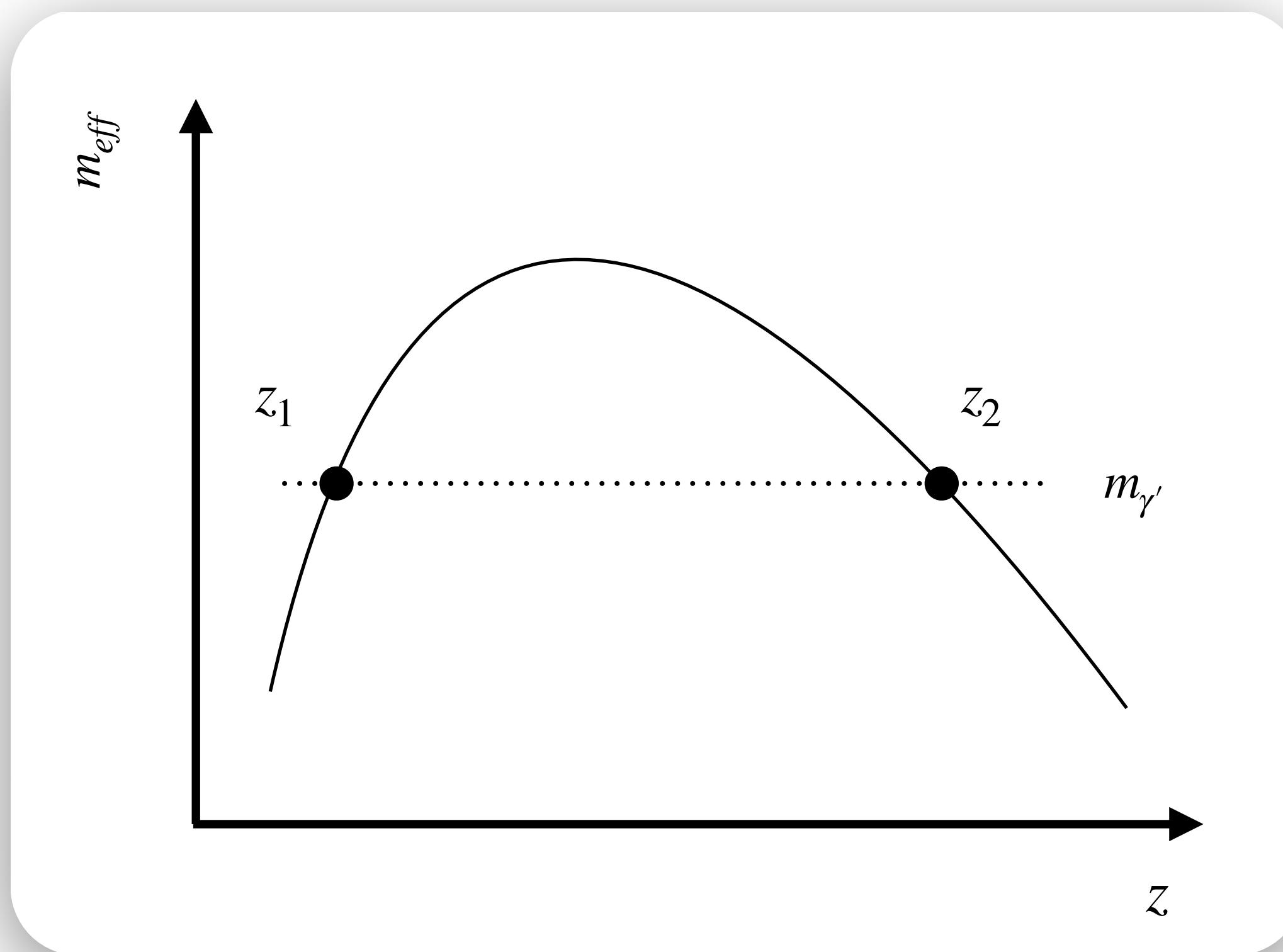
“Landau-Zener”



Non-monotonic profiles and multiple resonances

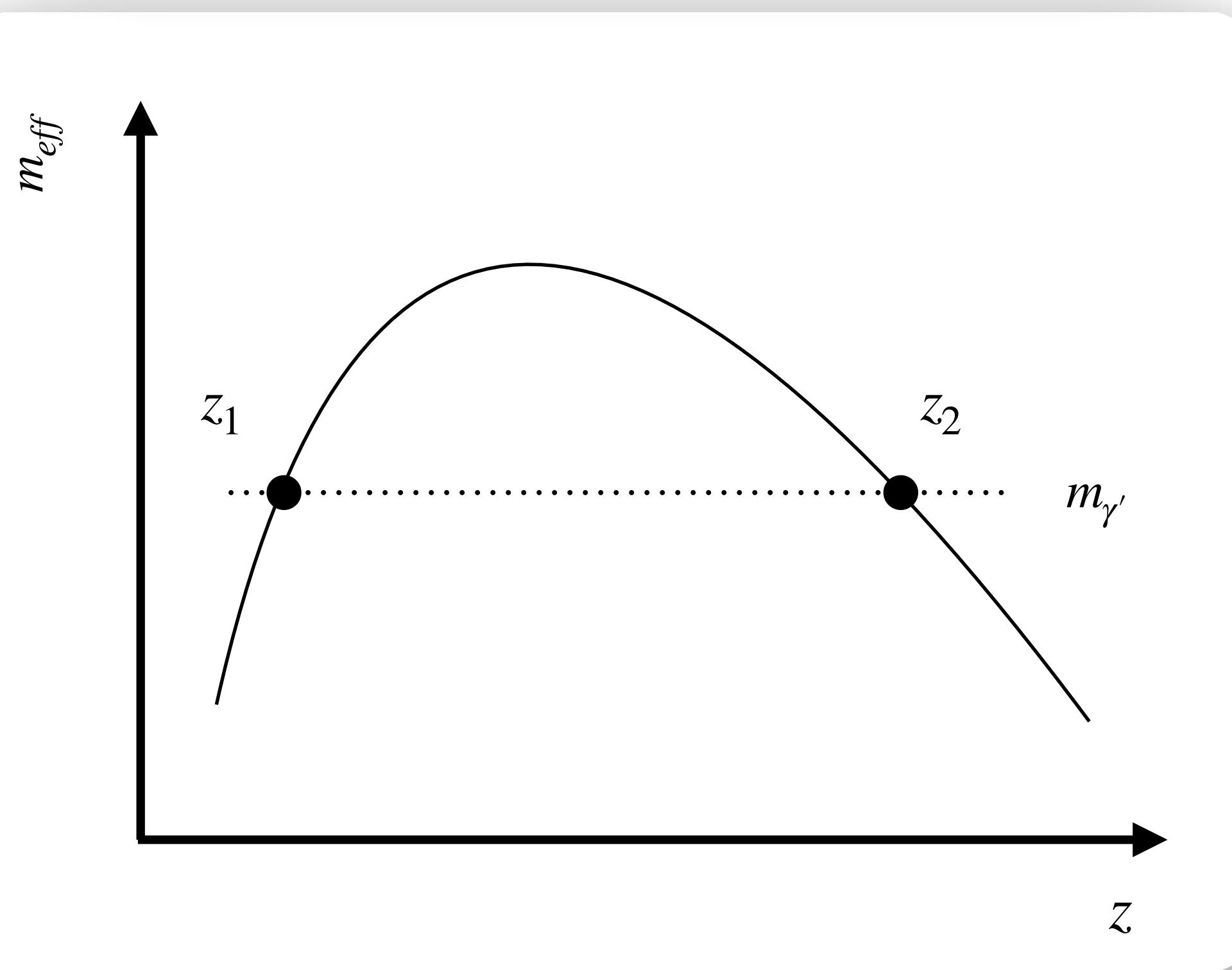


Non-monotonic profiles and multiple resonances



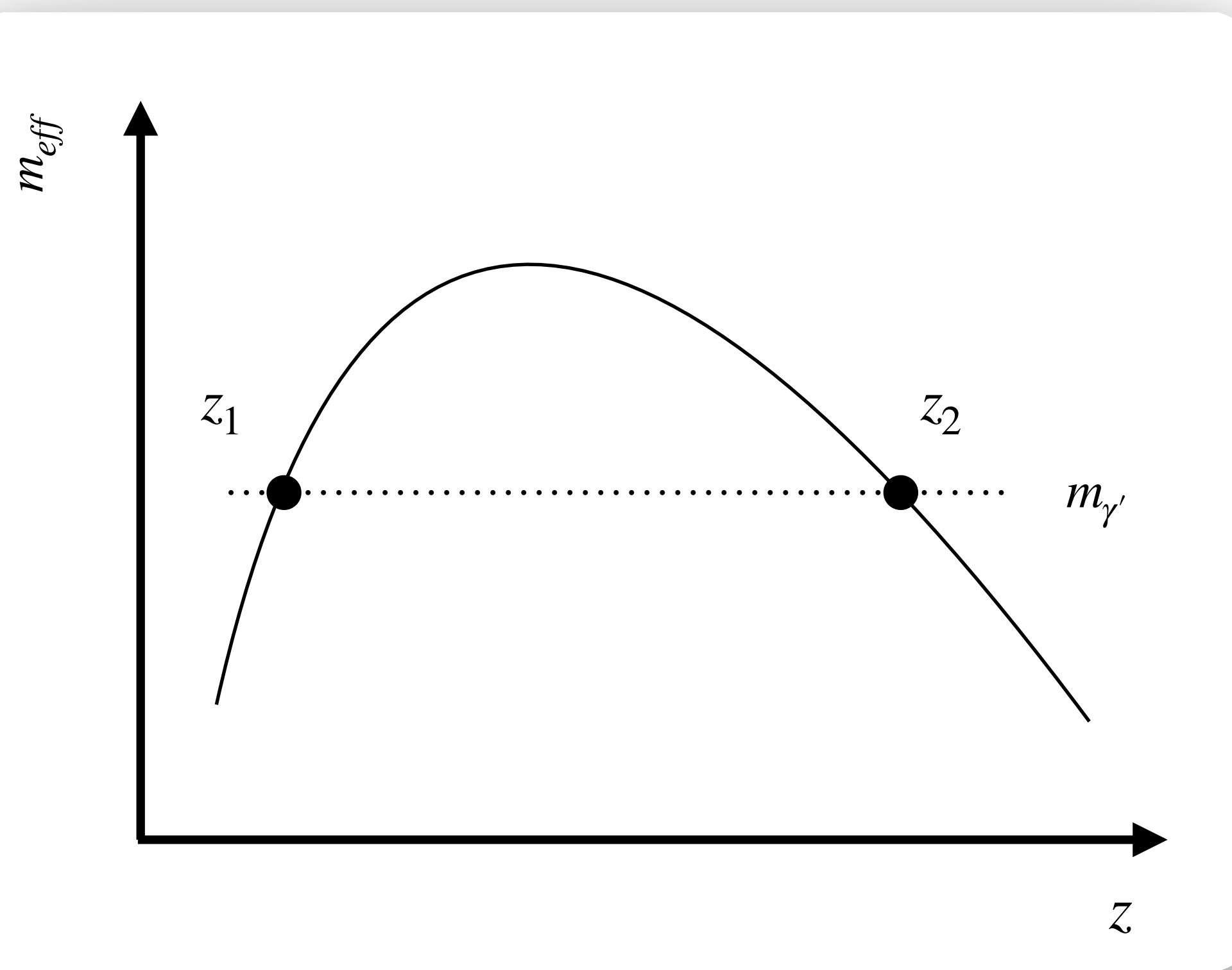
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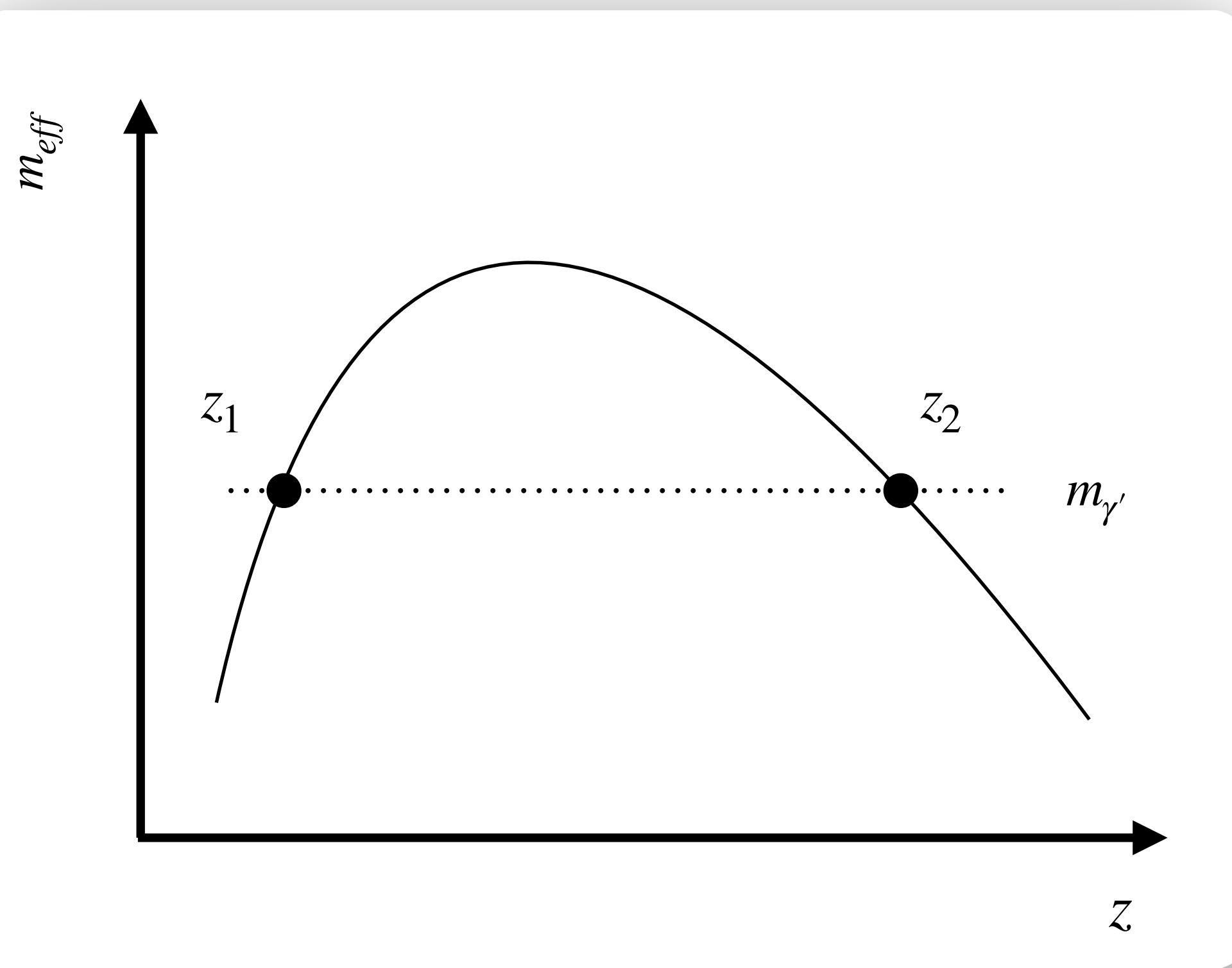
Non-monotonic profiles and multiple resonances

$$P_{\gamma \leftrightarrow \gamma'} \approx \epsilon^2 \left| \sum_n \sqrt{\frac{2\pi}{|\Phi^{(2)}(z_n)|}} \frac{m_{\gamma'}^2}{2\omega} e^{-i\Phi(z_n)} \right|^2$$



Non-monotonic profiles and multiple resonances

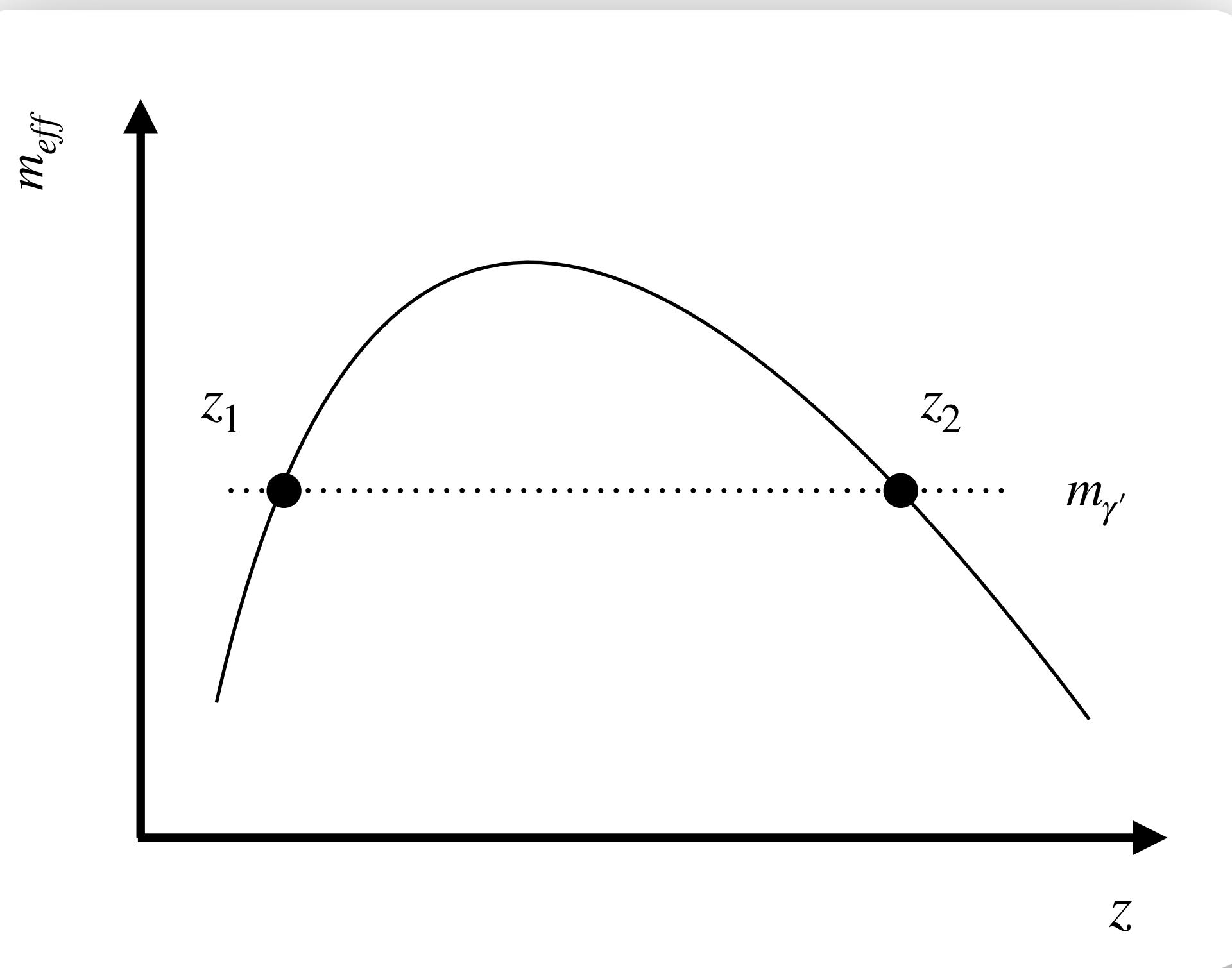
$$P_{\gamma \leftrightarrow \gamma'} = \epsilon^2 \left(\sum_n A_n^2 + 2 \sum_{n < k} A_n A_k \cos \Phi_{nk} \right)$$



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“Sum of LZ”

