Detecting New Nanometer-Range Forces Using Coherent Neutron Scattering

Zach Bogorad

Based on arXiv:2303.17744 (**ZB**, P. Graham, and G. Gratta)

New scalars can mediate macroscopic forces

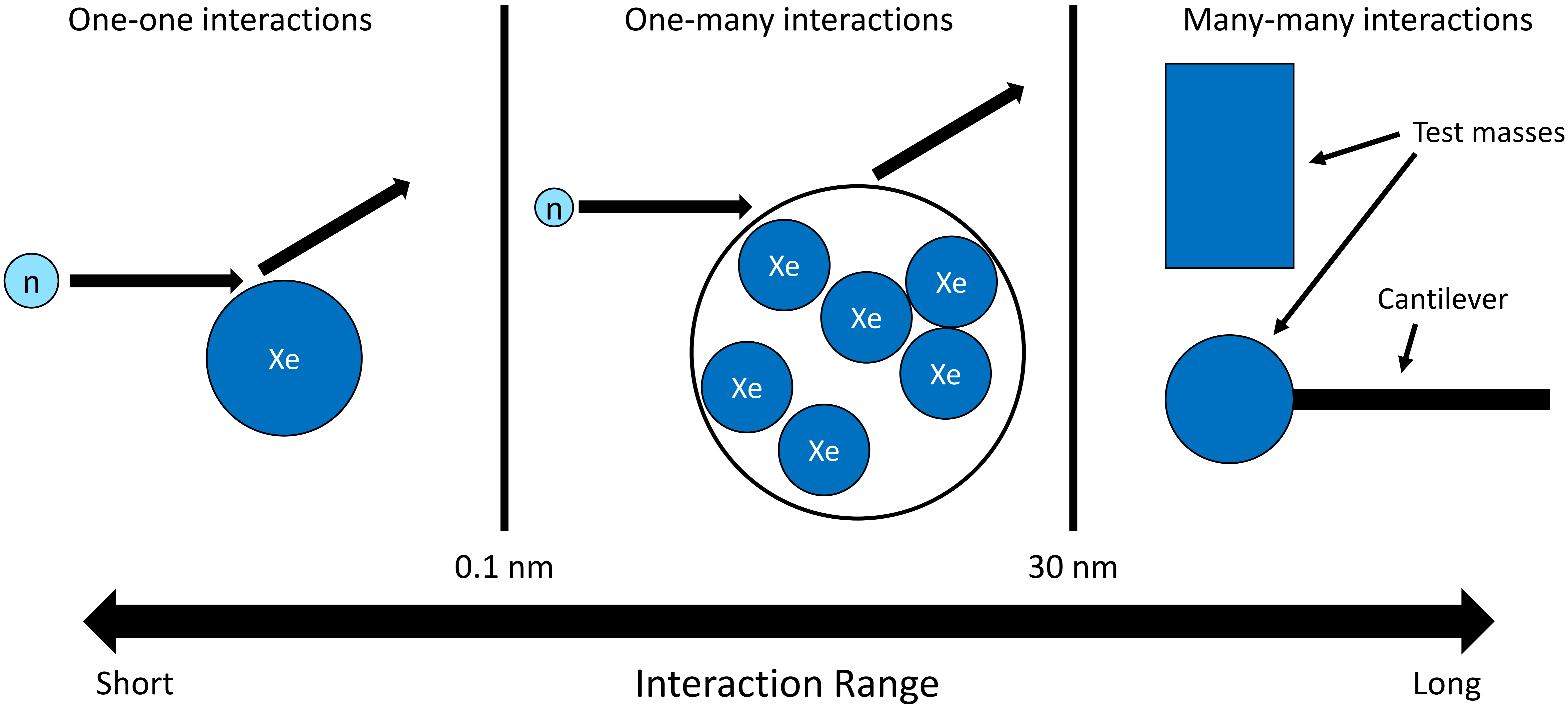
- Two possible fermion vertices:

$$\begin{array}{ll} \text{Scalar:} & \phi\bar{\psi}\psi \\ \text{Pseudoscalar:} & \phi\bar{\psi}i\gamma^5\psi \end{array}$$