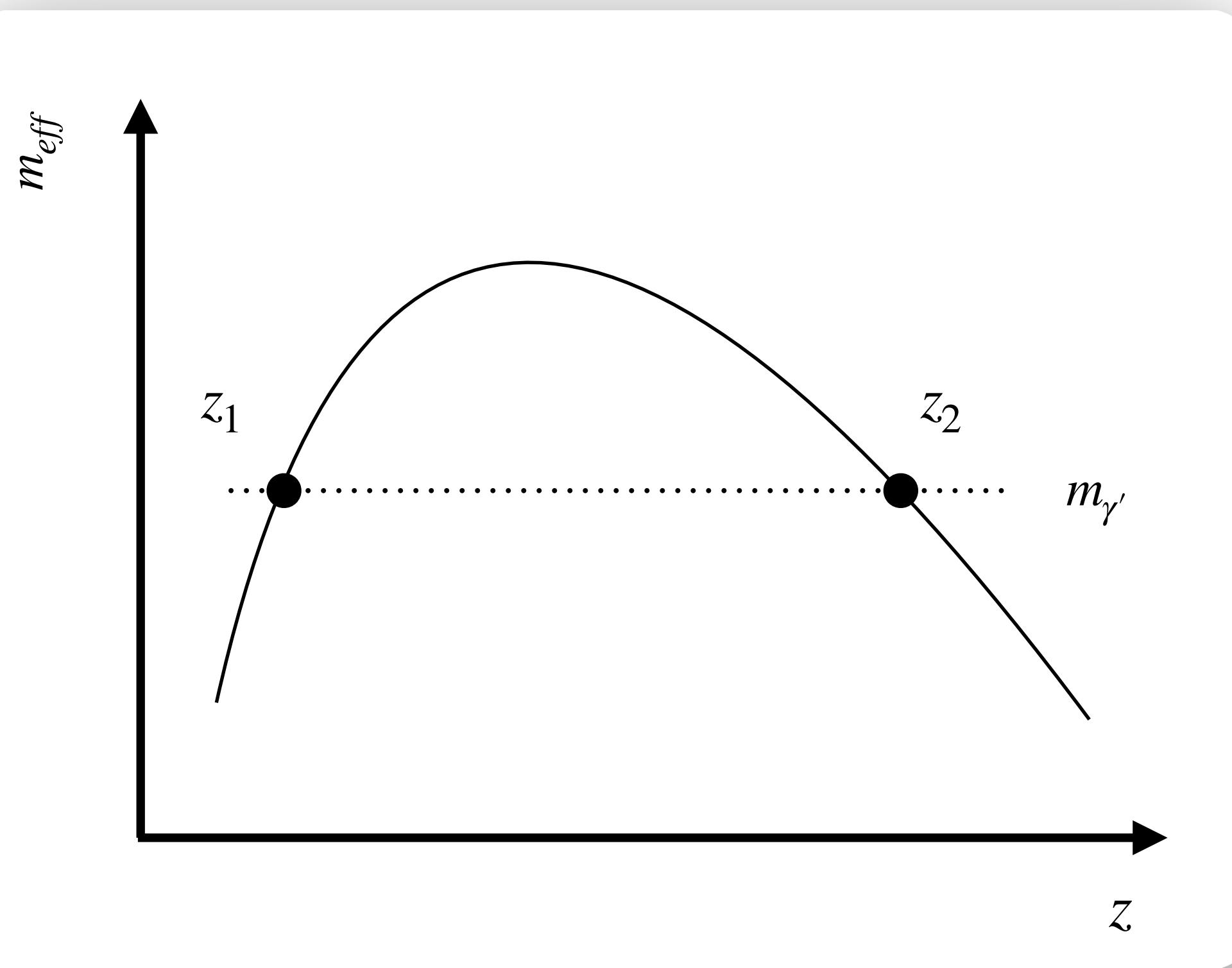


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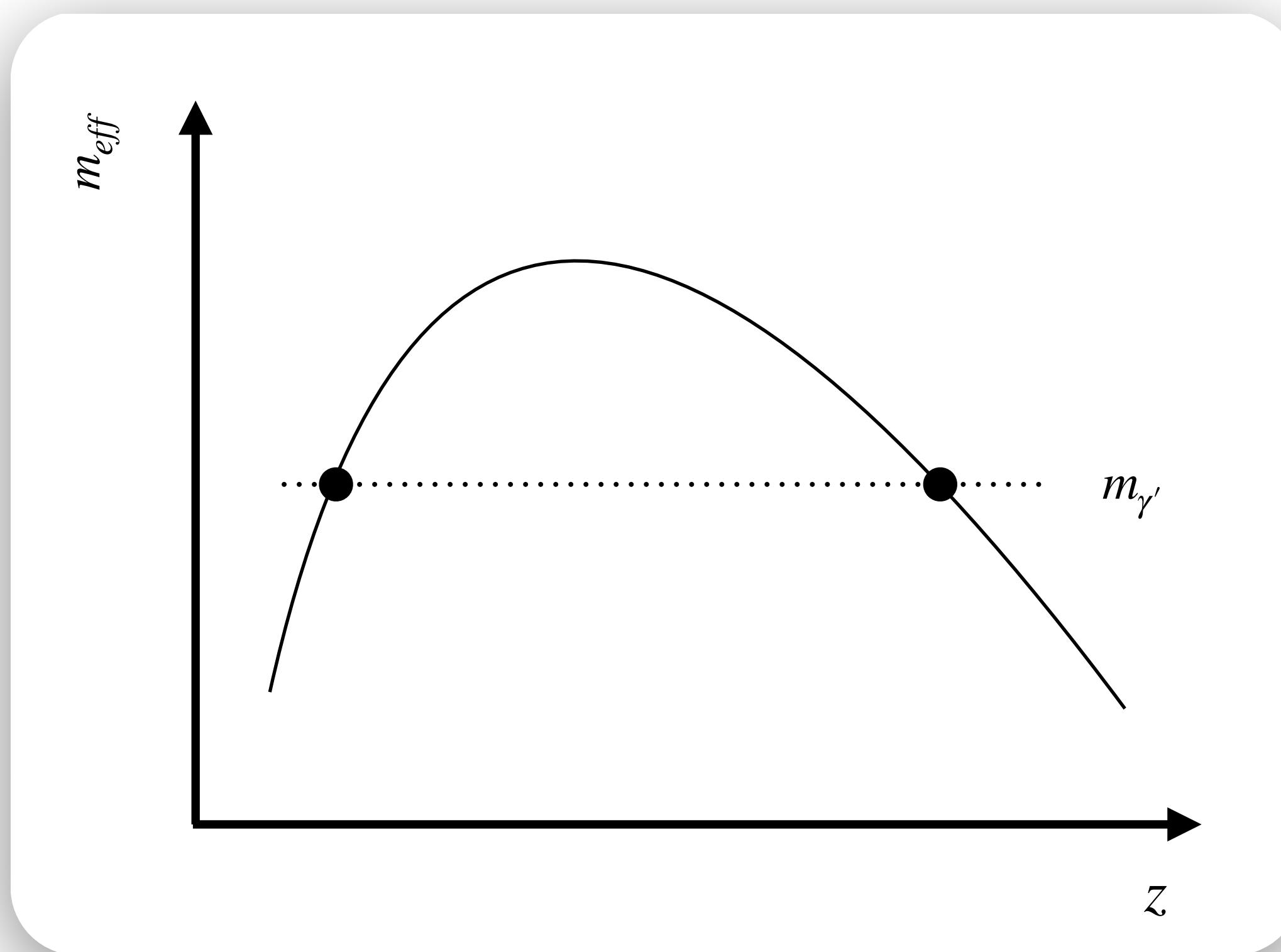
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“Phase effects”

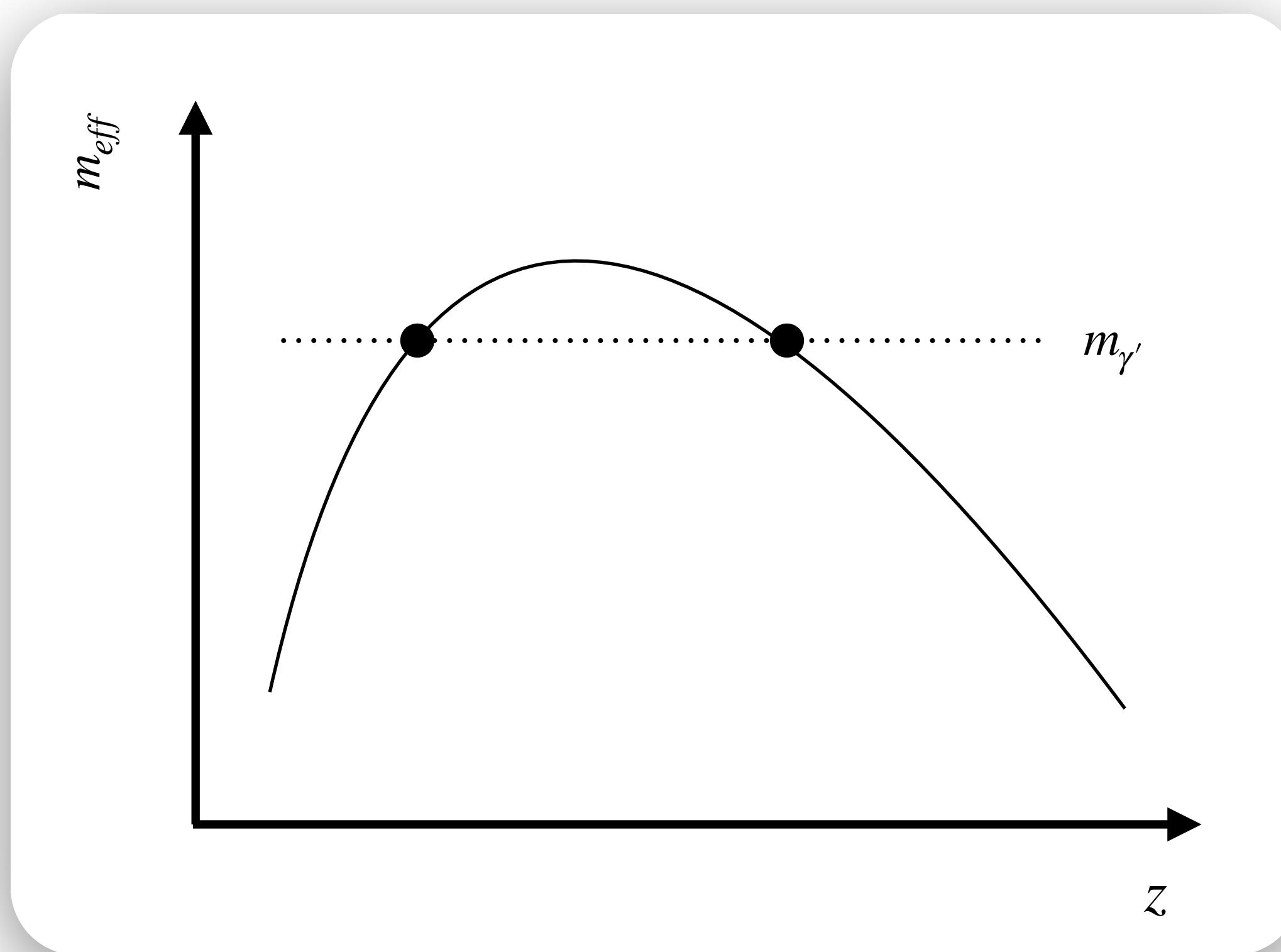
Dashgupta & Dighe (2007)



Breakdown of LZ

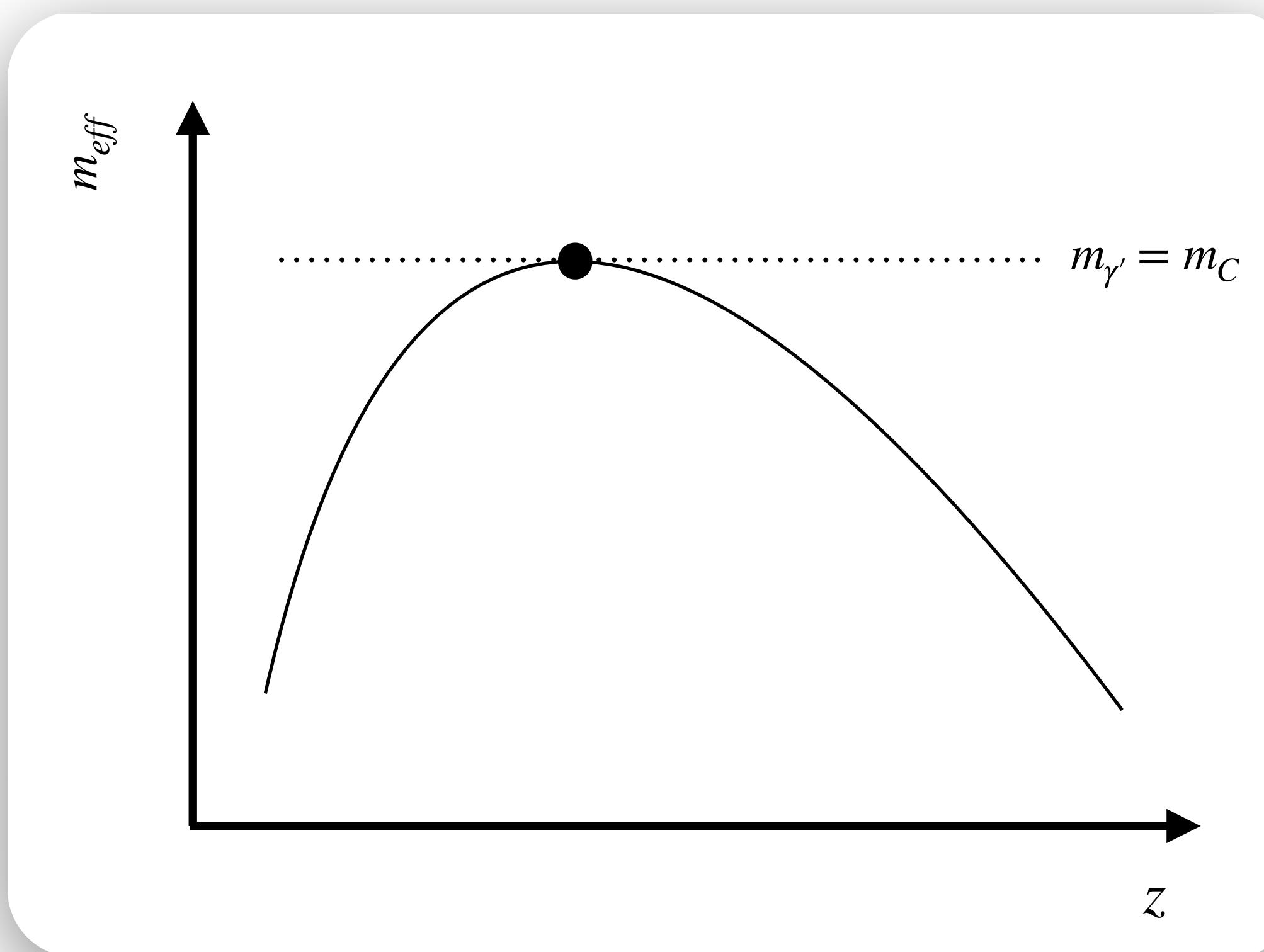


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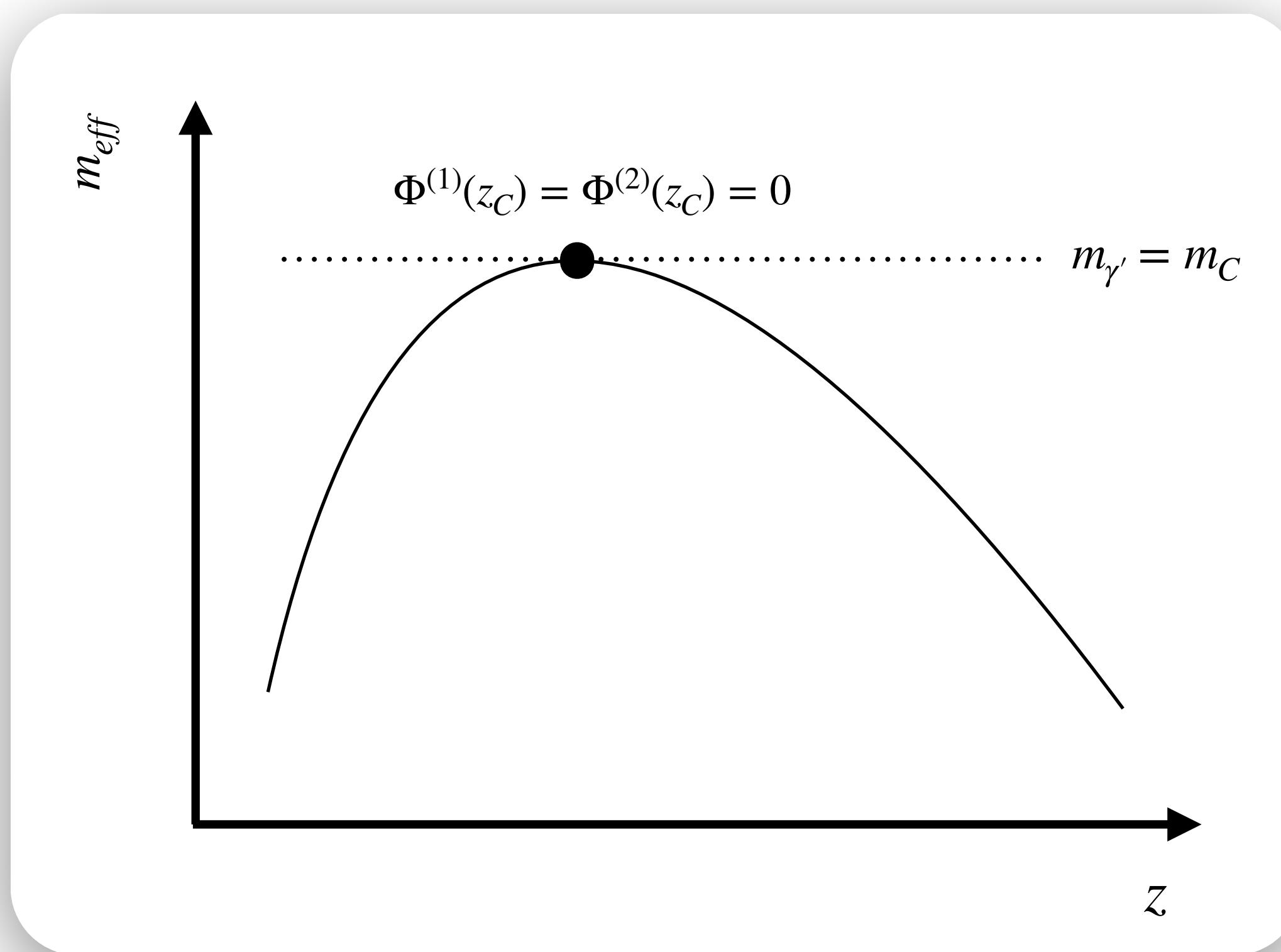


Breakdown of LZ

“Critical point”

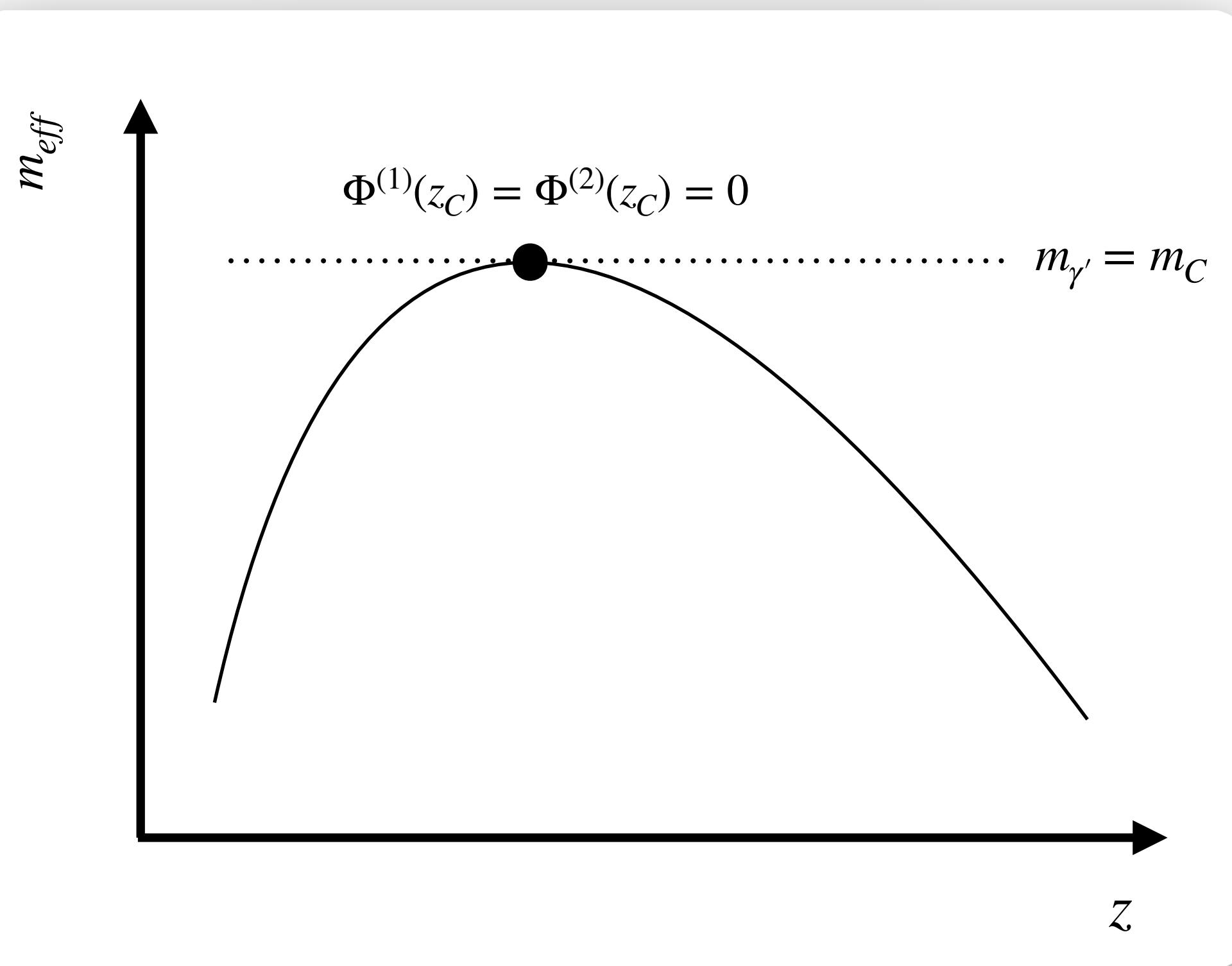


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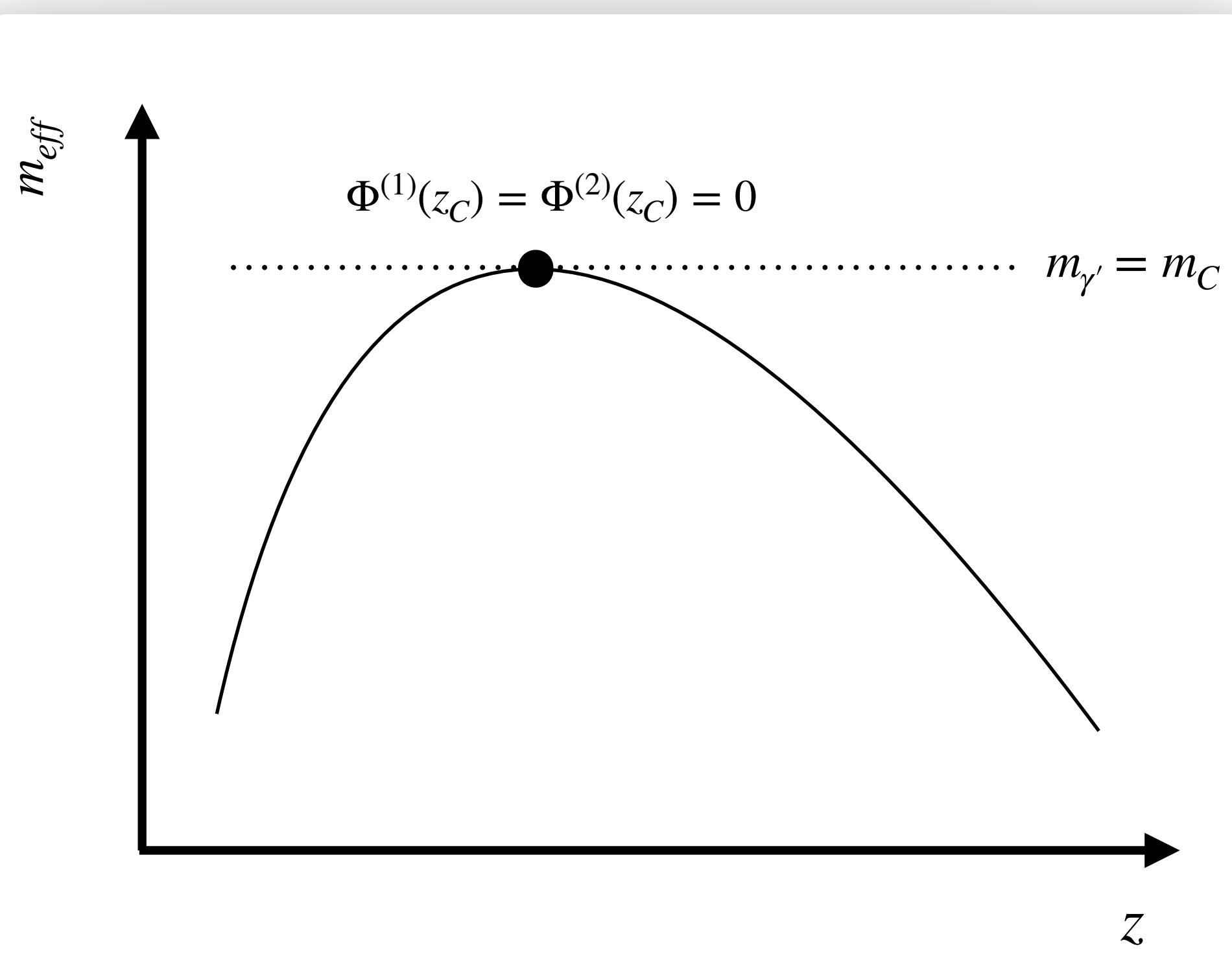
$$A_n \equiv \sqrt{\frac{2\pi}{|\Phi^{(2)}(z_n)|}} \left(\frac{m_{\gamma'}^2}{2\omega} \right)$$



Breakdown of LZ

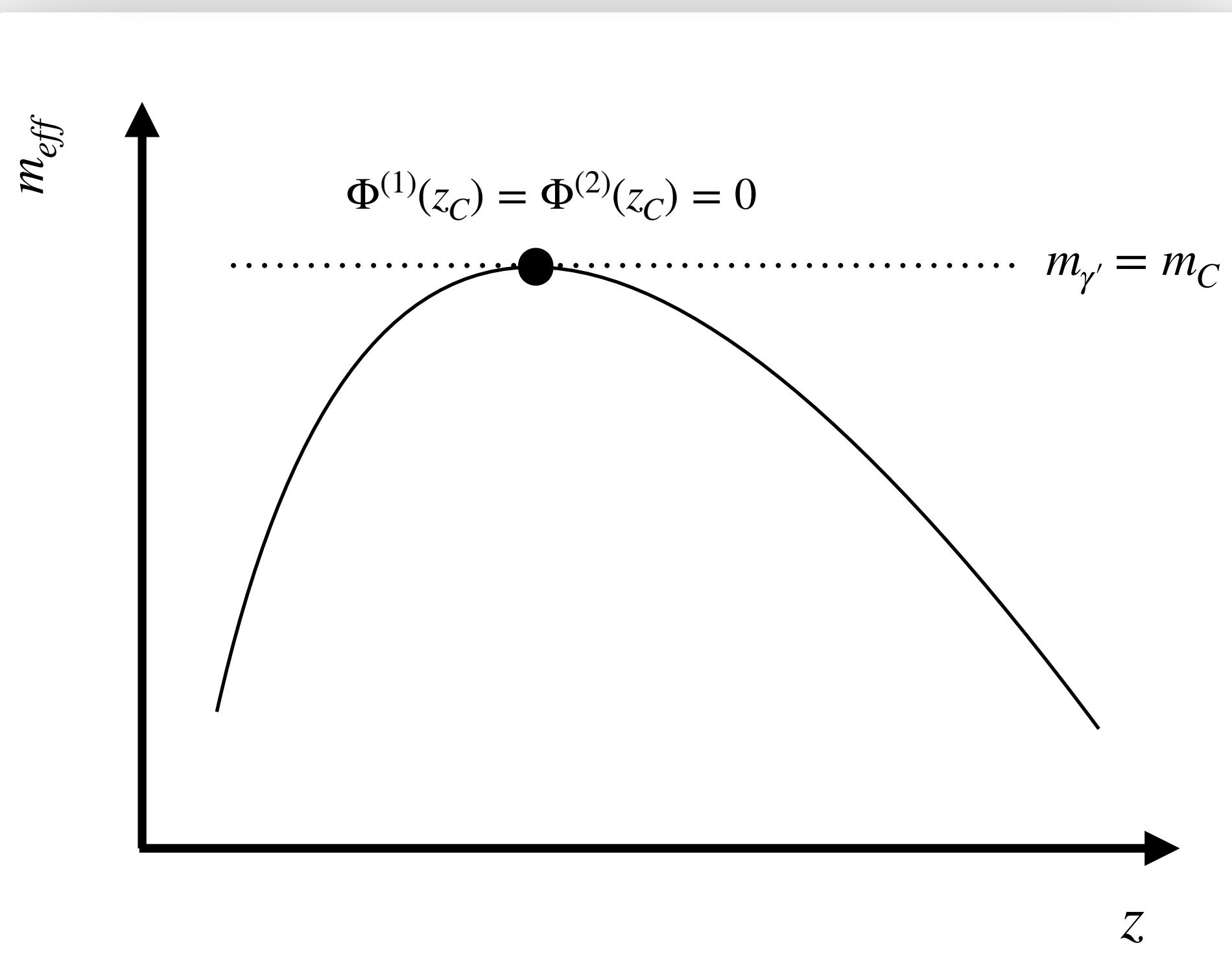
$$A_n \equiv \sqrt{\frac{2\pi}{|\Phi^{(2)}(z_n)|}} \left(\frac{m_{\gamma'}^2}{2\omega} \right) \rightarrow \infty$$

“Breakdown of LZ”



Breakdown of LZ

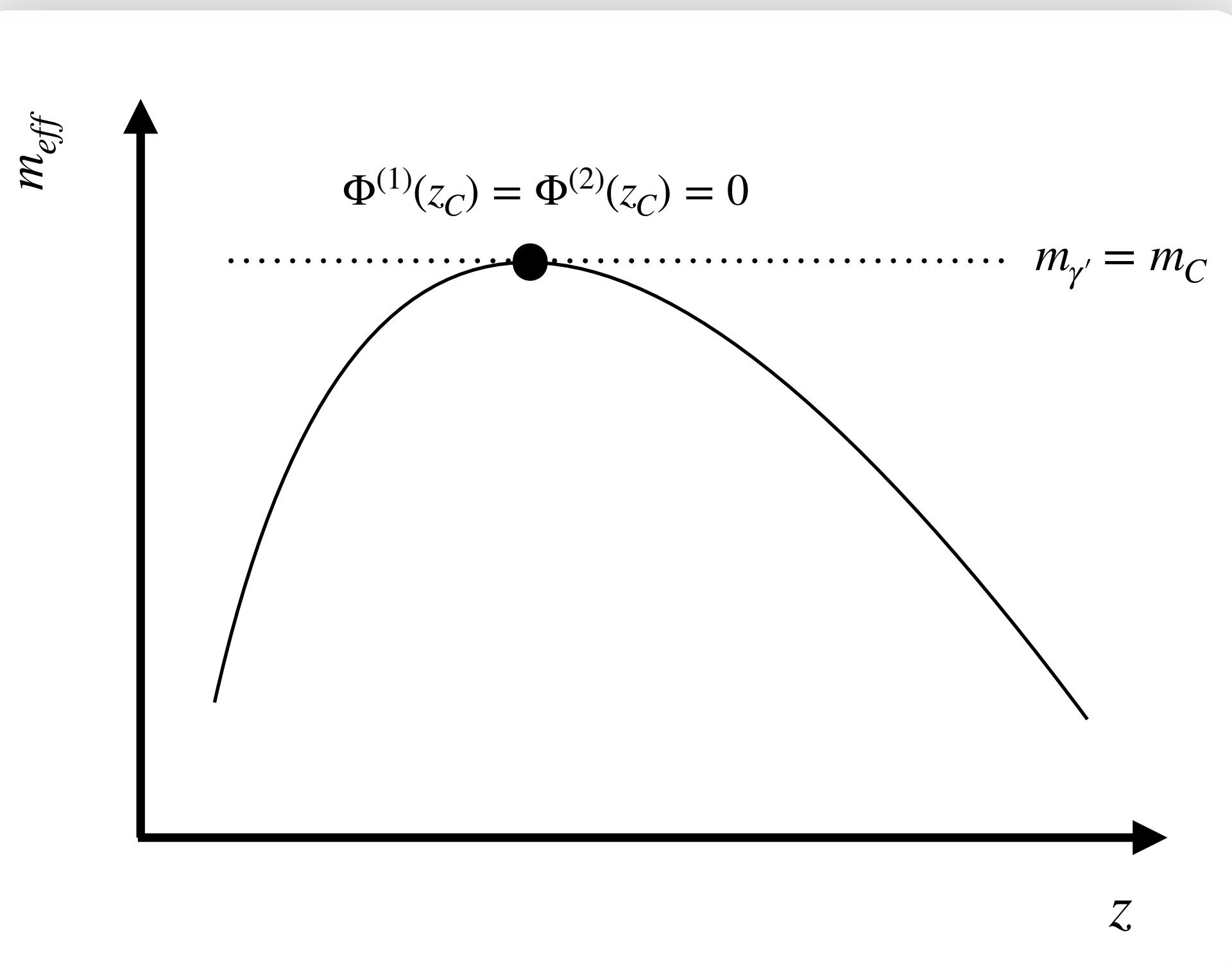
$$P_{\gamma \leftrightarrow \gamma'} = \epsilon^2 \left| \int_{z_i}^z dz' \frac{m_{\gamma'}^2}{2\omega} e^{-i\Phi(z')} \right|^2$$



Coalescing saddle points

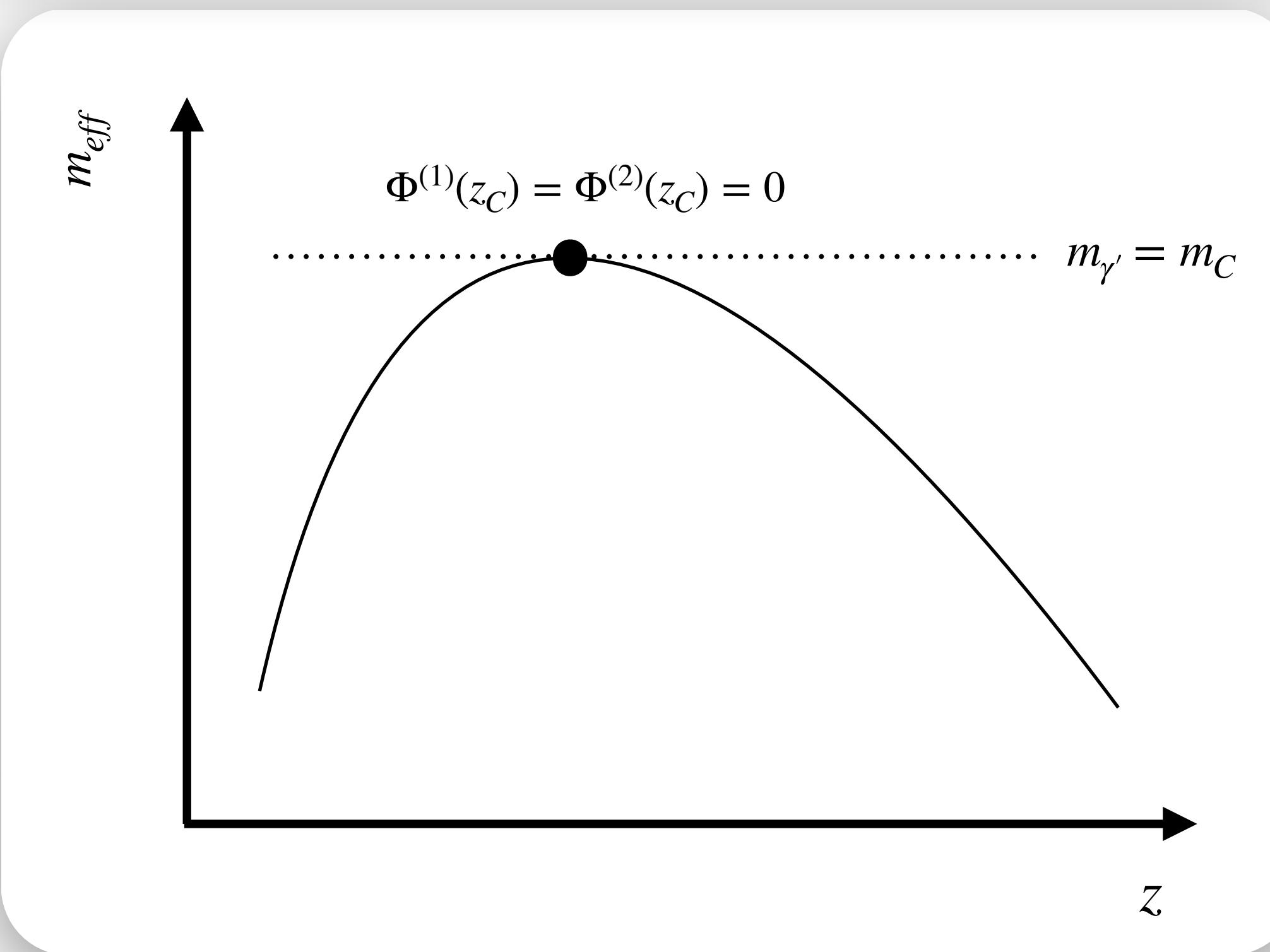
$$P_{\gamma \leftrightarrow \gamma'} \approx \epsilon^2 \left| 2\pi \left(\frac{2}{|\Phi^{(3)}(z_C)|} \right)^{1/3} \frac{m_{\gamma'}^2}{2\omega} (\text{Ai}(0) + i \# \text{Ai}^{(1)}(0)) \right|^2$$

Ai → Airy function



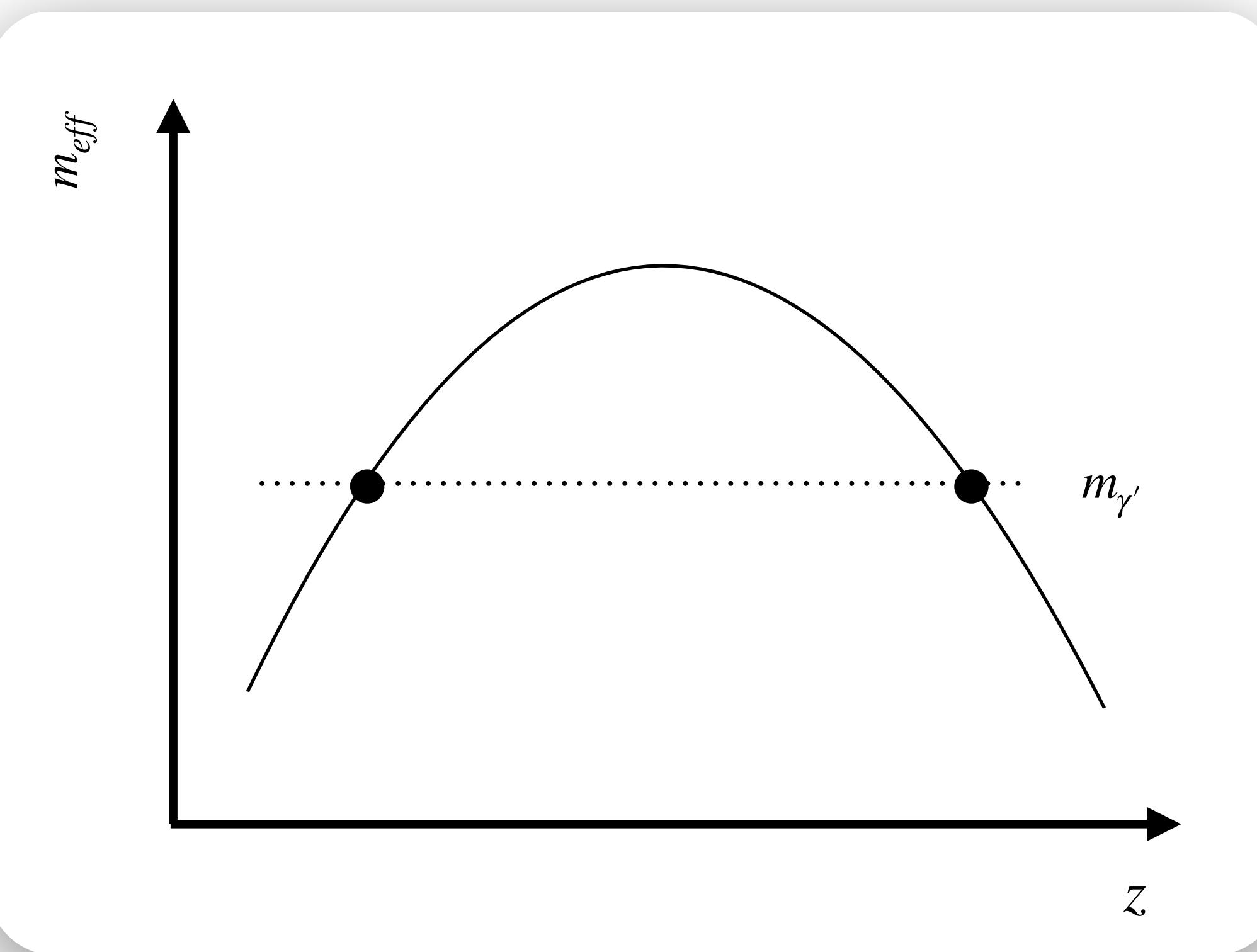
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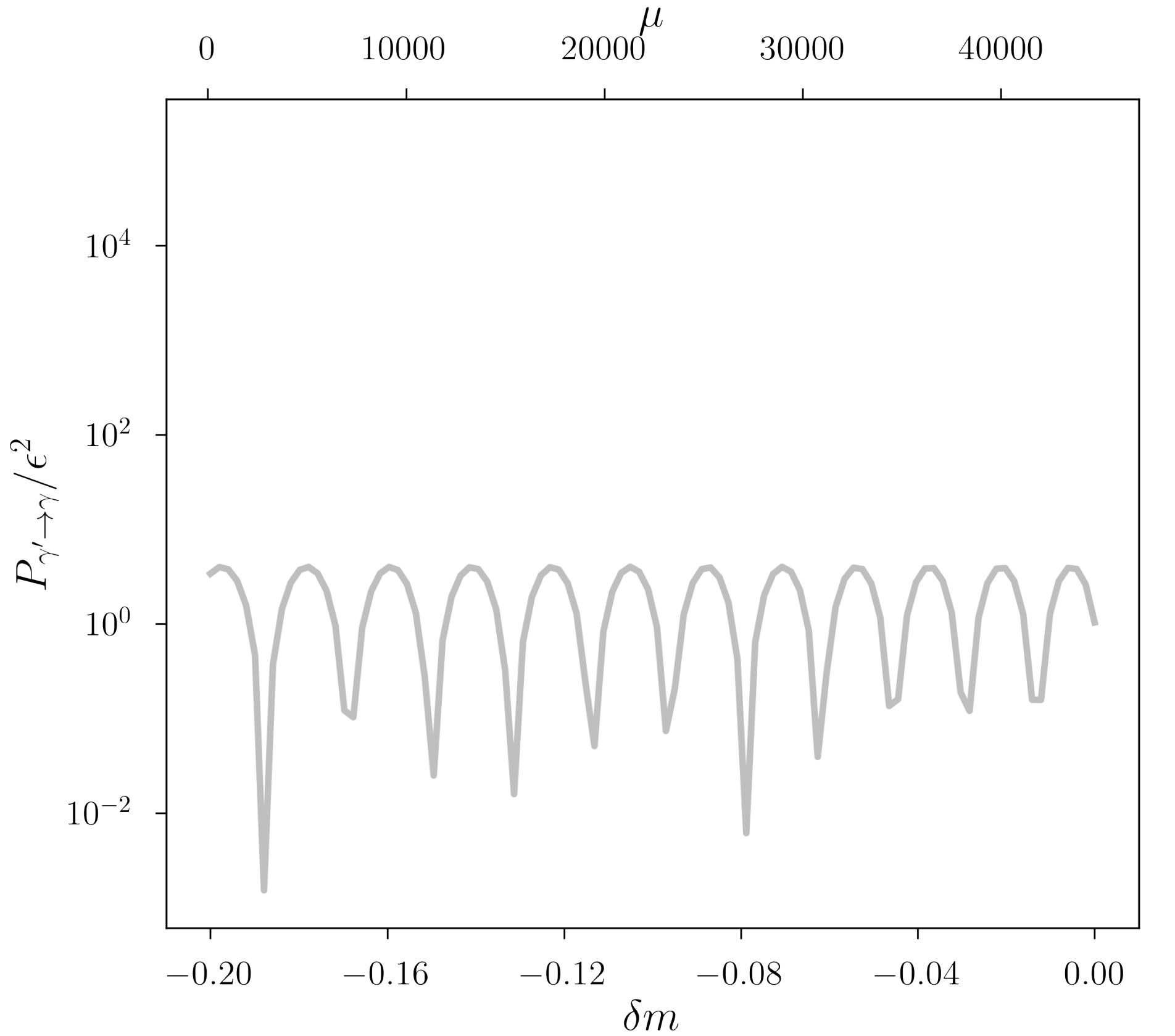
Toy model