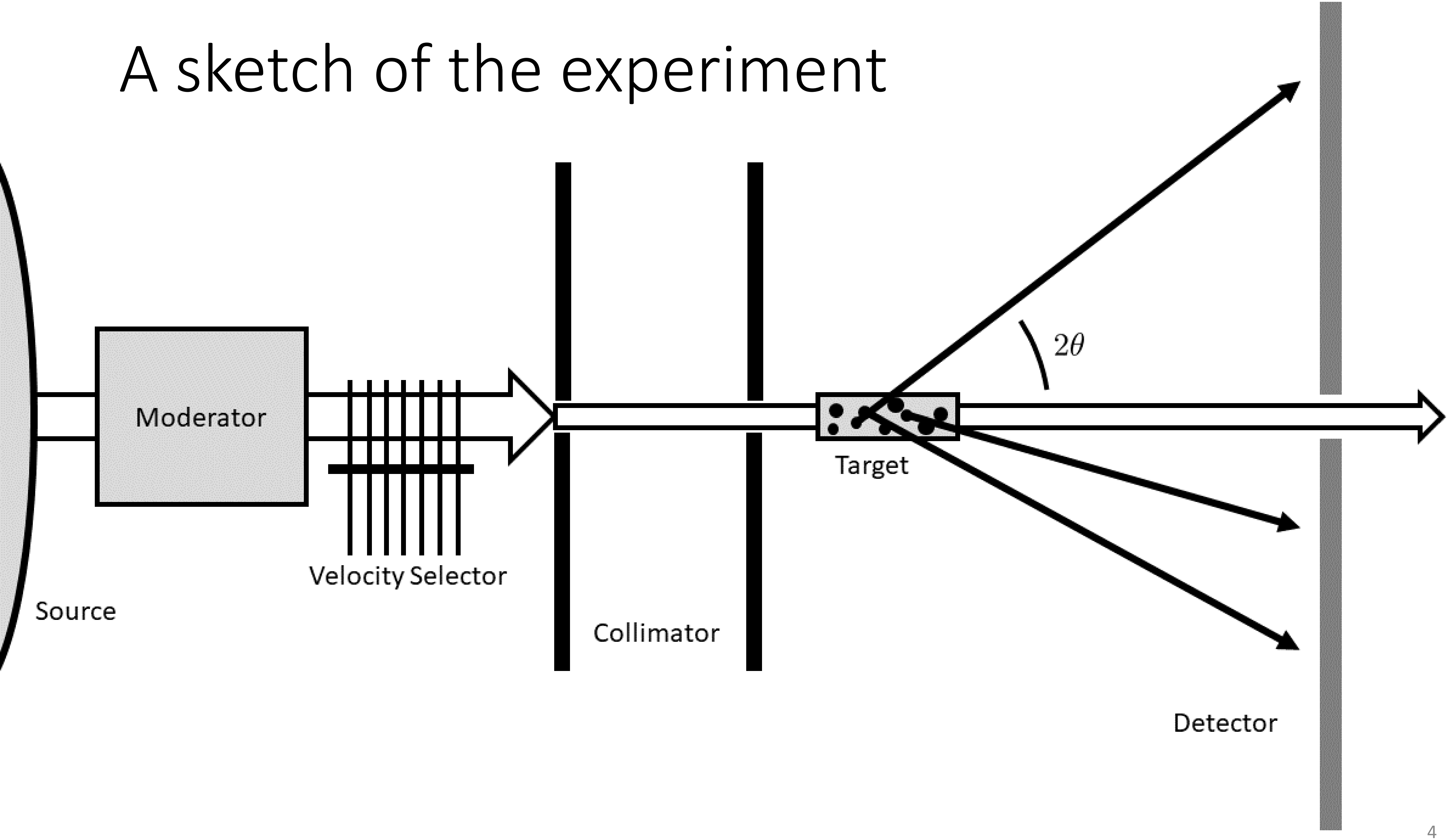
- In this talk we'll only care about the scalar-scalar potential:

$$V_{ss}(r) = -\frac{g_{s,1}g_{s,2}}{4\pi}\left(\frac{1}{r}\right)e^{-\mu r}$$

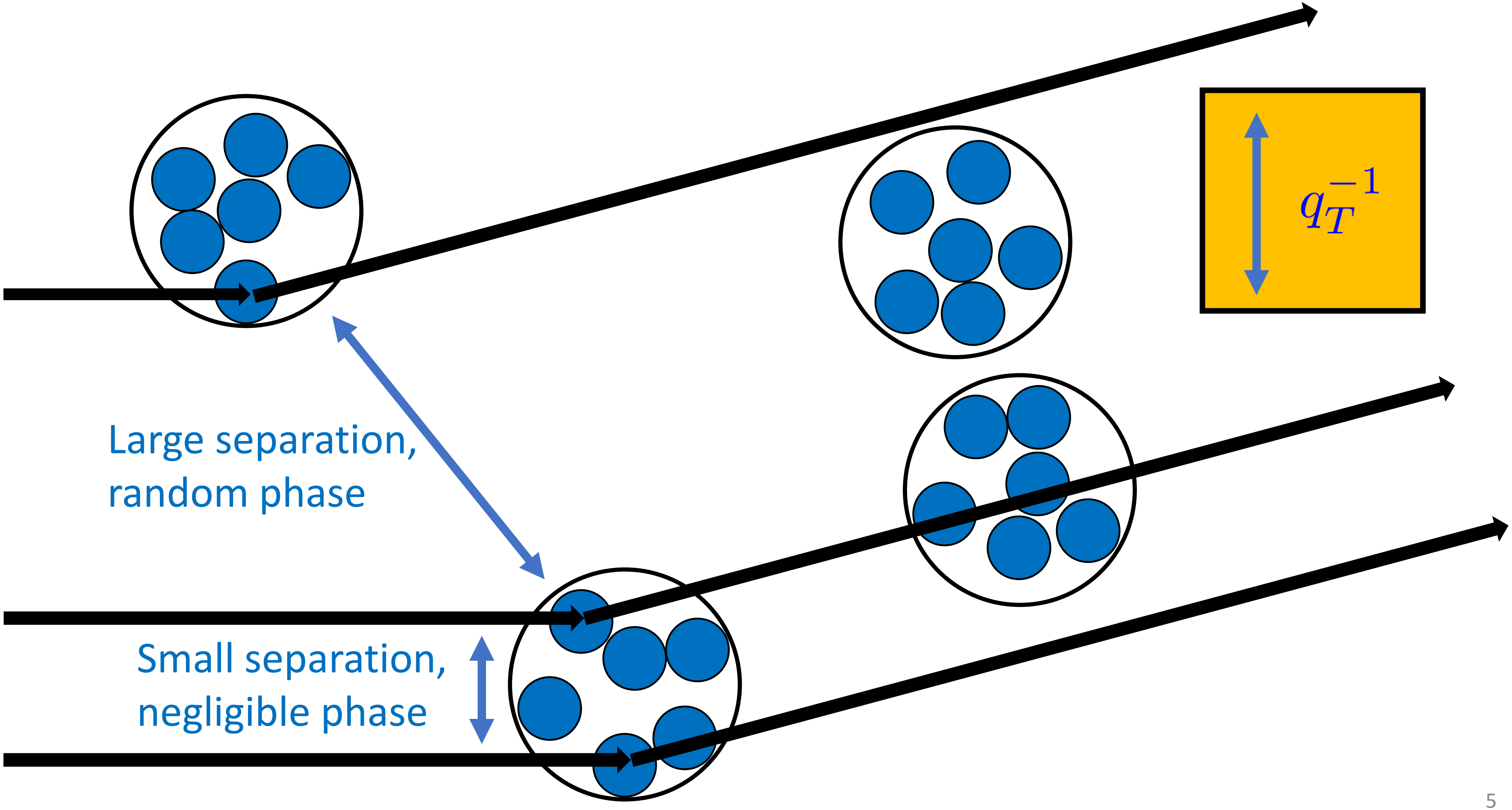
Single-particle versus collective interactions