$$m_{eff}^2(z) = b^2 \left[1 - \left(\frac{z}{a} - 1 \right)^2 \right]$$



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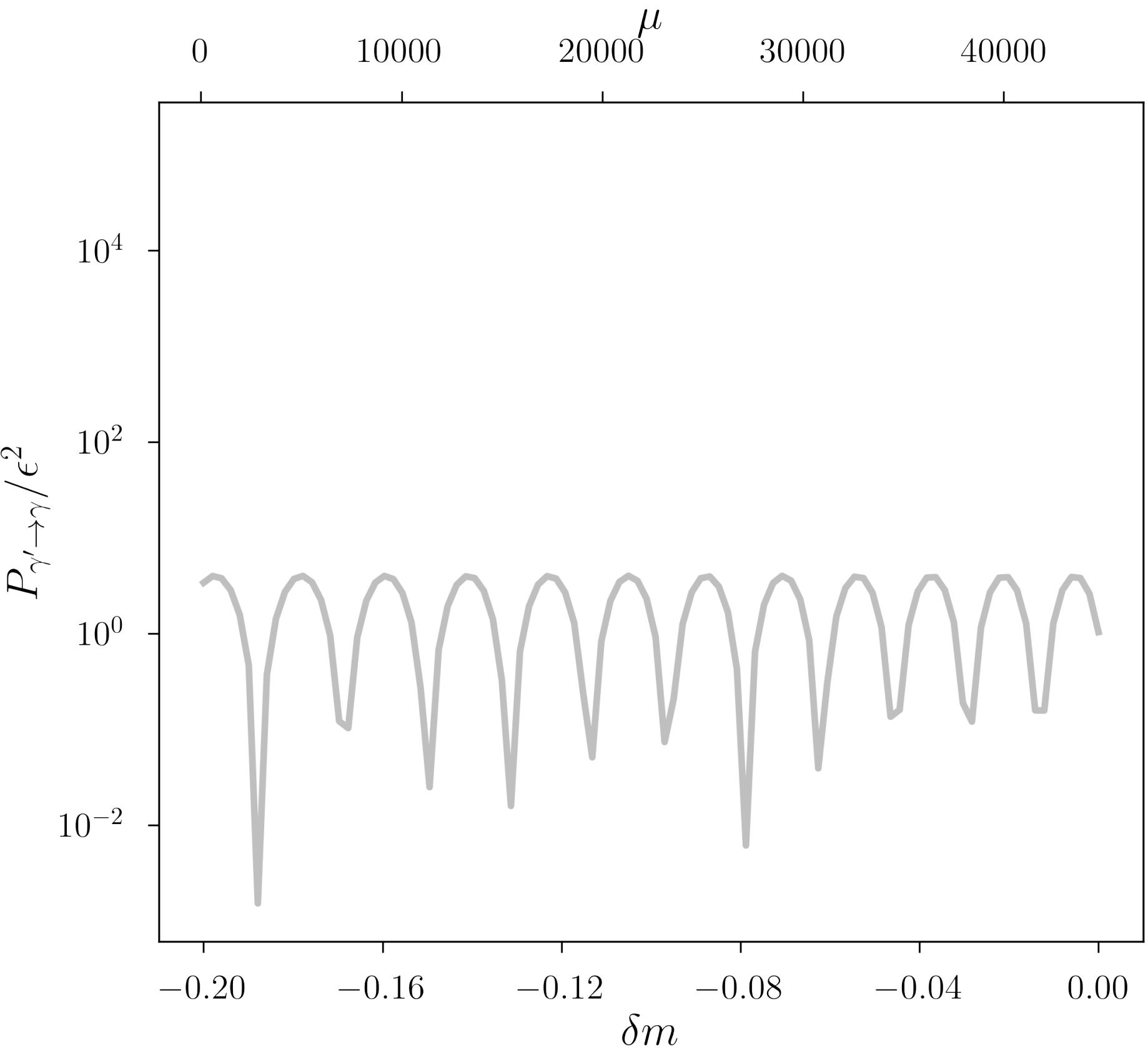
vacuum

$$\delta m = \frac{m_{\gamma'} - m_C}{m_C}$$

$a = 2000, b = 10$

Toy model

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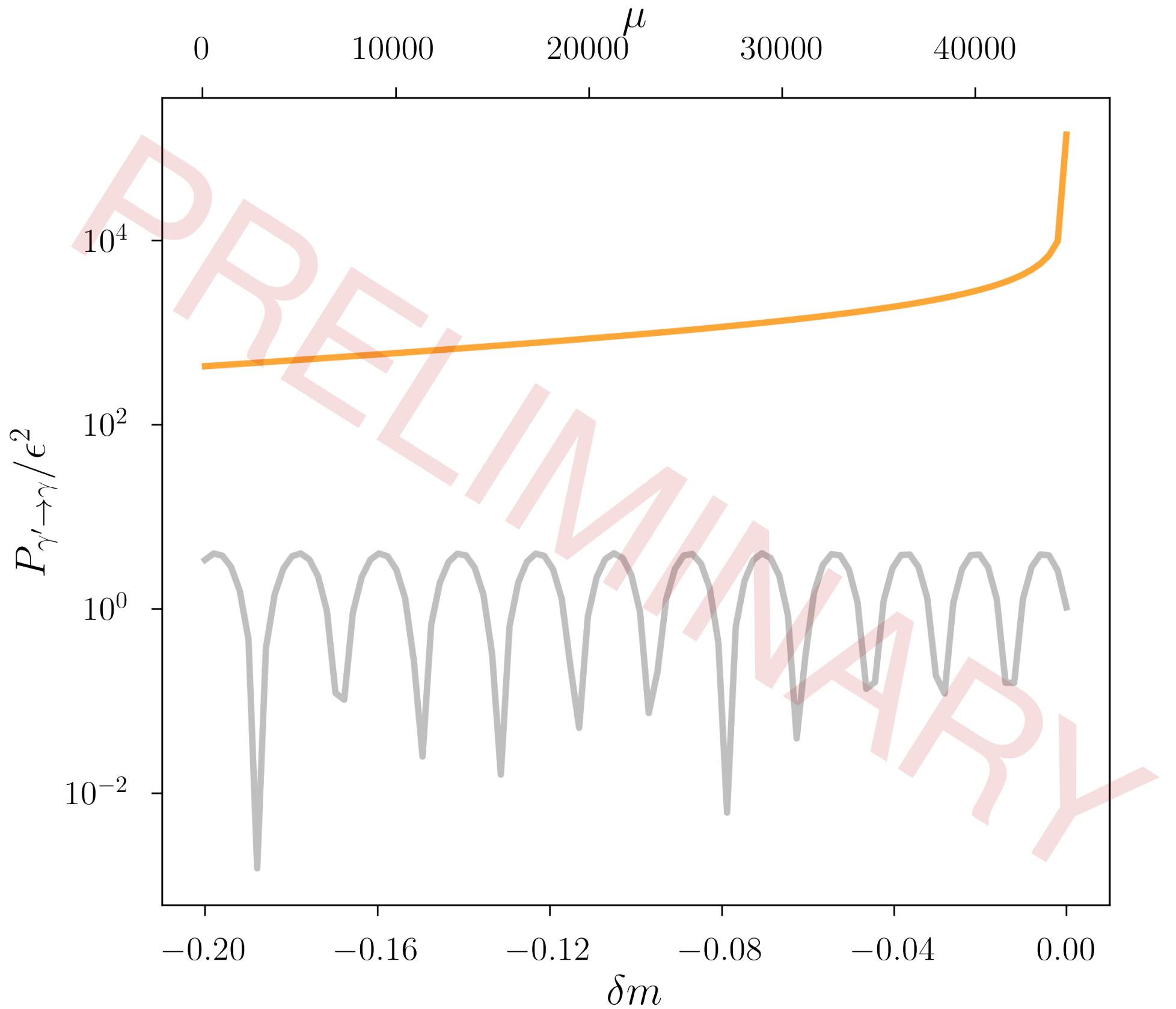
$$\mu \equiv \max(A_n)$$

“Resonance enhancement”

$$a = 2000, b = 10$$

Toy model

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vacuum

LZ

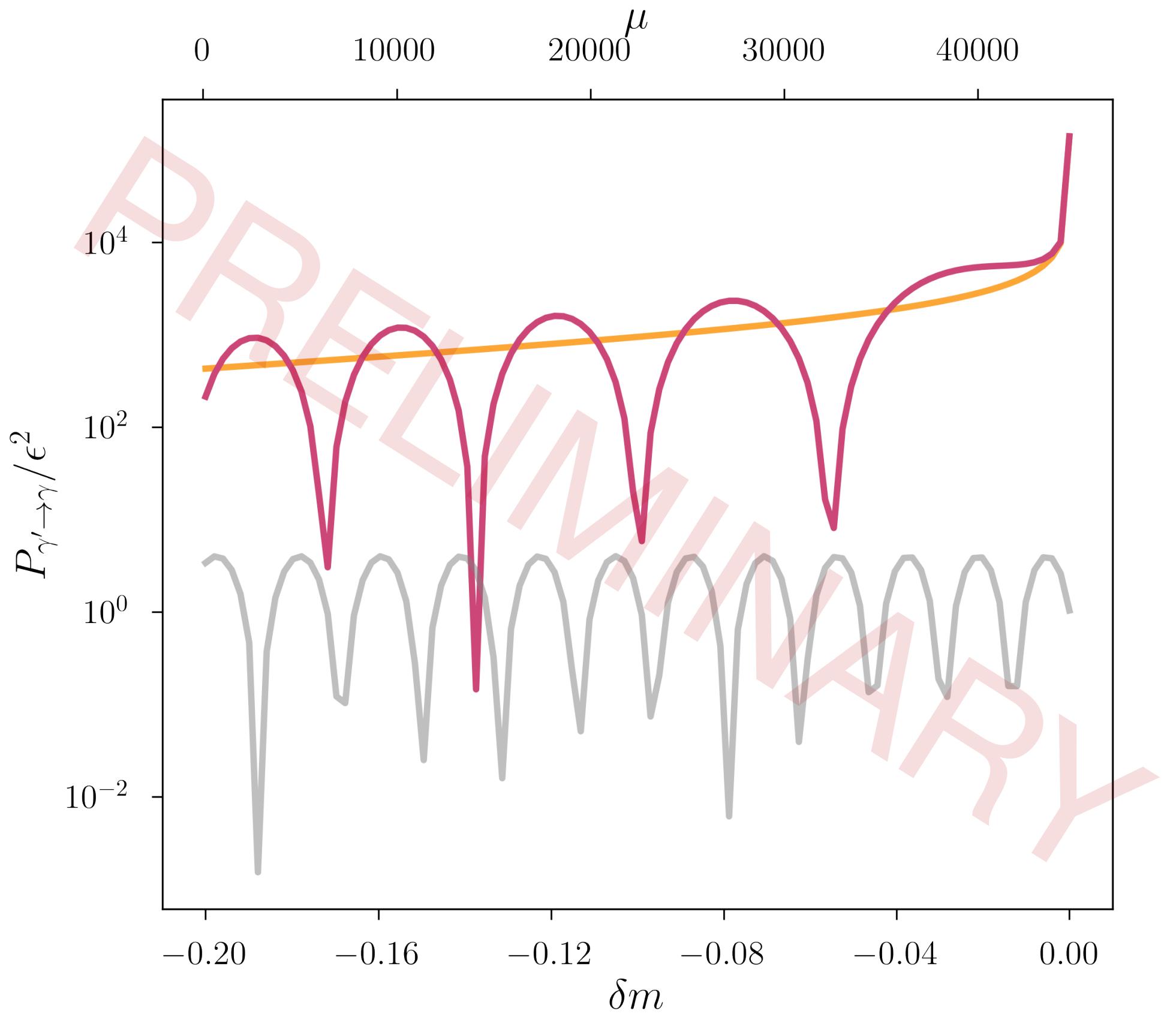
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vacuum

LZ

Phase

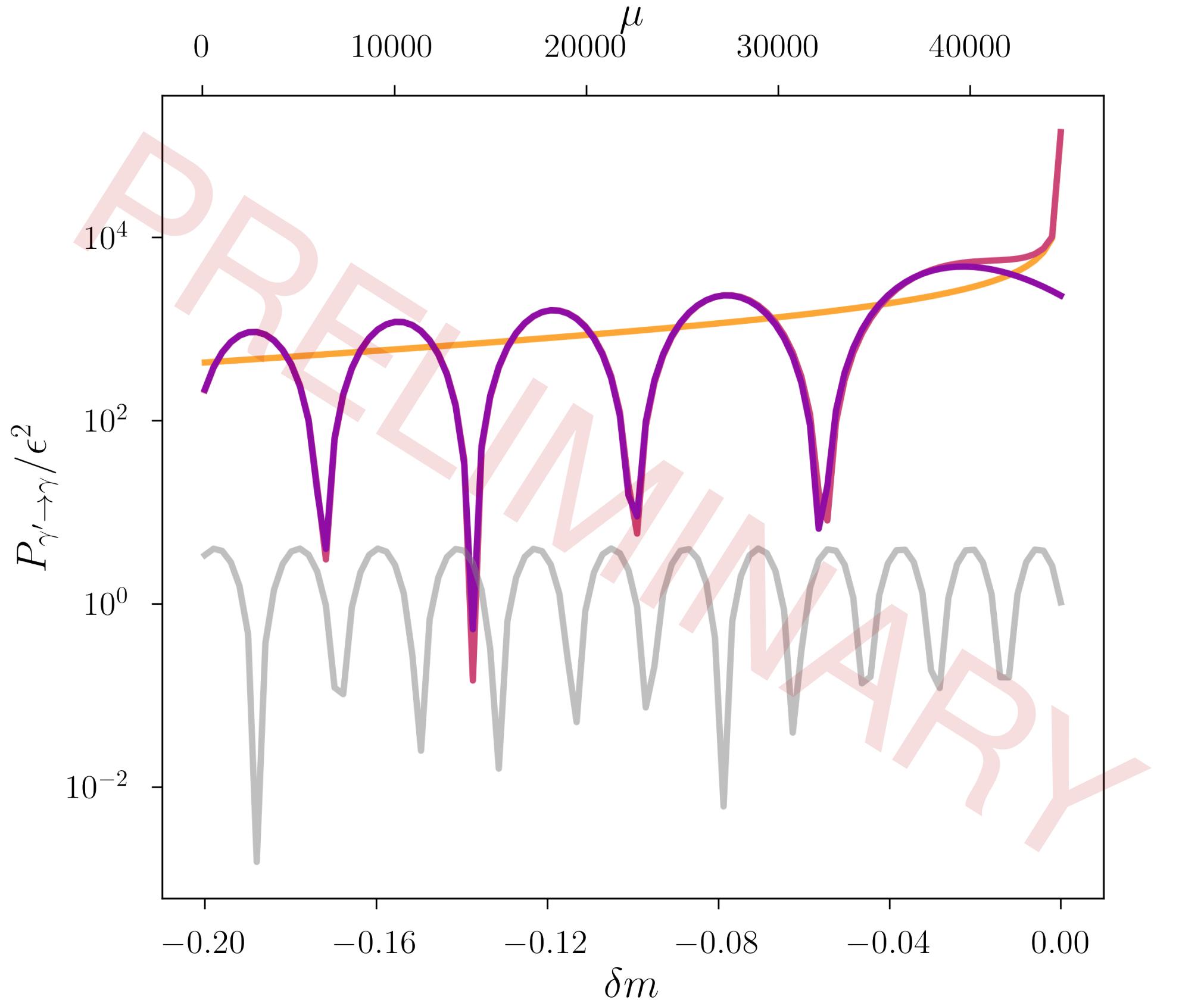
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This work

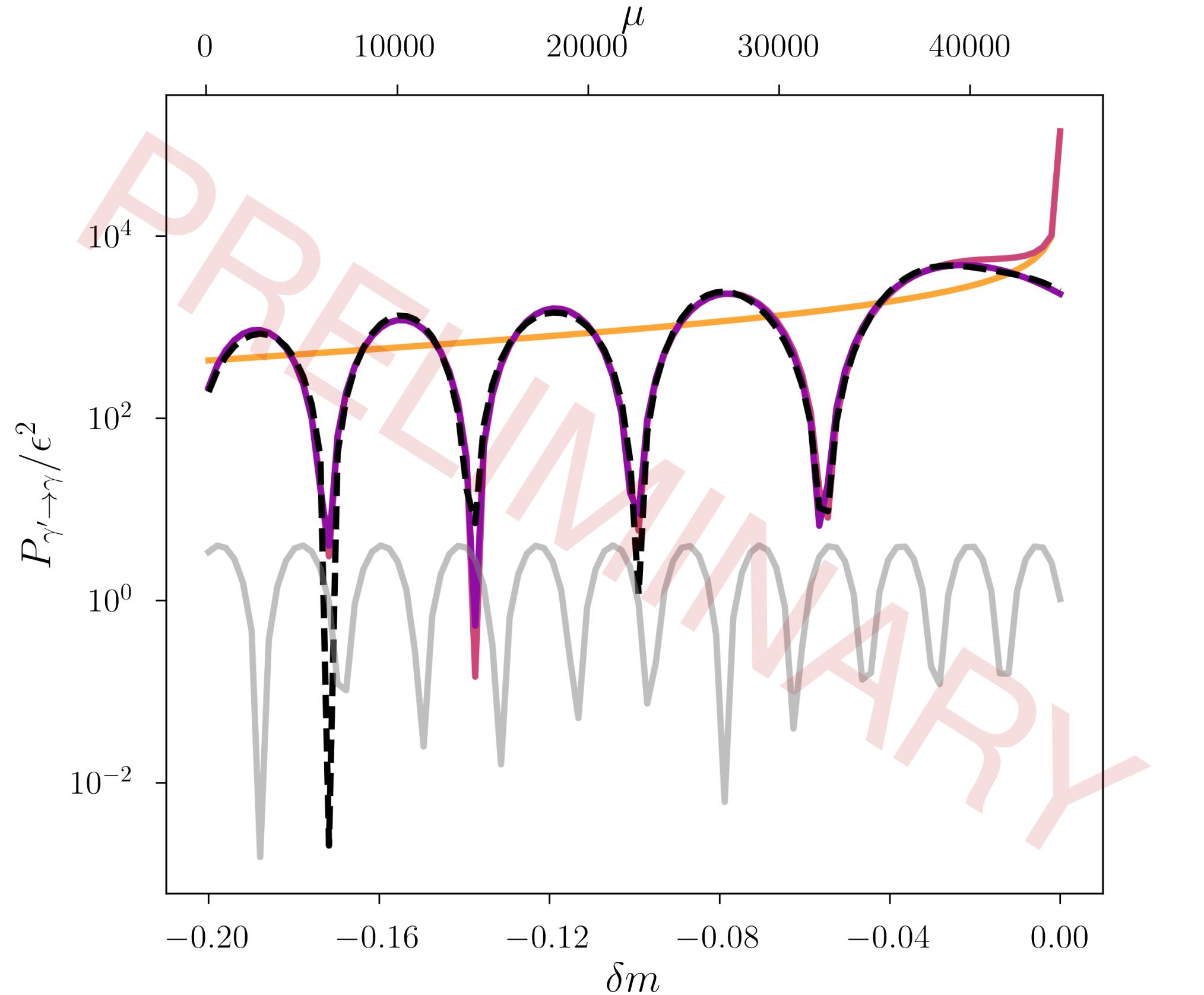
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Vacuum