A sketch of the experiment



Coherence from separated lengthscales



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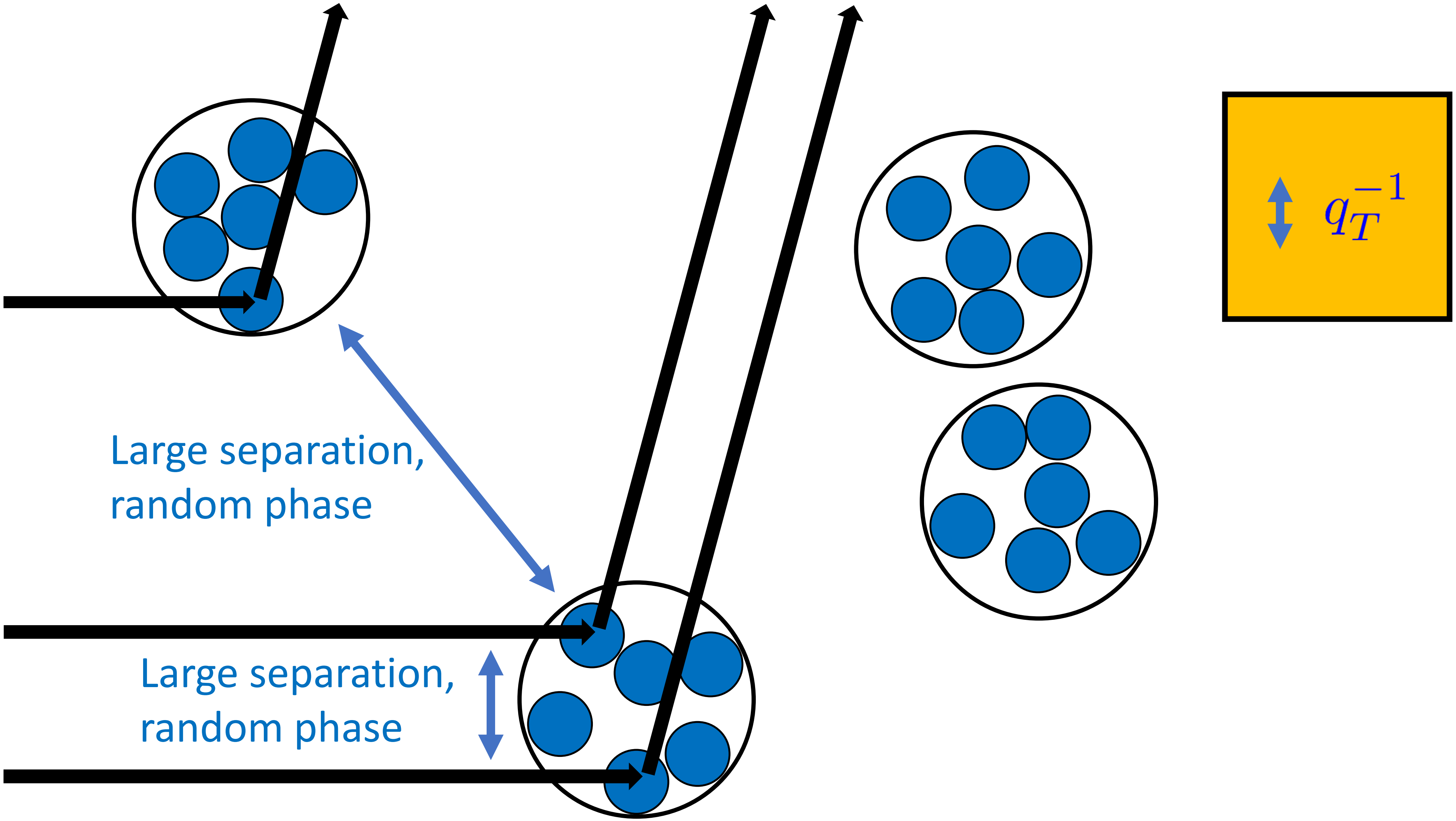
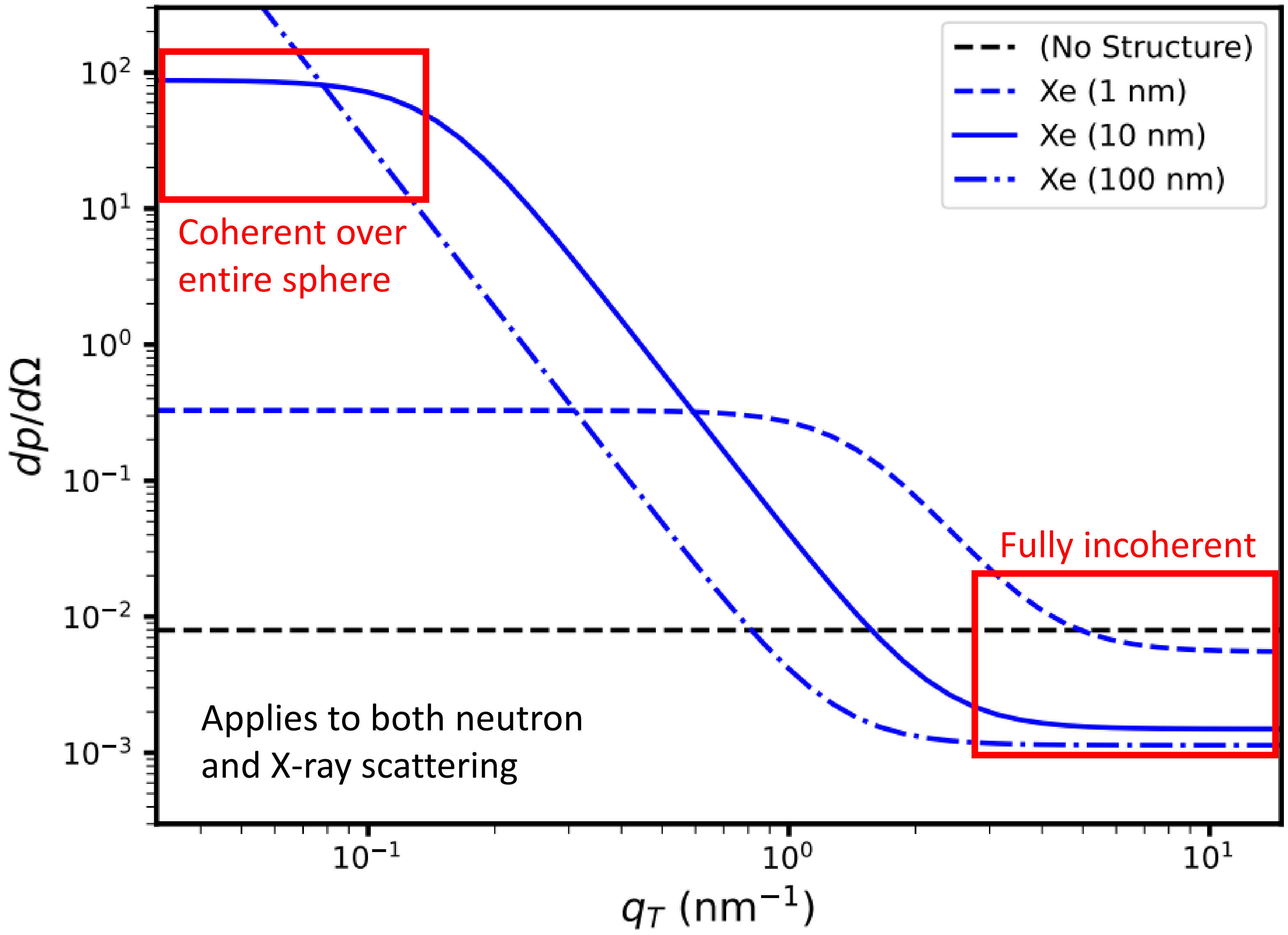


Structure factors: scaling behavior

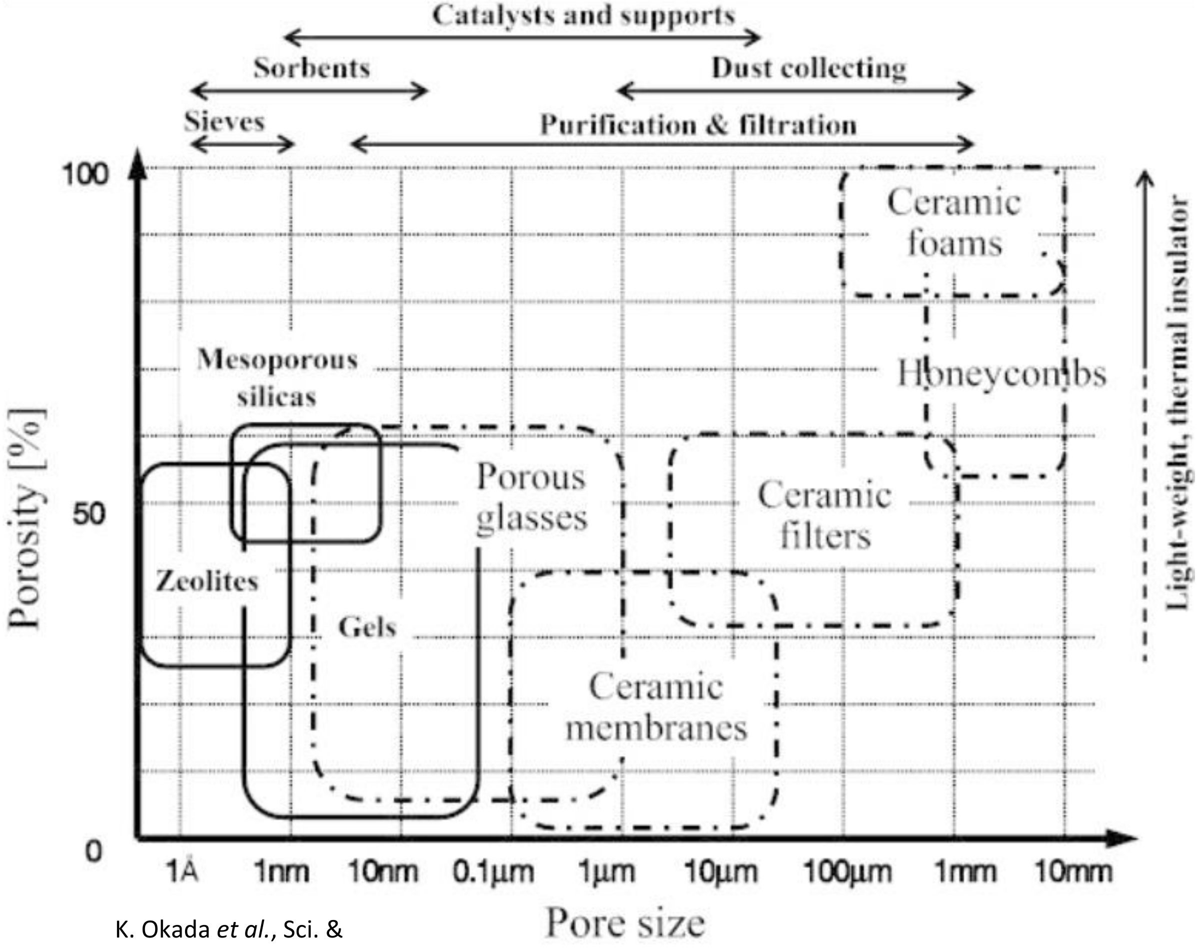


Target material candidates

- Single-material:
  - Noble “snow”
  - Aerosols
  - Boiling liquids

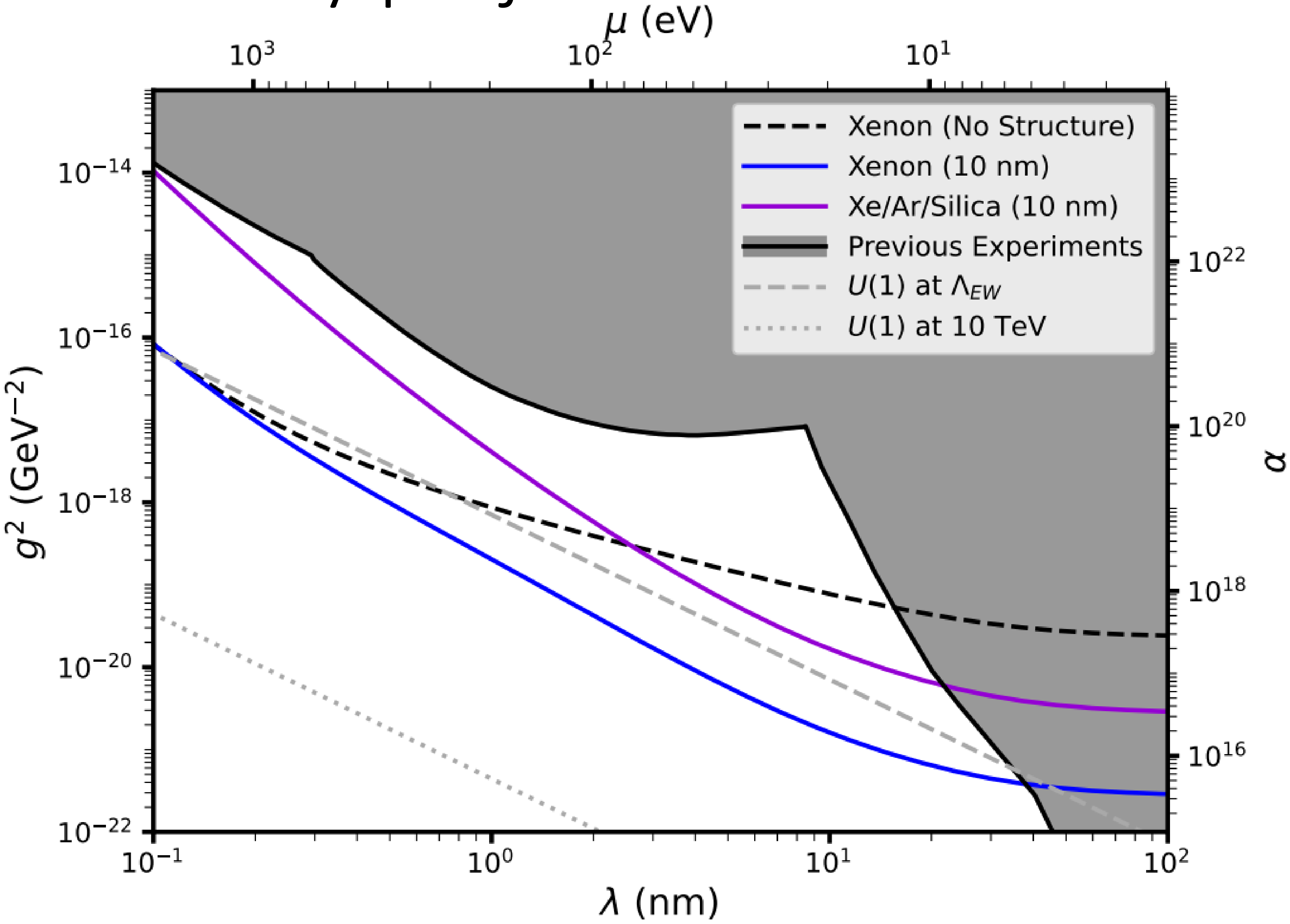
Image from https://www.youtube.com/watch?v=QtDPv637KHY&ab_channel=AttilaDobi , from Carter Hall’s group at UMD

- Two-material:



K. Okada *et al.*, Sci. & Tech. of Adv. Materials **12**, 064701 (2011)

Sensitivity projections



Astrophysical bounds are typically below the bottom of the plot, but are highly model-dependent

Thank you!