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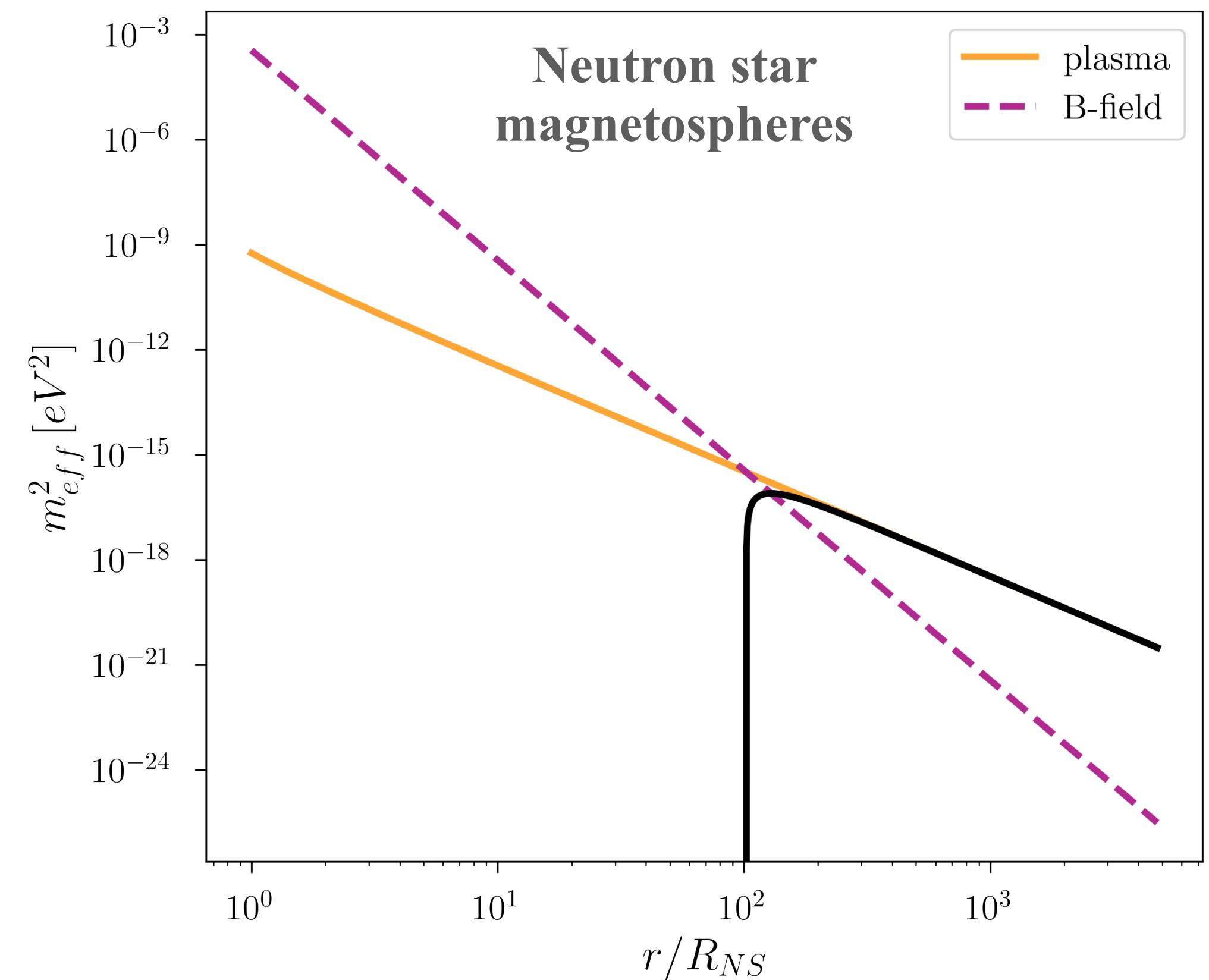
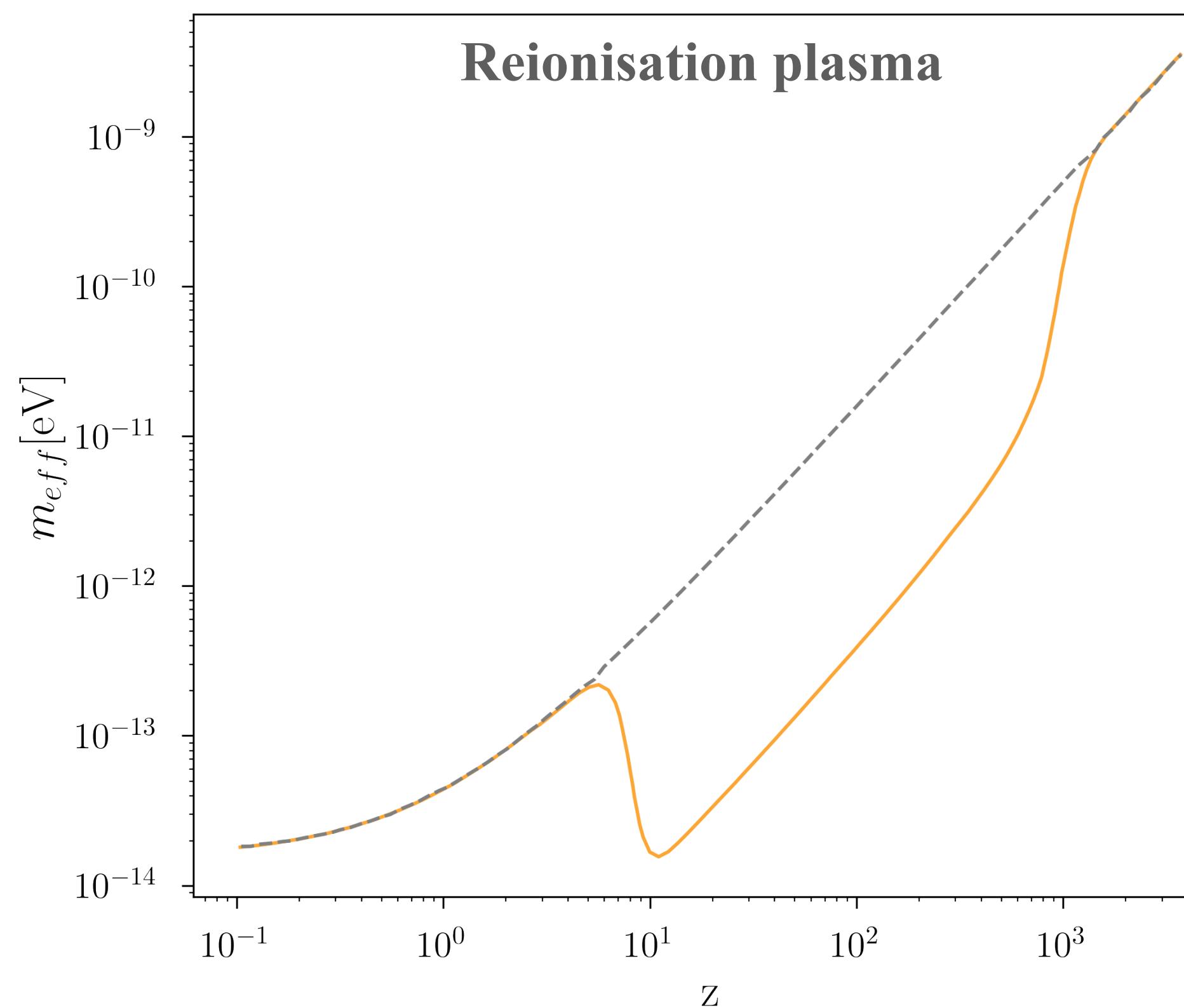
Numerical

$a = 2000, b = 10$

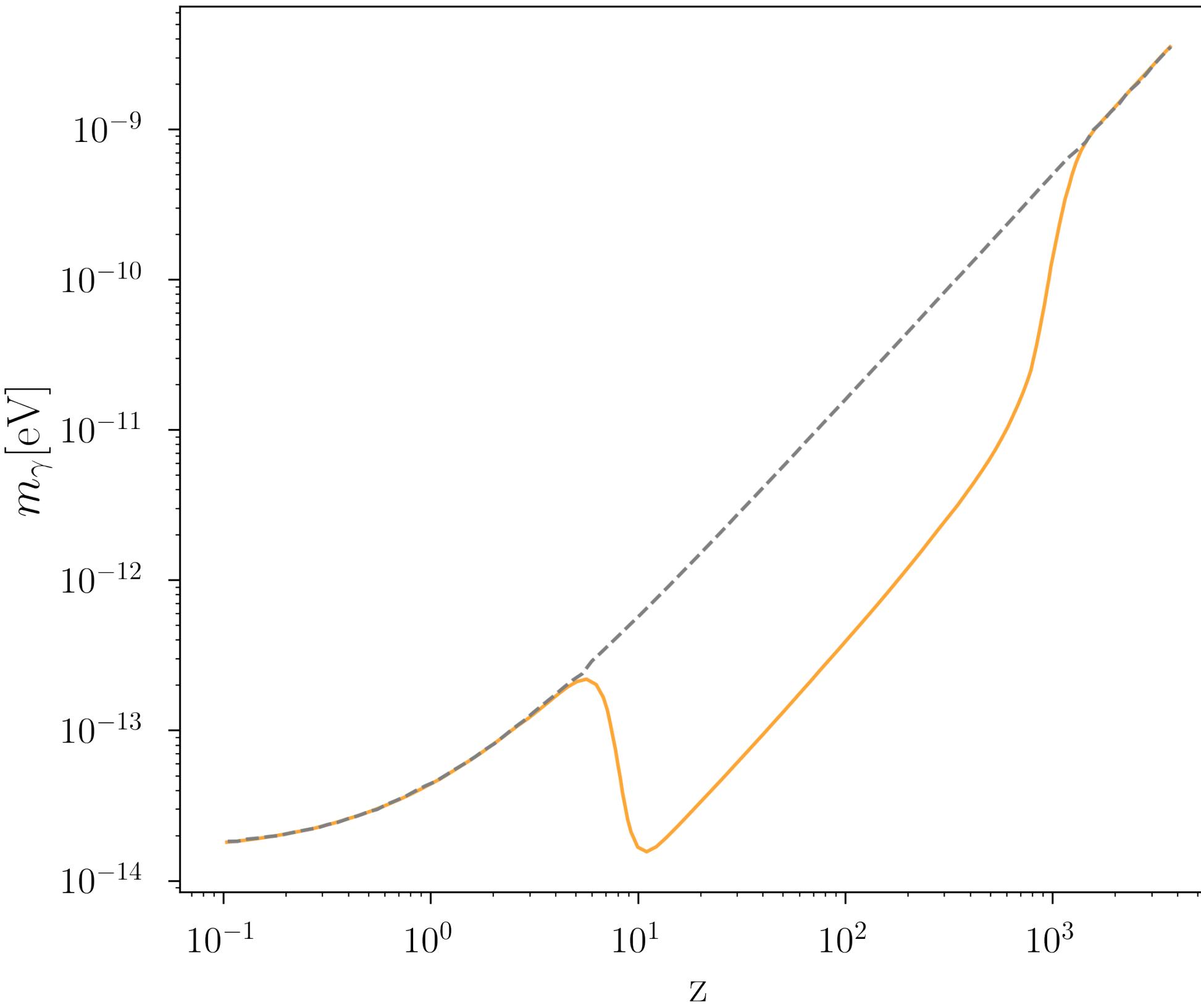
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Astrophysical examples

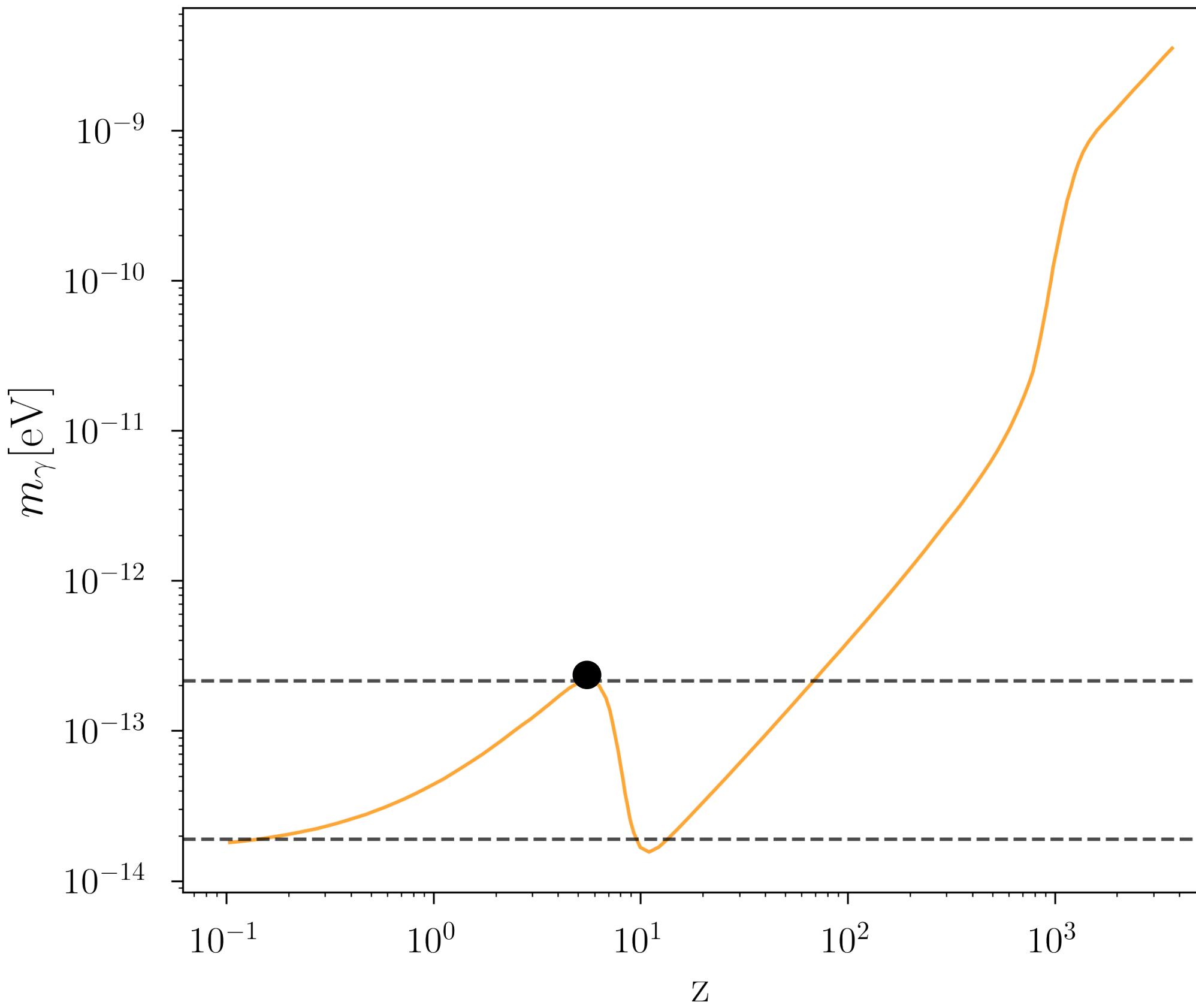


Reionisation plasma

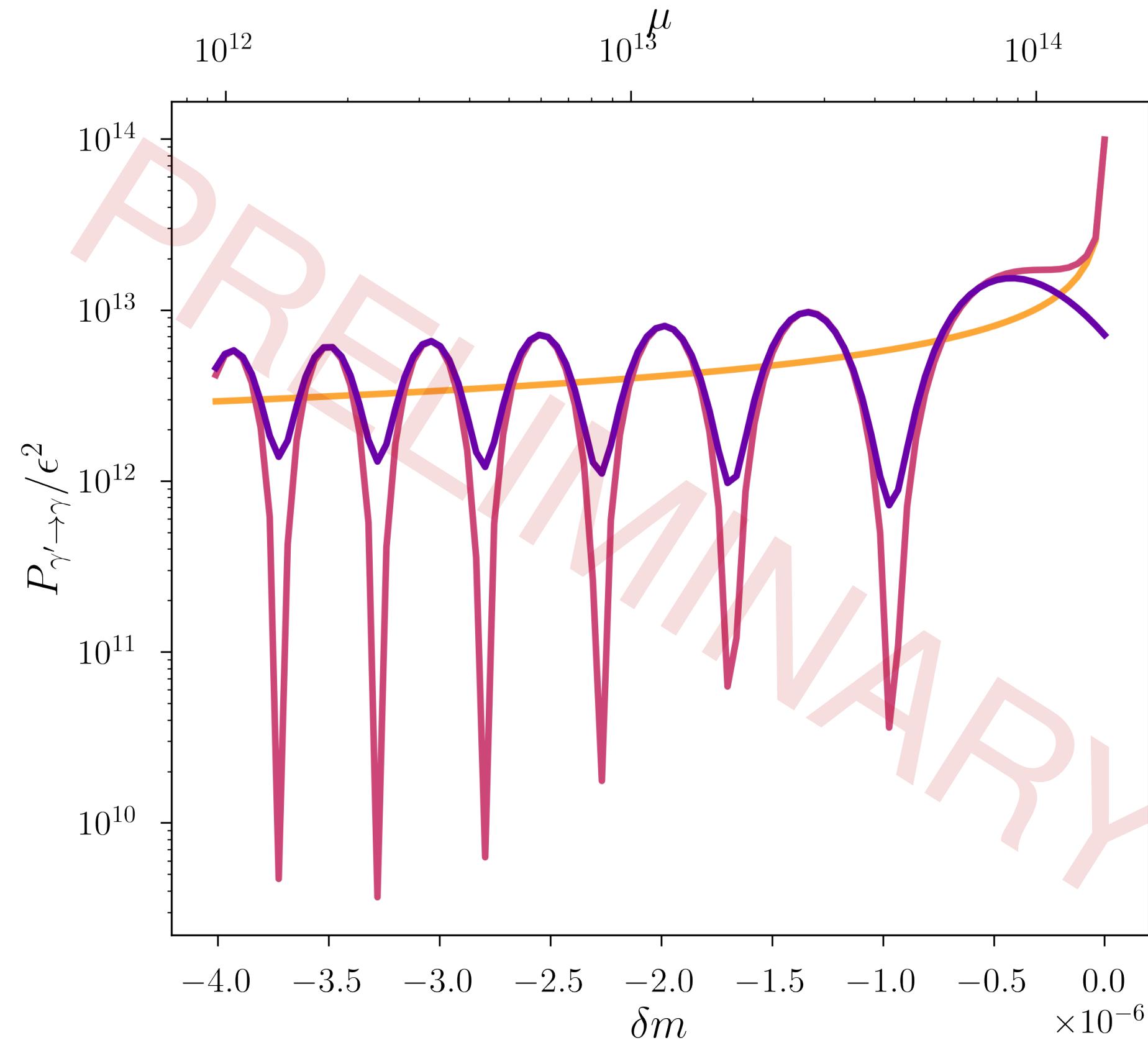


Mirizzi et al. (2009), Caputo et al. (2020)

Reionization plasma



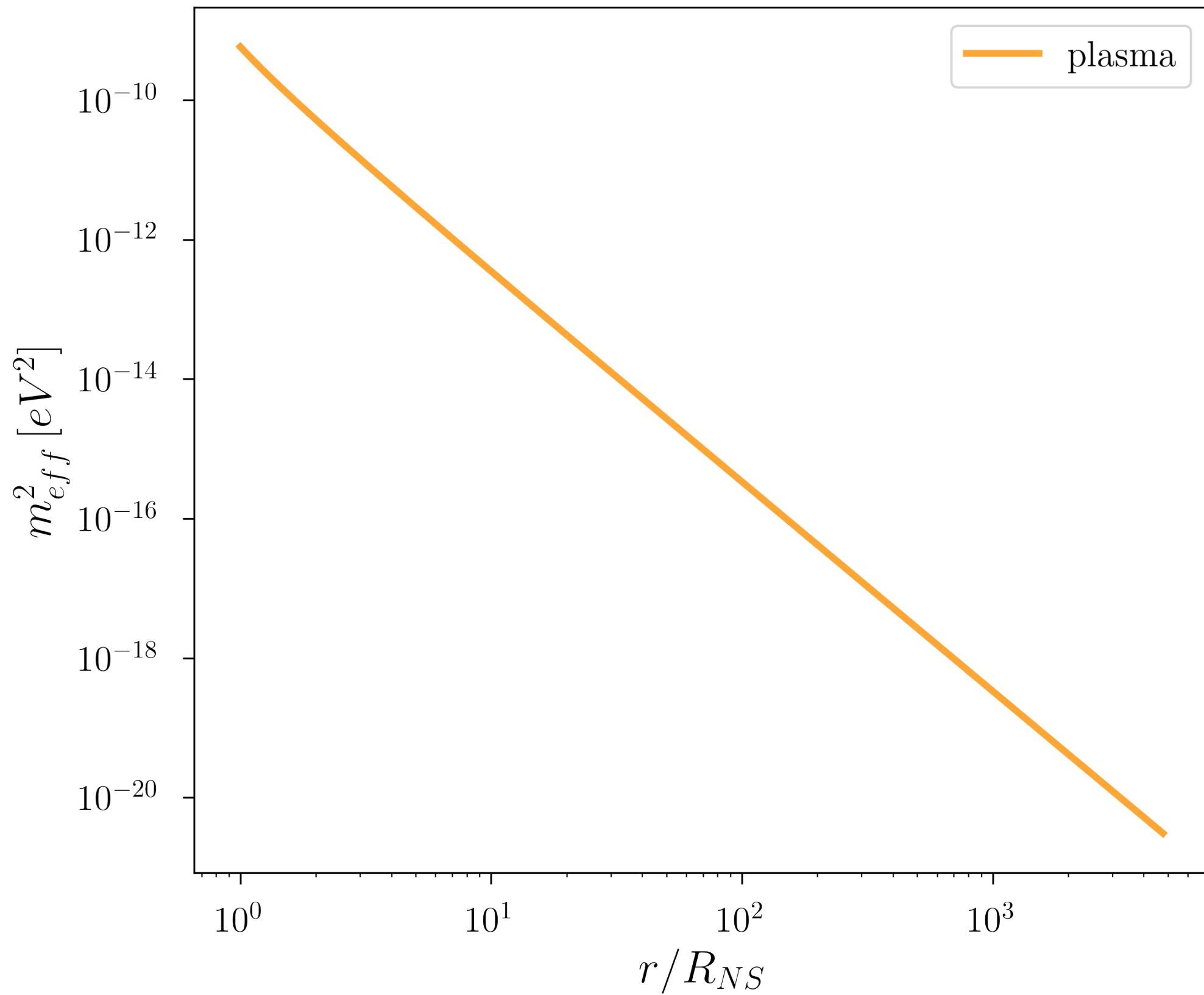
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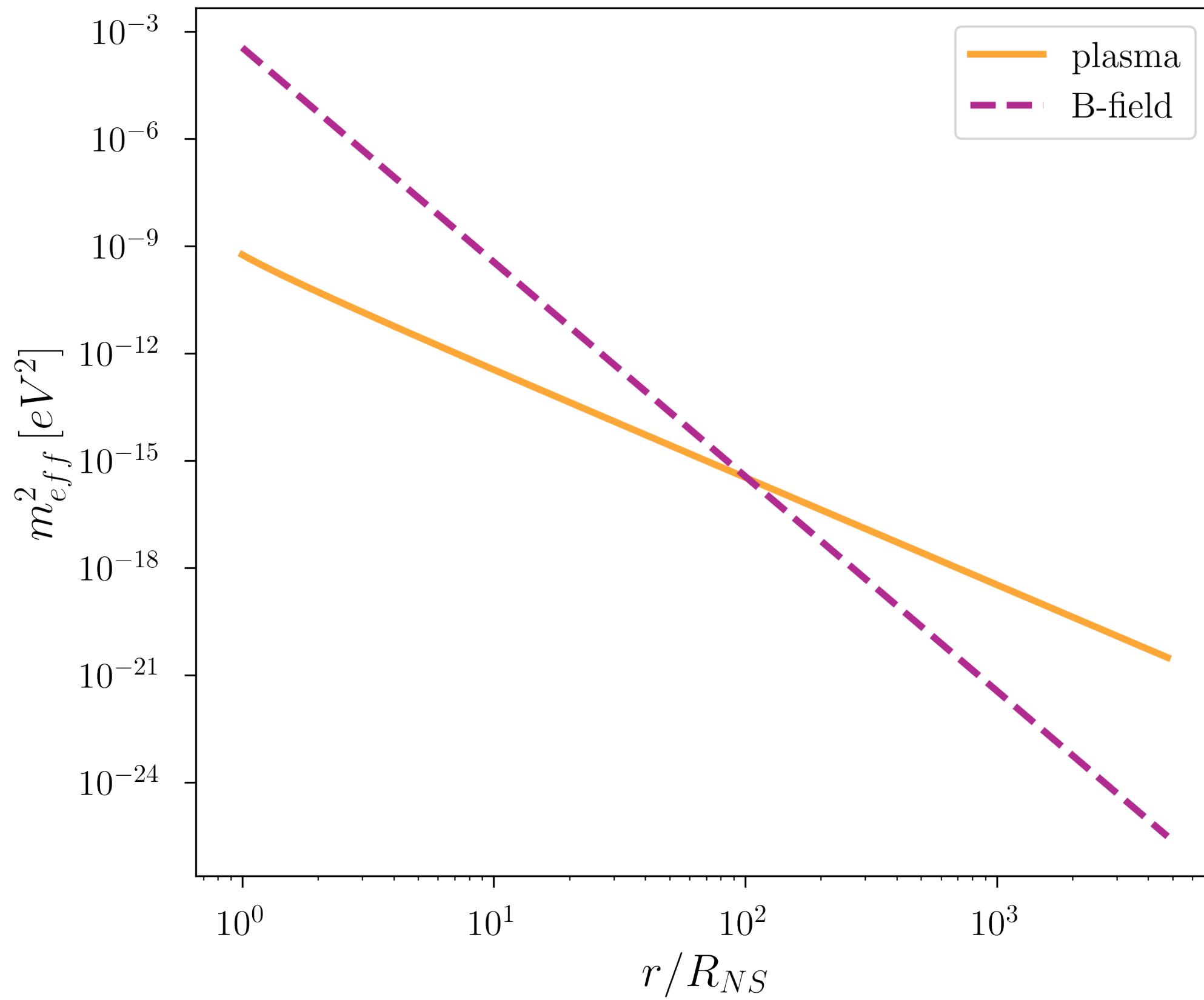
LZ Phase This work