Questions?

This talk is based on work supported by the Simons Investigator Award No. 824870, NSF Grant No. PHY-2014215, DOE HEP QuantISED Award No. 100495, the Gordon and Betty Moore Foundation Grant No. GBMF7946, and the U.S. Department of Energy (DOE), Office of Science, National Quantum Information Science Research Centers, Superconducting Quantum Materials and Systems Center (SQMS) under contract No. DE-AC02-07CH11359. ZB is supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1656518 and by the Robert and Marvel Kirby Fellowship and the Dr. HaiPing and Jianmei Jin Fellowship from the Stanford Graduate Fellowship Program. GG is supported, in part, by DoE grant DE-SC0017970.

Backup Slides

Outline

- New forces: motivation and previous experiments
- Single-material targets: how they work and possible implementation
- Two-material targets: challenges and possible implementation

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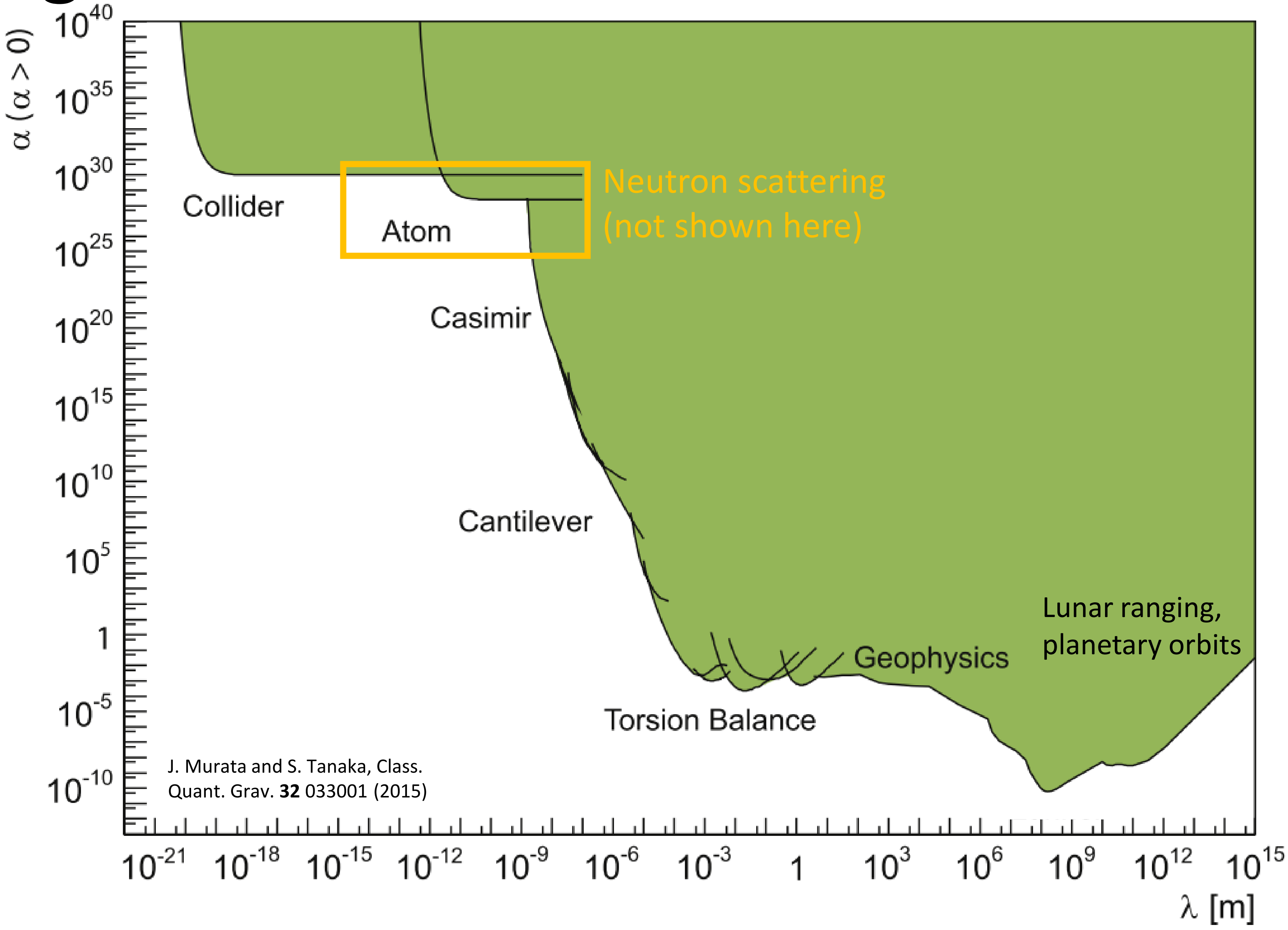
Existing limits on new forces