Neutron star magnetospheres

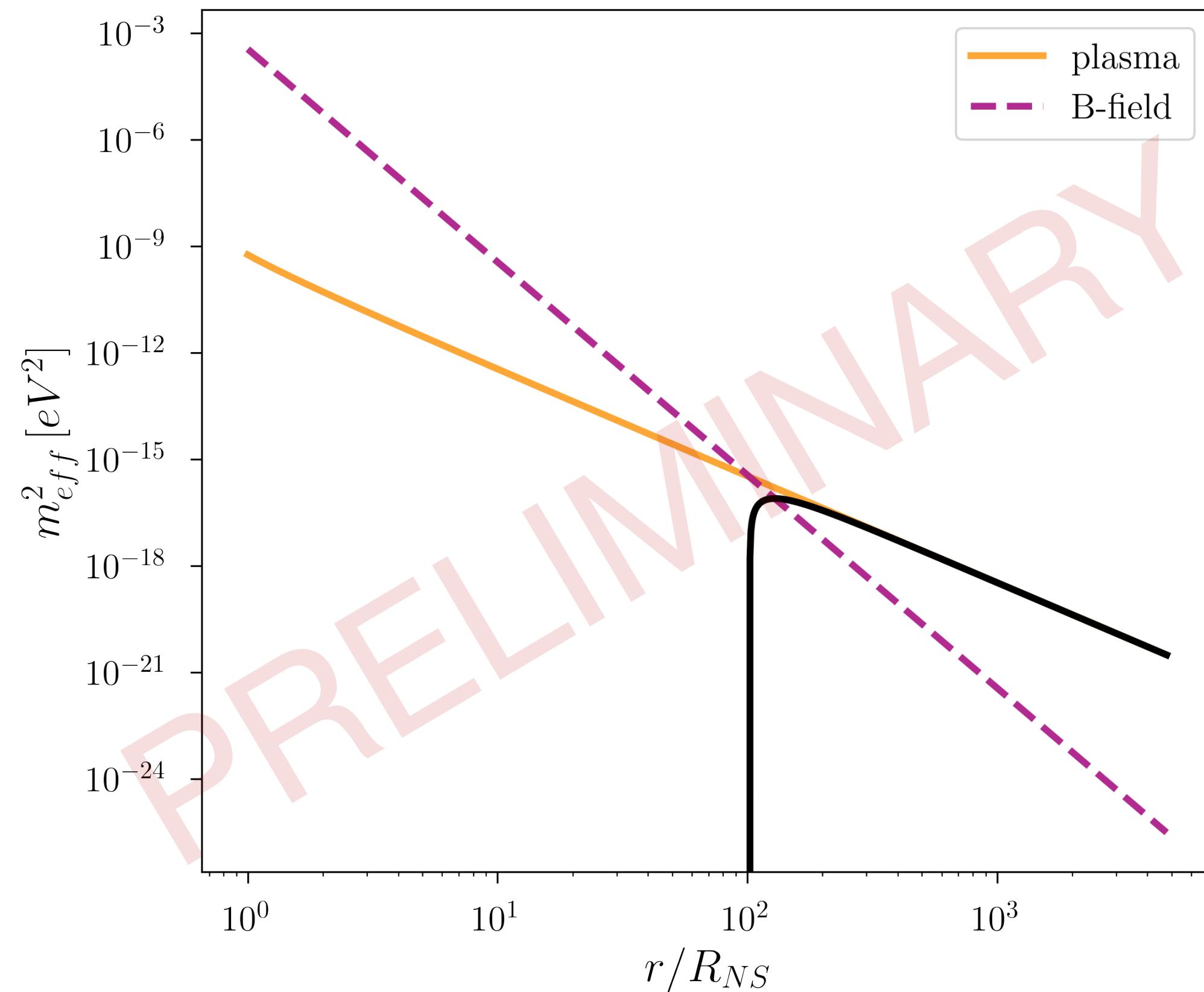
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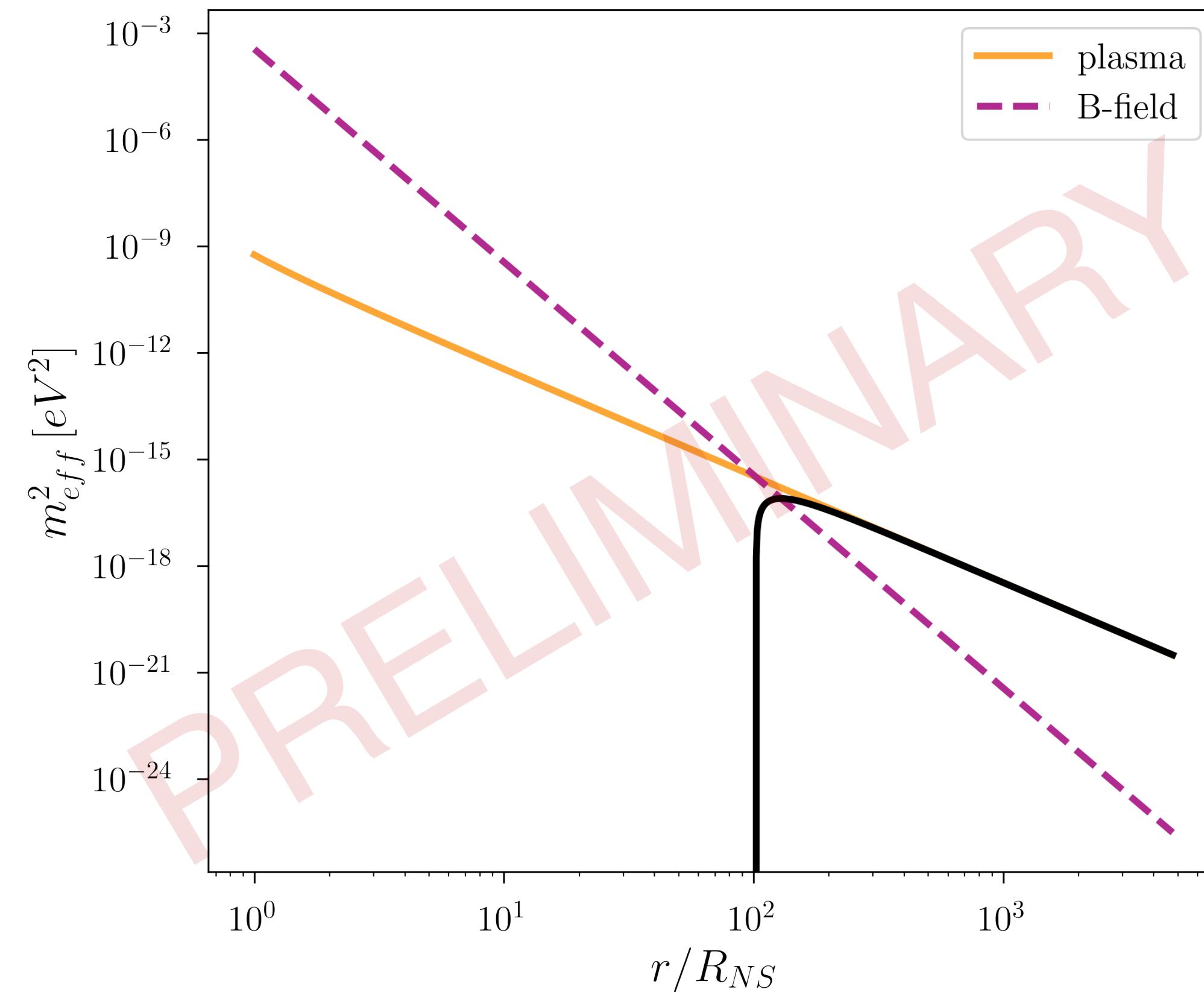


Neutron star magnetospheres



$$B_0/B_{crit} = 1, P = 1\text{sec}, \omega = 1\text{eV}$$

Neutron star magnetospheres

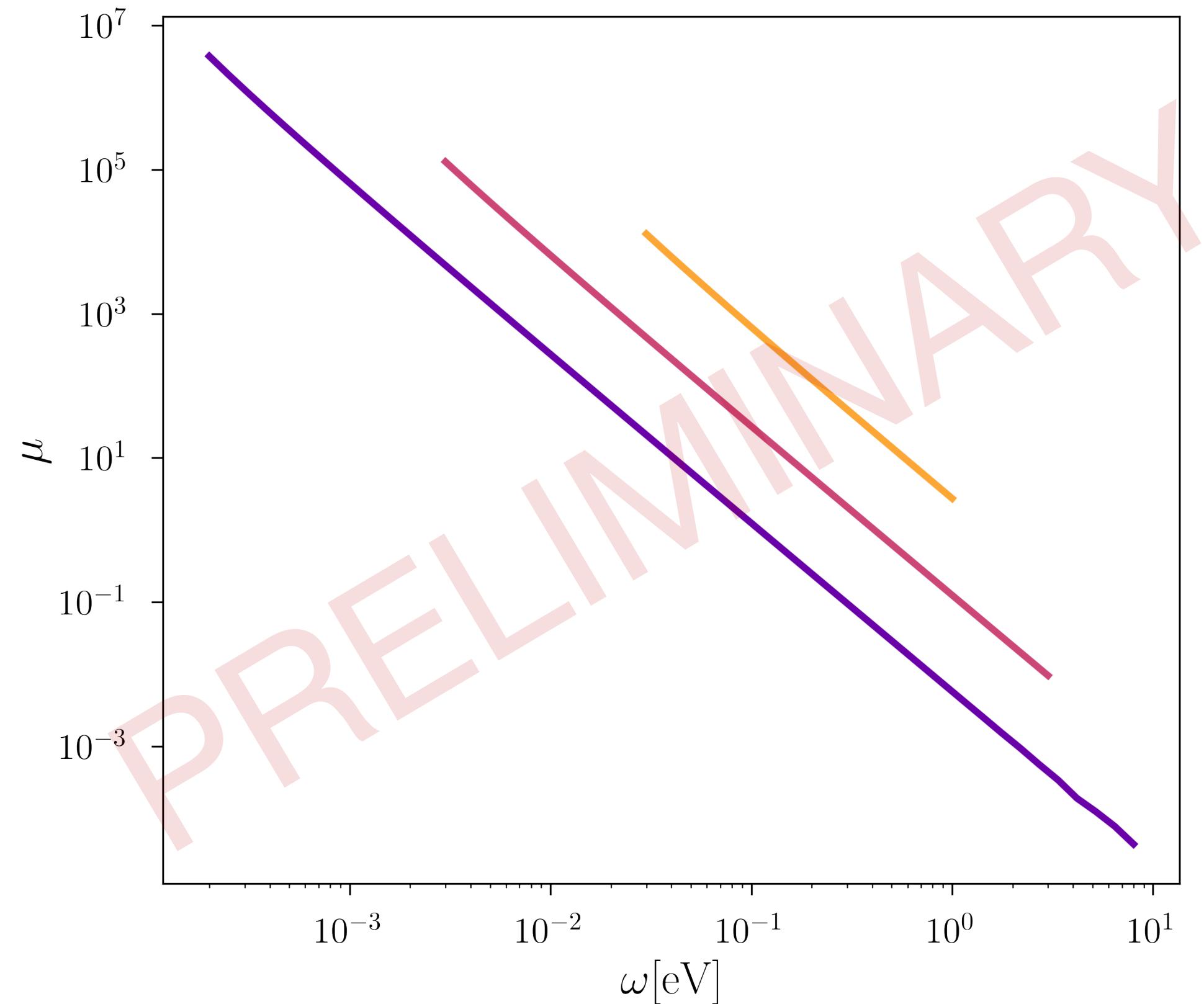


$B_0/B_{crit} = 1, P = 1\text{ sec}, \omega = 1\text{ eV}$



$\mu \approx 1$

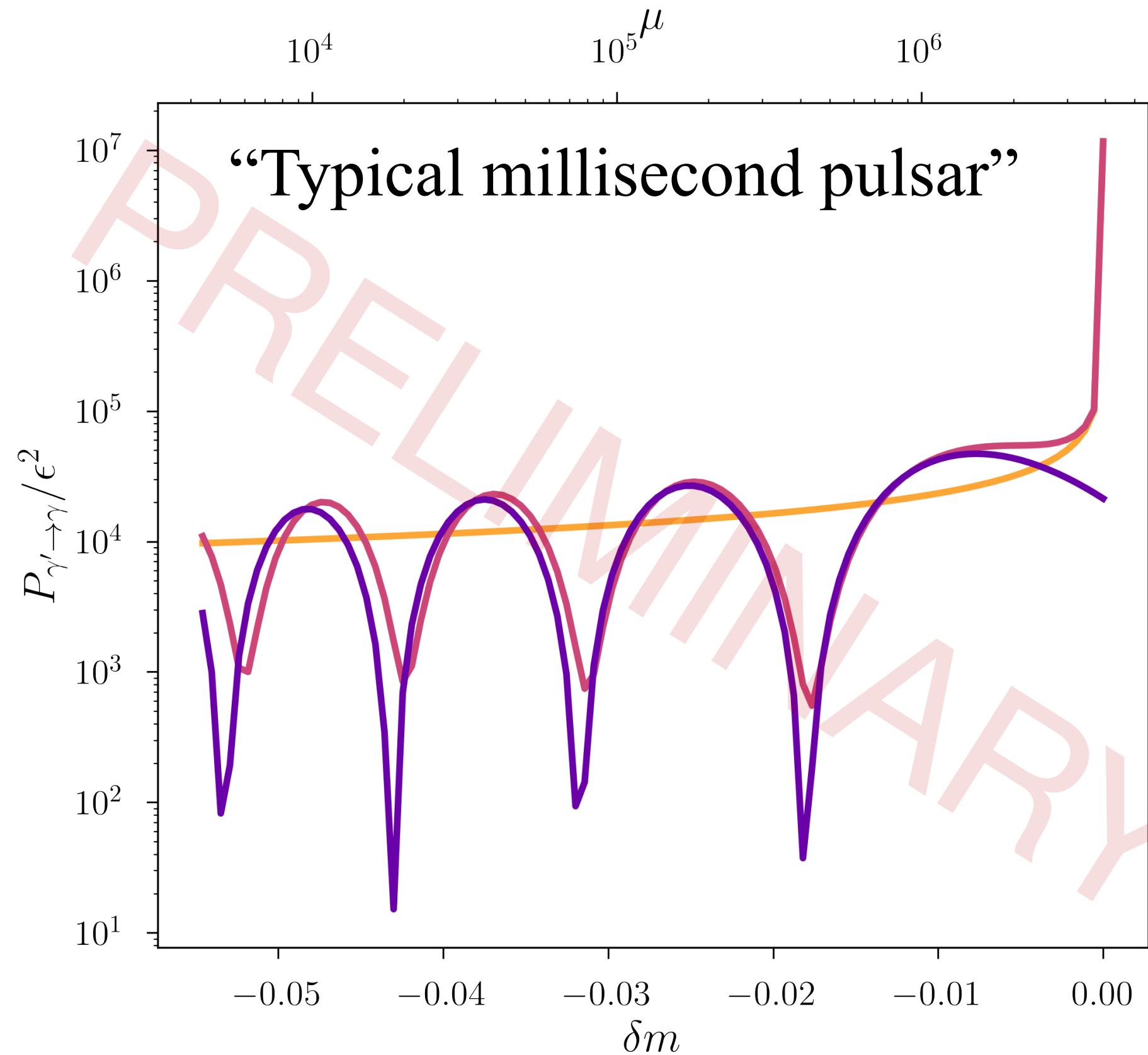
Neutron star magnetospheres



$\mu \rightarrow$ resonance enhancement

- Yellow bar: $B_0/B_{crit} = 1, P = 1\text{ms}$
- Pink bar: $B_0/B_{crit} = 10, P = 10\text{ms}$
- Purple bar: $B_0/B_{crit} = 100, P = 100\text{ms}$

Neutron star magnetospheres



LZ

Phase

This work

$B_0/B_{crit} = 1, P = 10 \text{ ms}, \omega = 0.01 \text{ eV}$

Summary

- Non-monotonic potential profiles are ubiquitous in astrophysics. More examples - supernova shockwave, solar chromosphere etc.

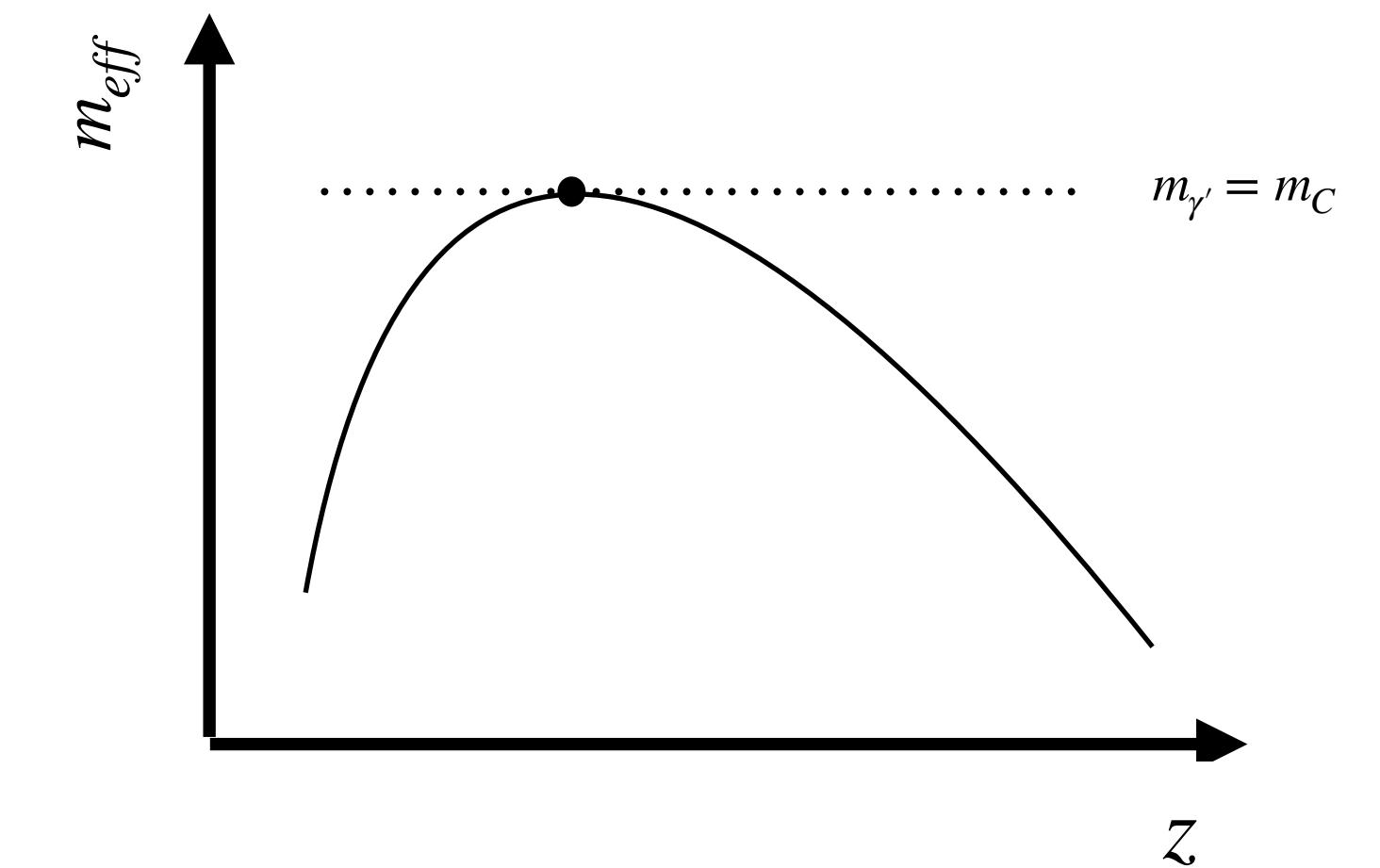
Summary

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Summary

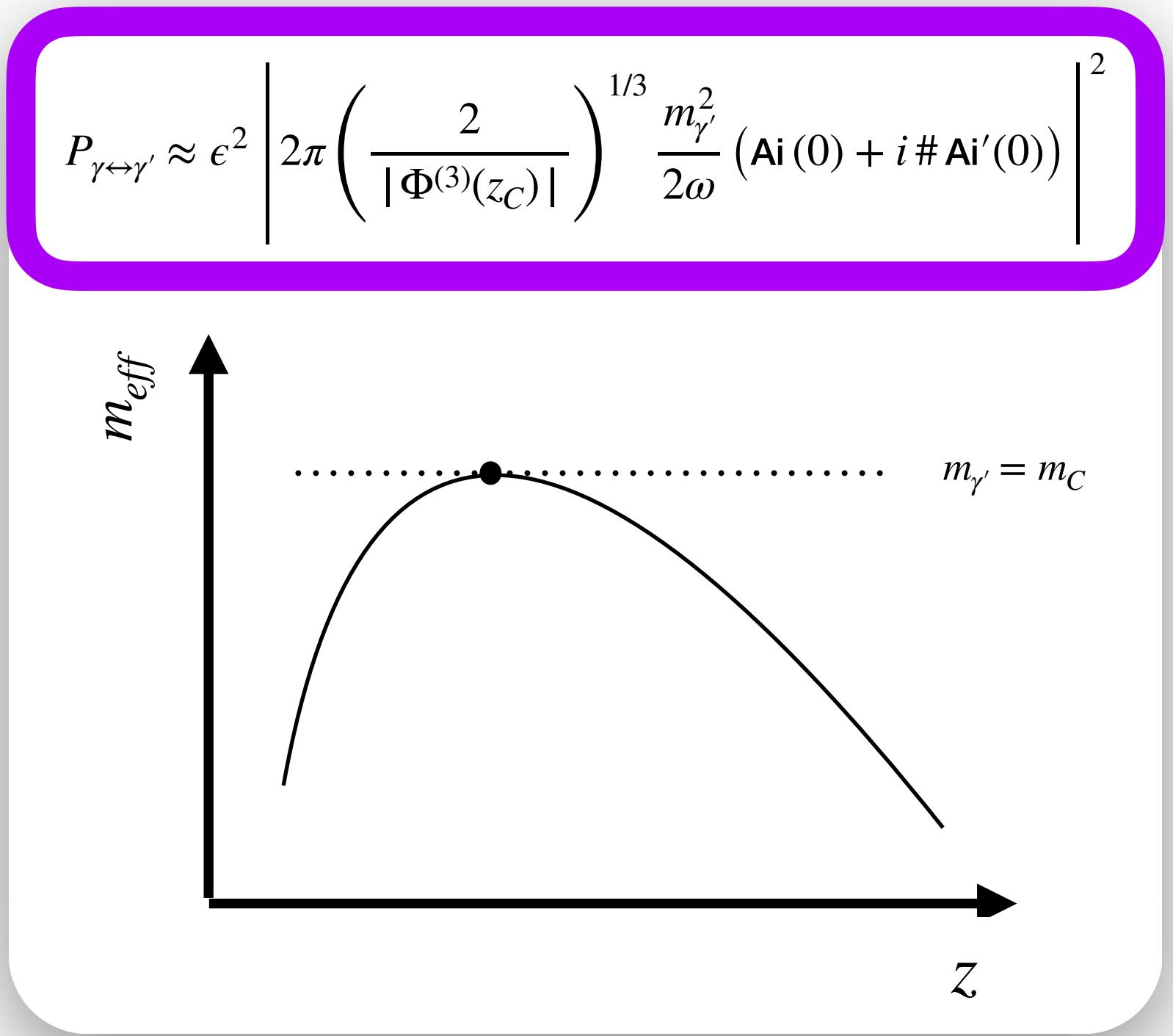
- Non-monotonic potential profiles are ubiquitous in astrophysics. More examples - supernova shockwave, solar chromosphere etc.
- The usual Landau-Zener formula breaks down near critical points.
- Our expression for coalescing saddle point provides an accurate prescription for evaluating the conversion probability.

$$P_{\gamma \leftrightarrow \gamma'} \approx \epsilon^2 \left| 2\pi \left(\frac{2}{|\Phi^{(3)}(z_C)|} \right)^{1/3} \frac{m_{\gamma'}^2}{2\omega} (\text{Ai}(0) + i\# \text{Ai}'(0)) \right|^2$$



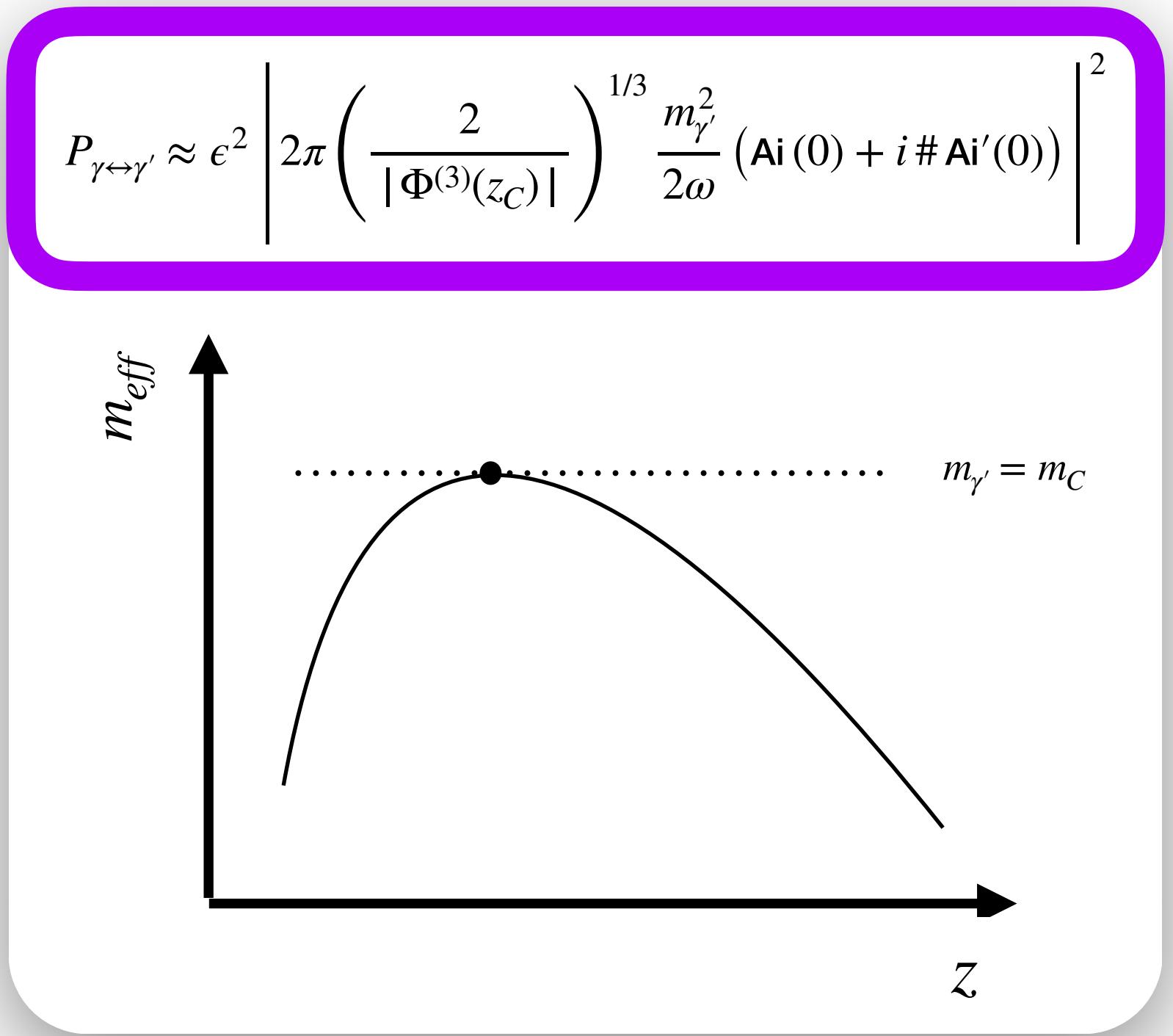
Summary

- Non-monotonic potential profiles are ubiquitous in astrophysics. More examples - supernova shockwave, solar chromosphere etc.
- The usual Landau-Zener formula breaks down near critical points.
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- The NS parameter space has a very wide range of μ values and is an excellent candidate for dark-photon convertor.



Summary

- Non-monotonic potential profiles are ubiquitous in astrophysics. More examples - supernova shockwave, solar chromosphere etc.
- The usual Landau-Zener formula breaks down near critical points.
- Our expression for coalescing saddle point provides an accurate prescription for evaluating the conversion probability.
- The NS parameter space has a very wide range of μ values and is an excellent candidate for dark-photon convertor.
- Moreover, it can be used for neutrino oscillations, axion-photon conversions, etc.



Thank you!



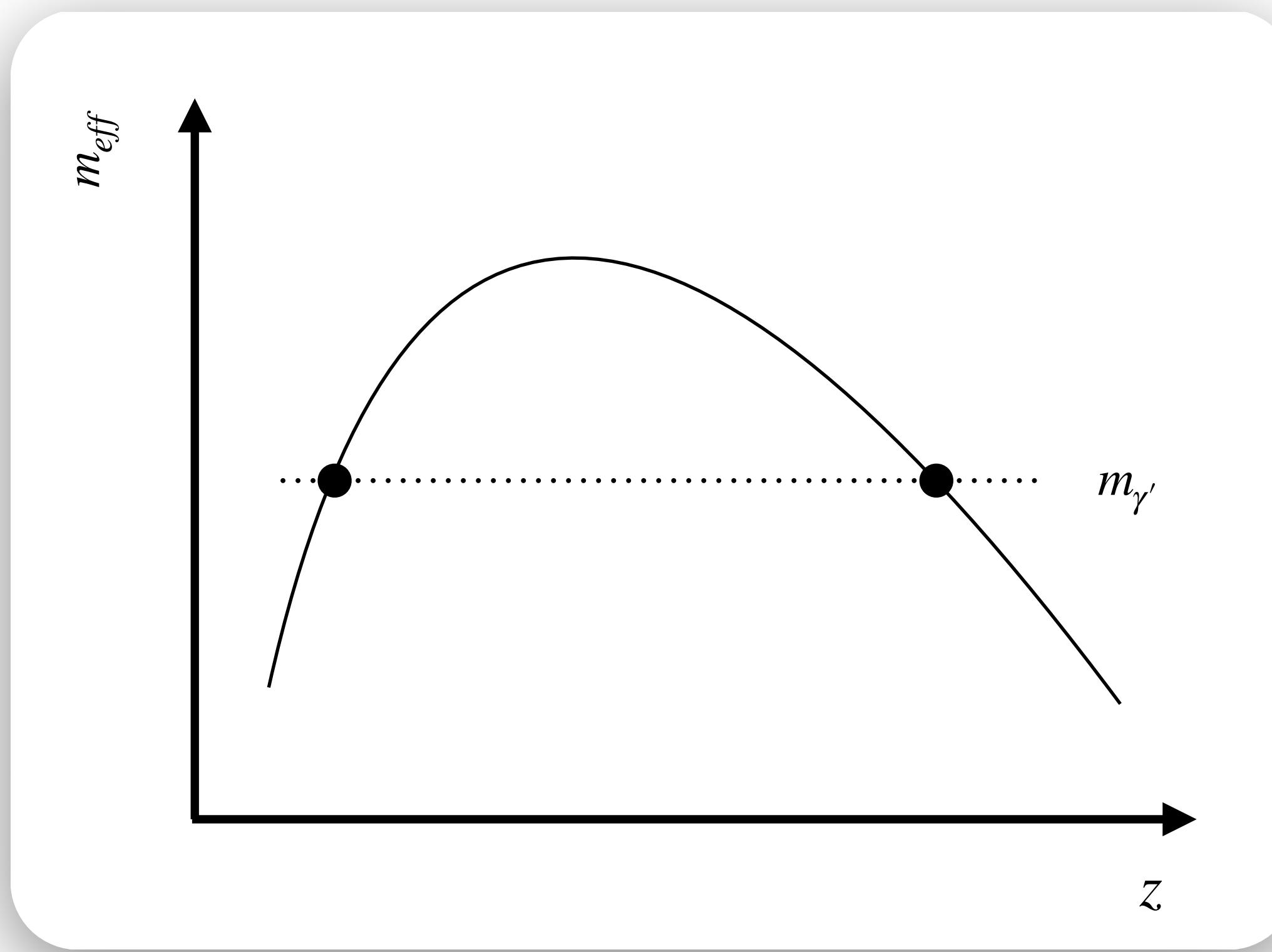
Stationary phase approximation

$$\Phi(m, z) = \Phi(m, z_0) + \Phi^{(1)}(m, z_0)(z - z_0) + \frac{1}{2!}\Phi^{(2)}(m, z_0)(z - z_0)^2 + \frac{1}{3!}\Phi^{(3)}(m, z_0)(z - z_0)^3 + \dots$$

- At critical point, $m = m_C$

$$\Phi^{(1)}(m_C, z_0) = \Phi^{(2)}(m_C, z_0) = 0$$

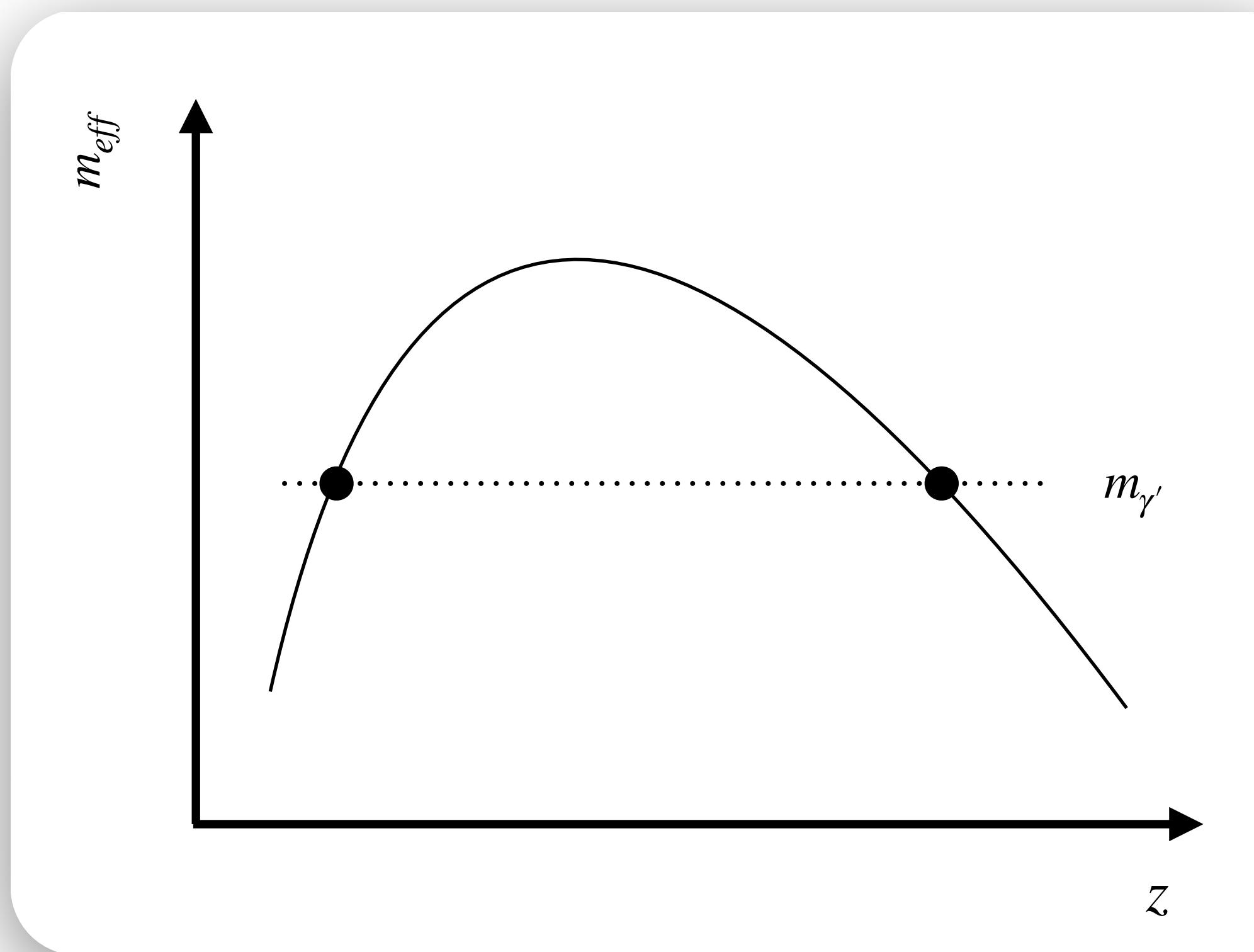
Sudden approximation



Sudden approximation

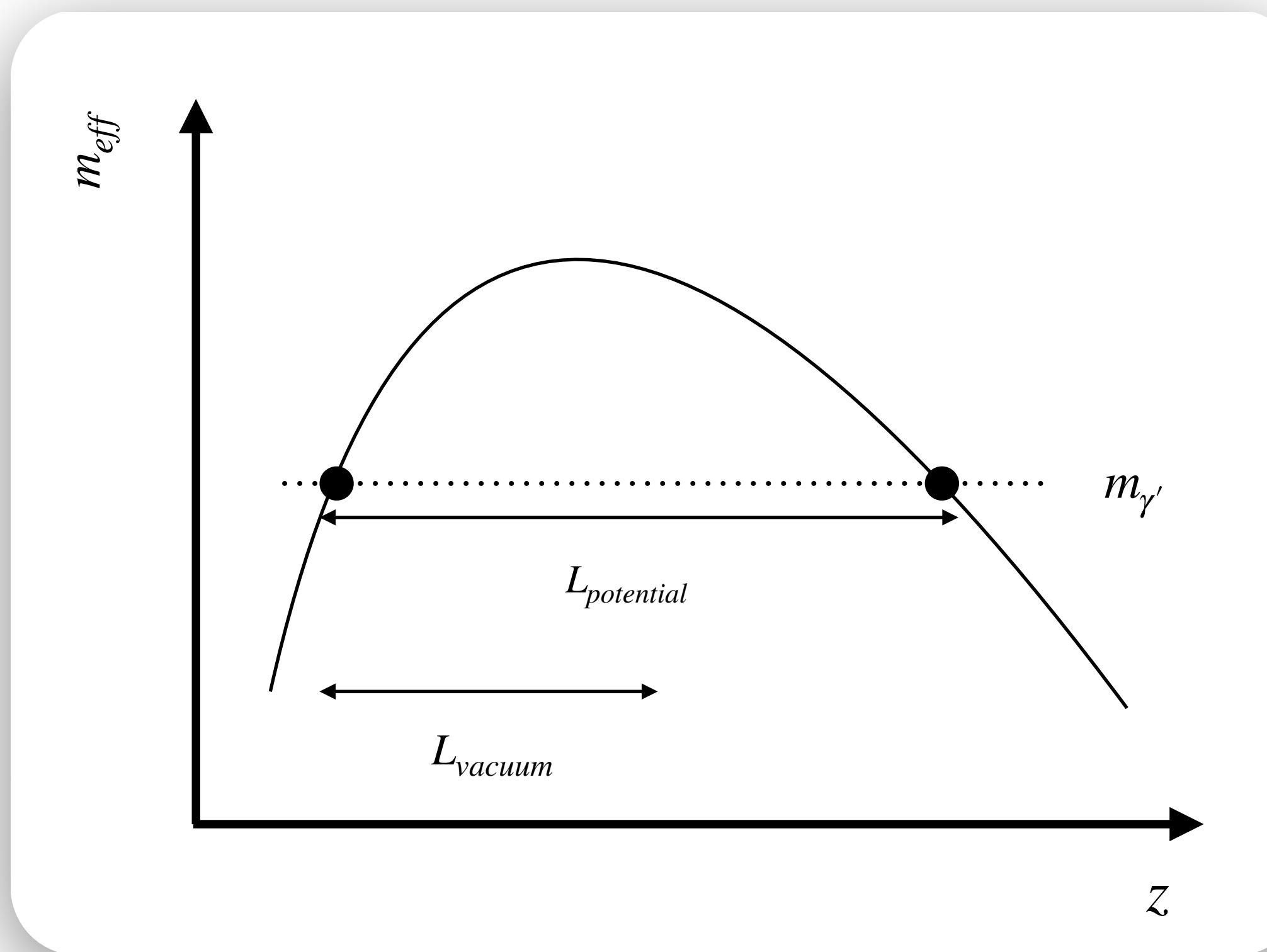
$$\mu \equiv \max(A_n)$$

“Resonance enhancement



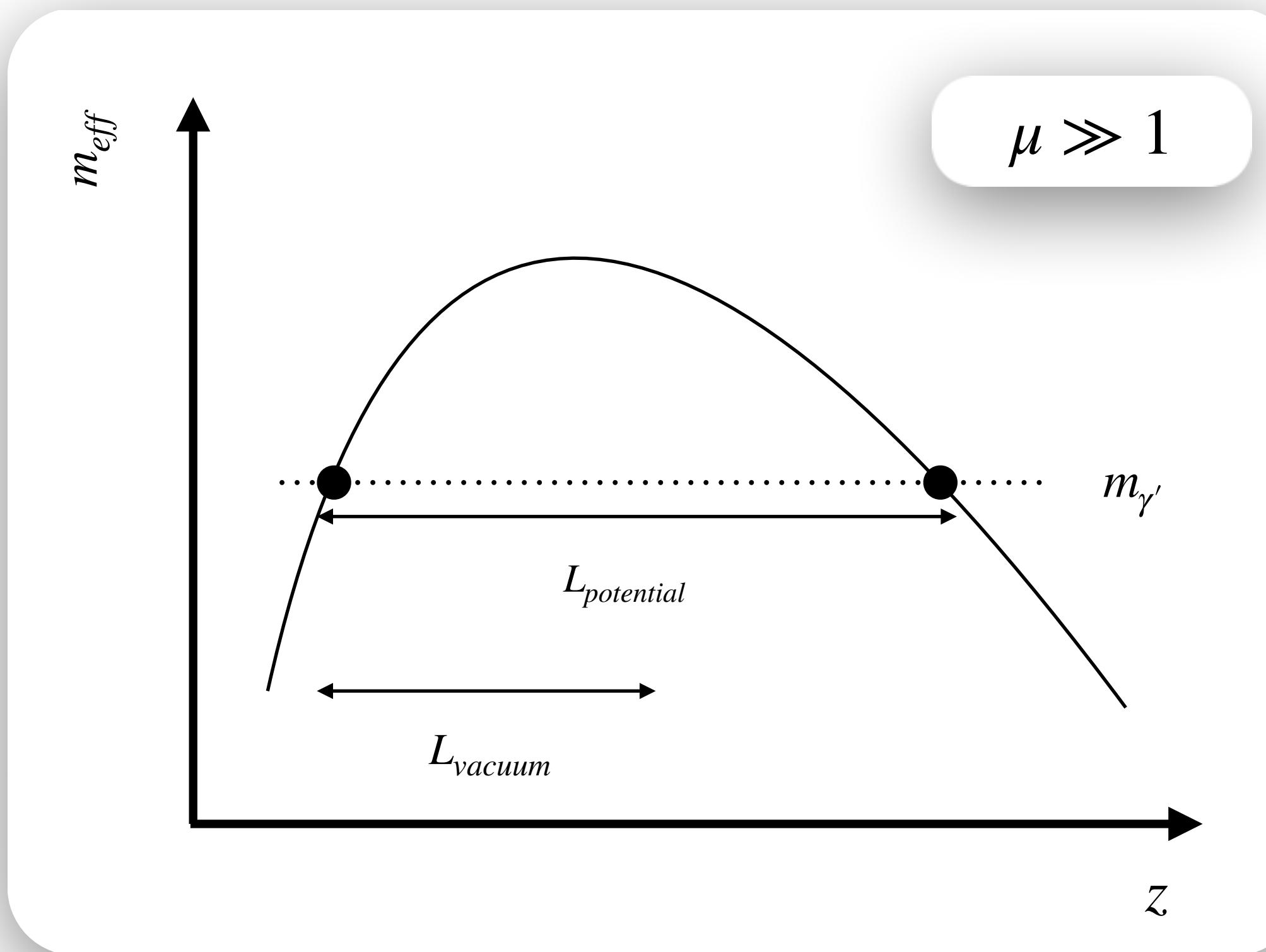
Sudden approximation

$$\mu \sim \frac{L_{potential}}{L_{vacuum}}$$



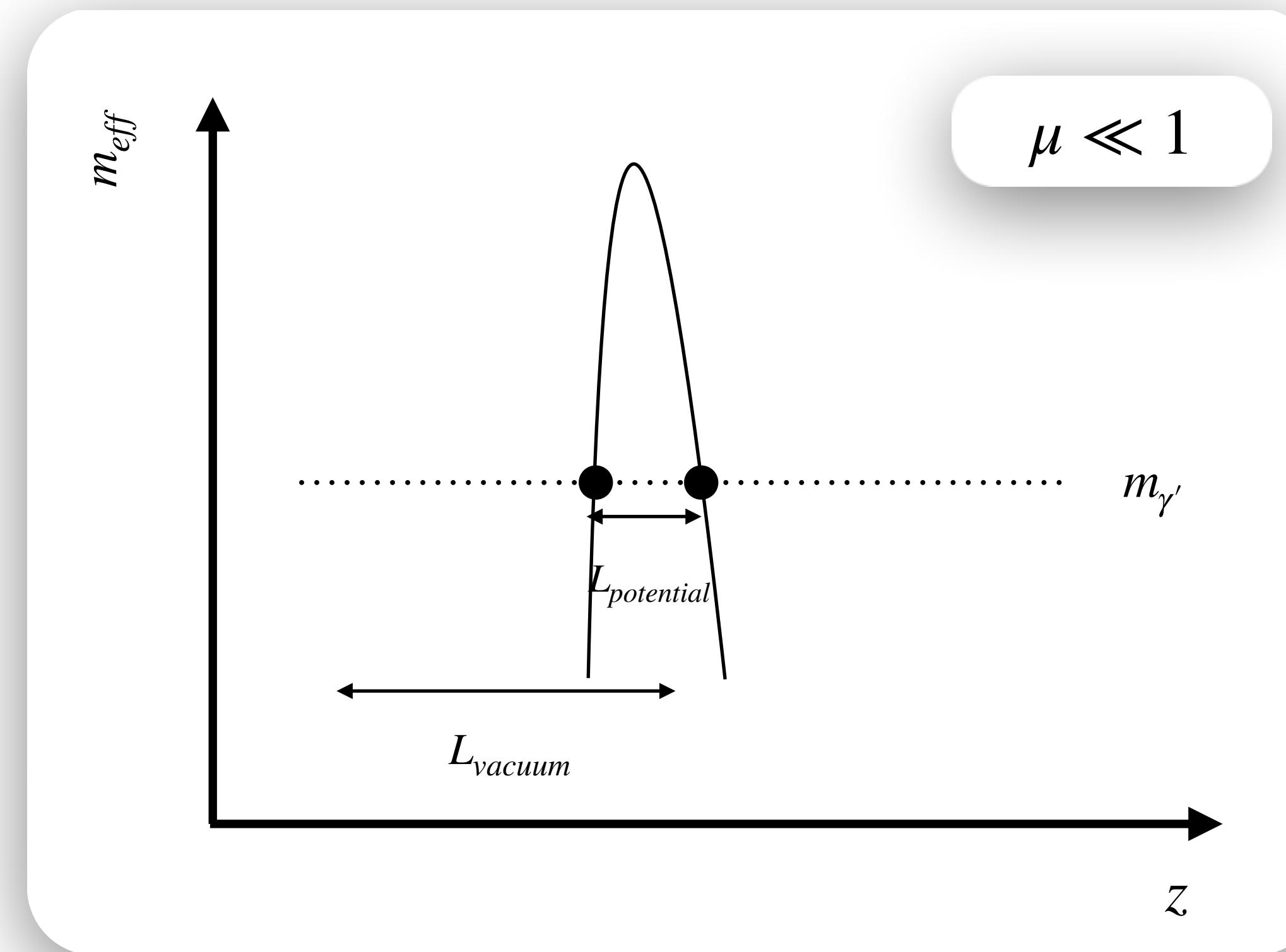
Sudden approximation

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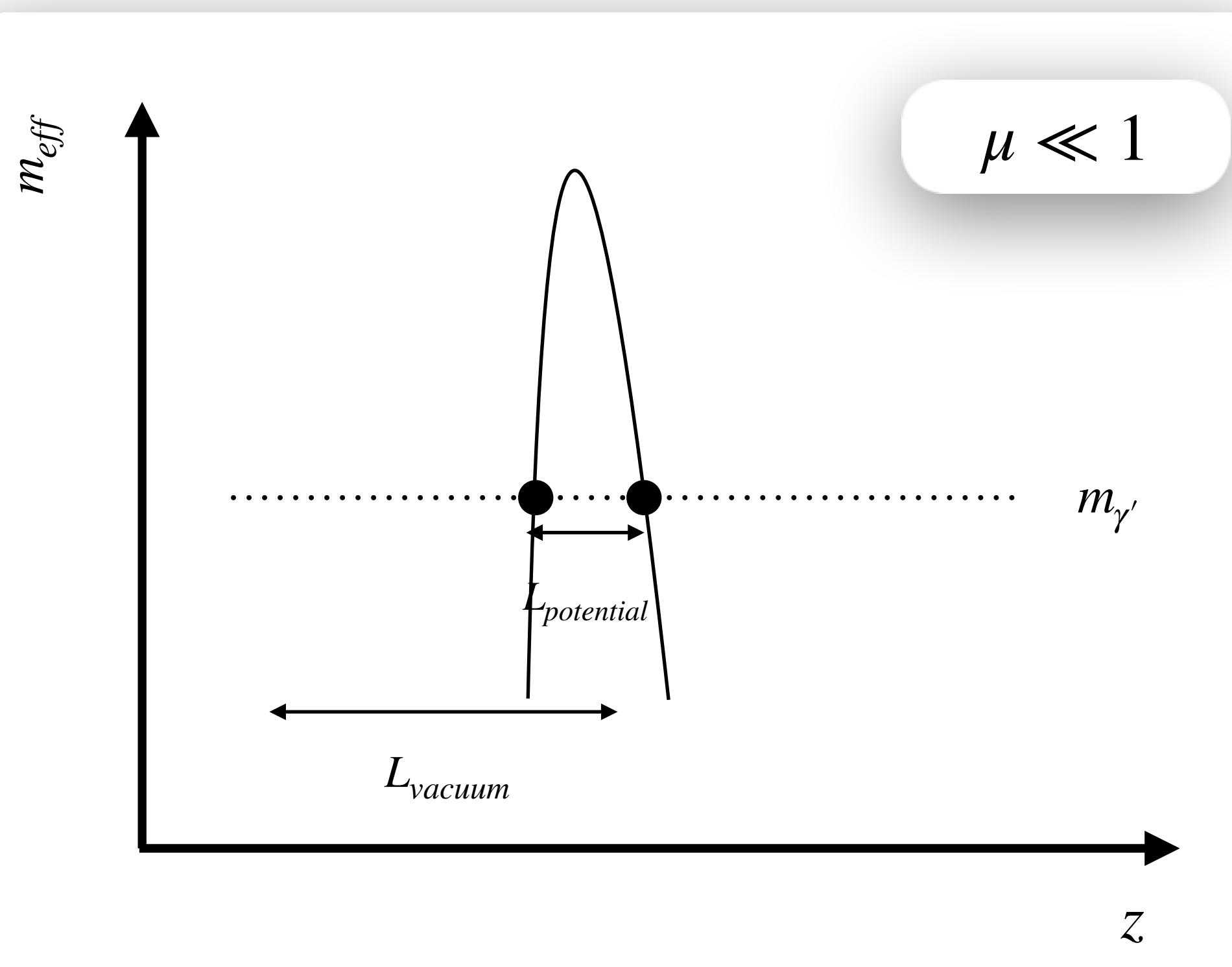
Sudden approximation

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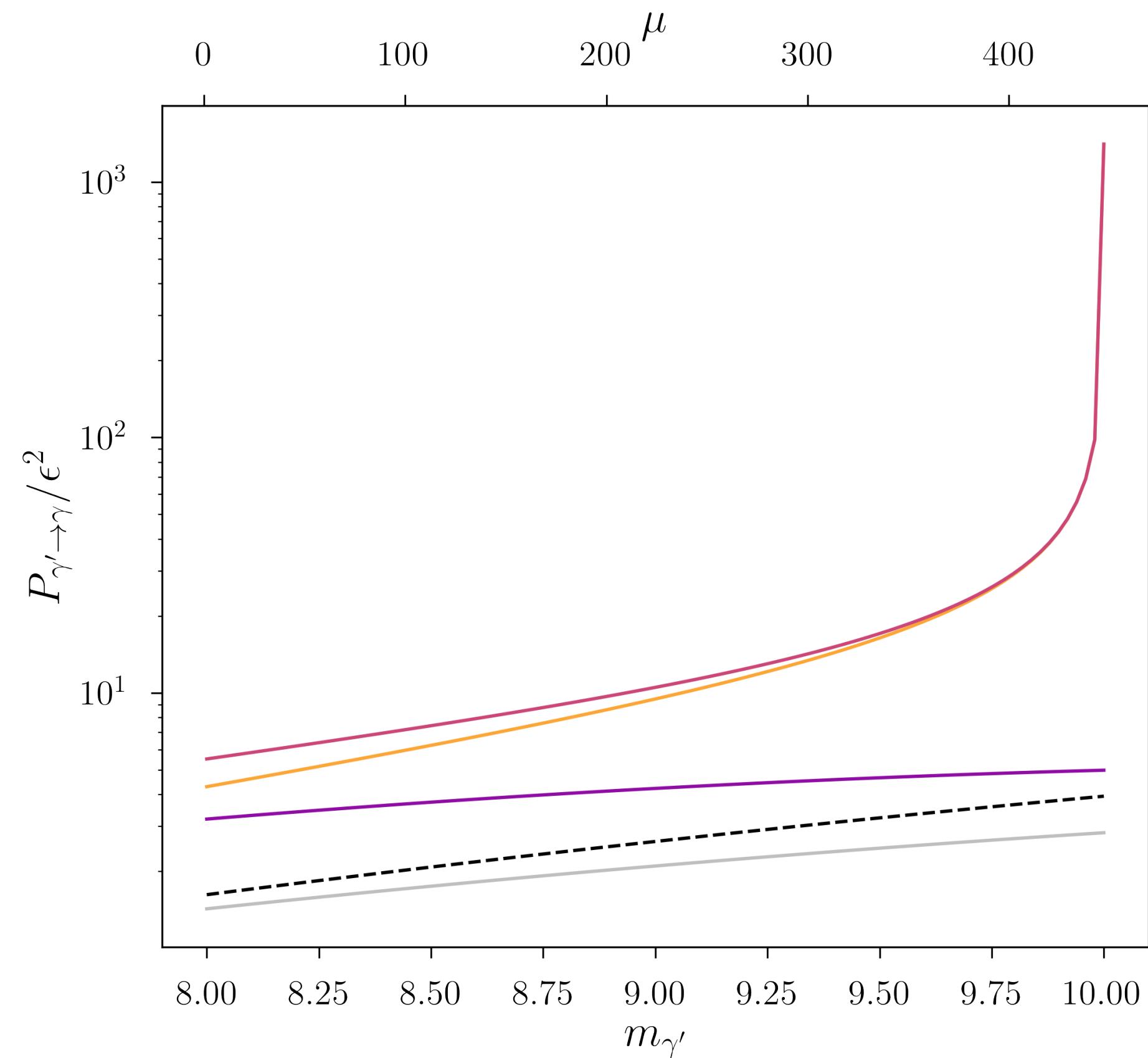


Sudden approximation

$\mu \ll 1 \longrightarrow P_{\gamma \leftrightarrow \gamma'} = \text{vacuum osc.}$



Toy model



■ Vacuum

■ LZ

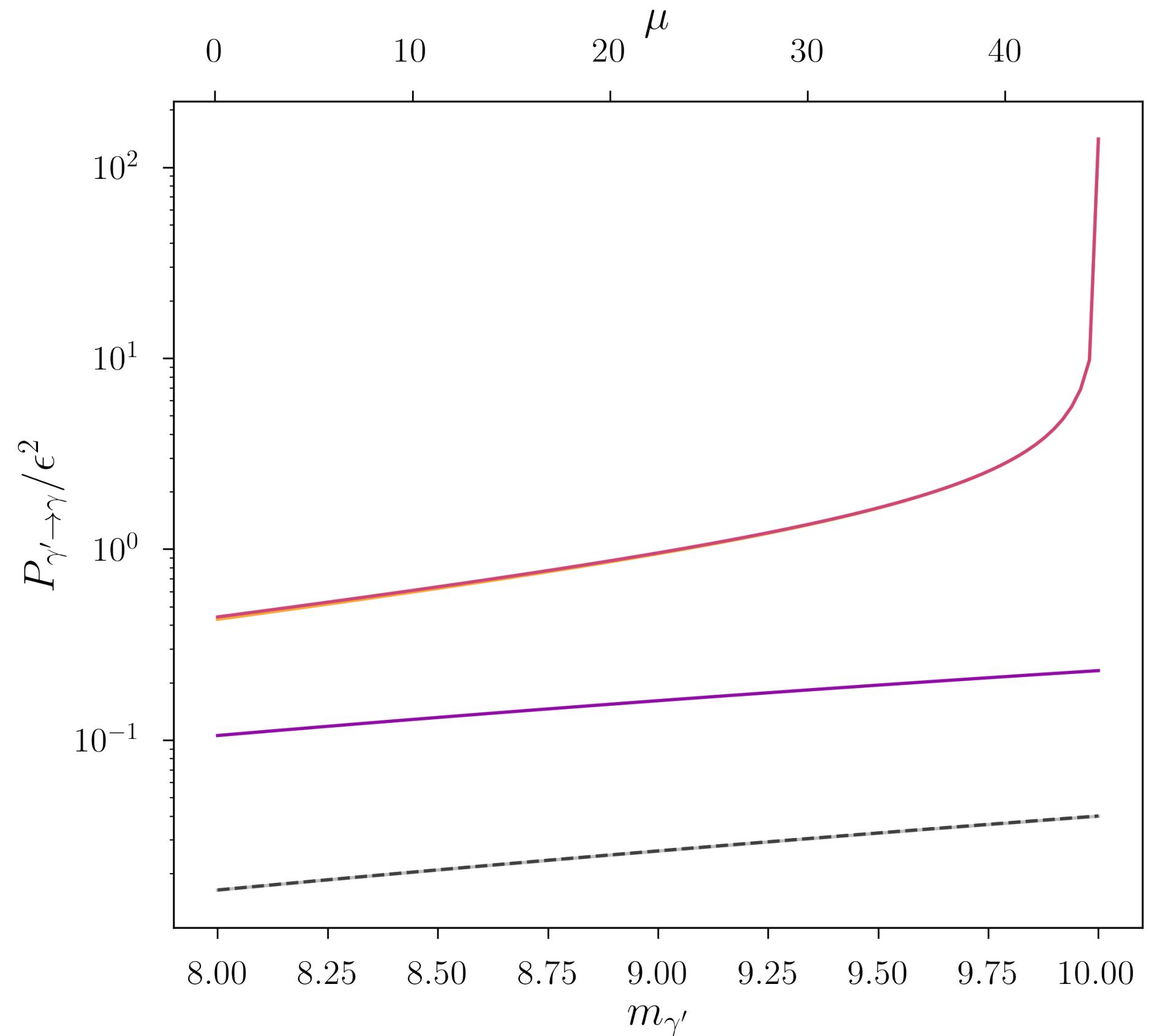
■ Phase

■ This work

■ ■ ■ Numerical

$a = 20, b = 10$

Toy model



Vacuum

LZ

Phase

This work

Numerical

$a = 2, b = 10$

Neutron star magnetospheres

- Effective mass induced by plasma

$$m_{eff}^2 = \frac{4\pi\alpha\rho_{GJ}}{em_e}$$

- Effective mass induced by large external magnetic fields

$$m_{eff}^2 = -\frac{7\alpha}{45\pi} \left(\frac{B_{ext}}{B_{crit}} \right)^2 \omega^2$$

- B_{ext} is dominated by the dipole component

Non-monotonic profiles and multiple resonances

$$\left| \int_{z_i}^{z_f} dz' \Delta_{\gamma'}(z') e^{-i\Phi(m_{\gamma'}, z')} \right|^2 \approx \left| \sum_n \sqrt{\frac{2\pi}{|\Phi^{(2)}(m_{\gamma'}, z_n)|}} \Delta_{\gamma'}(z_n) e^{-i\Phi(m_{\gamma'}, z_n) - i\sigma_n \frac{\pi}{4}} \right|^2$$

$$P_{\gamma \rightarrow \gamma'}(m_{\gamma'}) \approx 4\pi^2 \epsilon^2 \Delta_{\gamma'}^2(z_C) \left(\frac{2}{|\Phi_C^{(3)}(m_{\gamma'})|} \right)^{2/3} \left\{ \text{Ai}(-\zeta) + i\sigma_1 \left(\frac{2}{|\Phi_C^{(3)}(m_{\gamma'})|} \right)^{1/3} \left[\frac{\omega'_C}{\omega_C} - \frac{1}{6} \frac{\Phi_C^{(4)}(m_{\gamma'})}{\Phi_C^{(3)}(m_{\gamma'})} \right] \text{Ai}'(-\zeta) \right\}^2$$

$$\zeta(m_{\gamma'}) = \left(\frac{2}{|\Phi^{(3)}(z_C, m)|} \right)^{1/3} \Phi^{(1)}(z_C, m)$$