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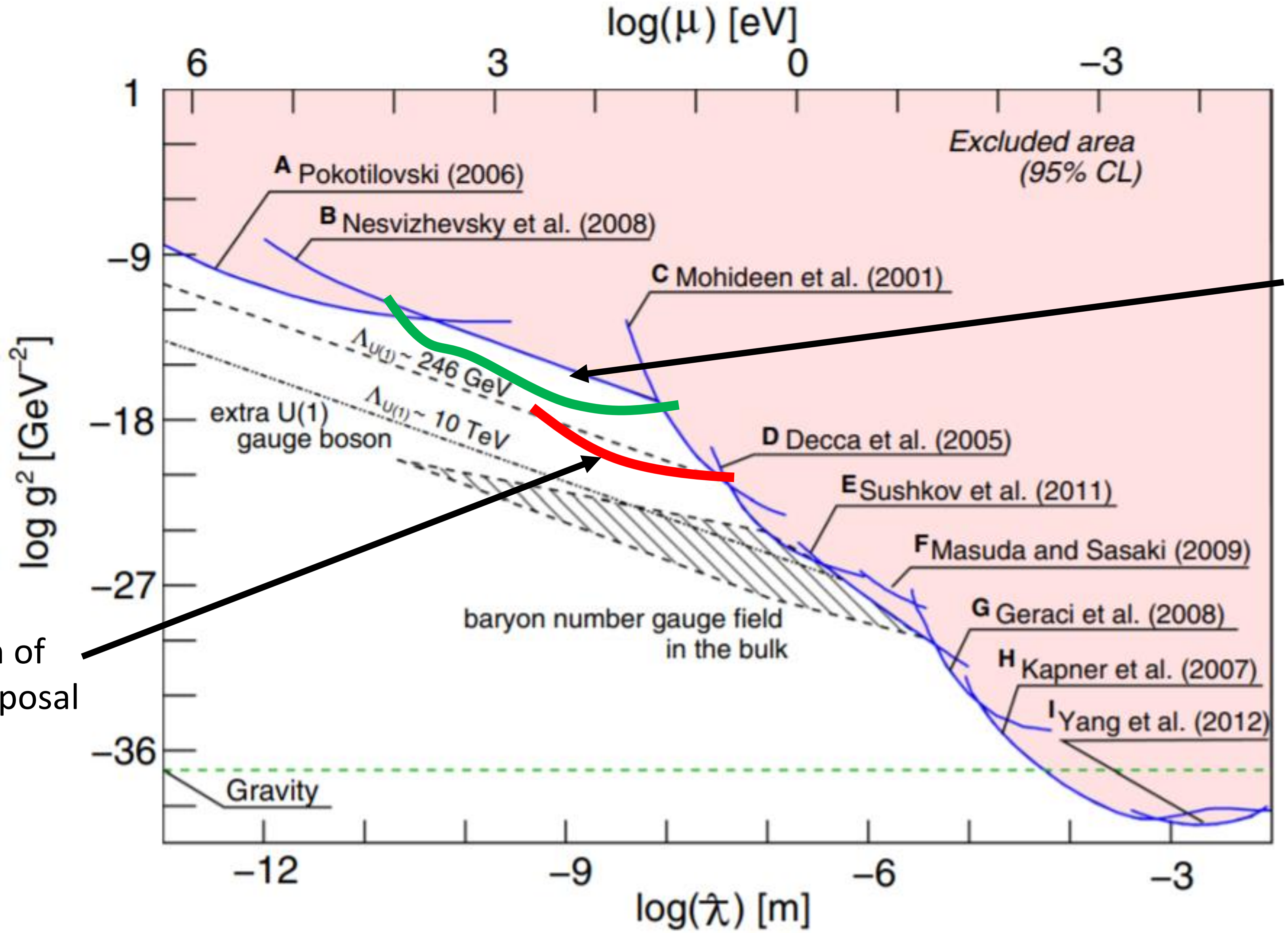
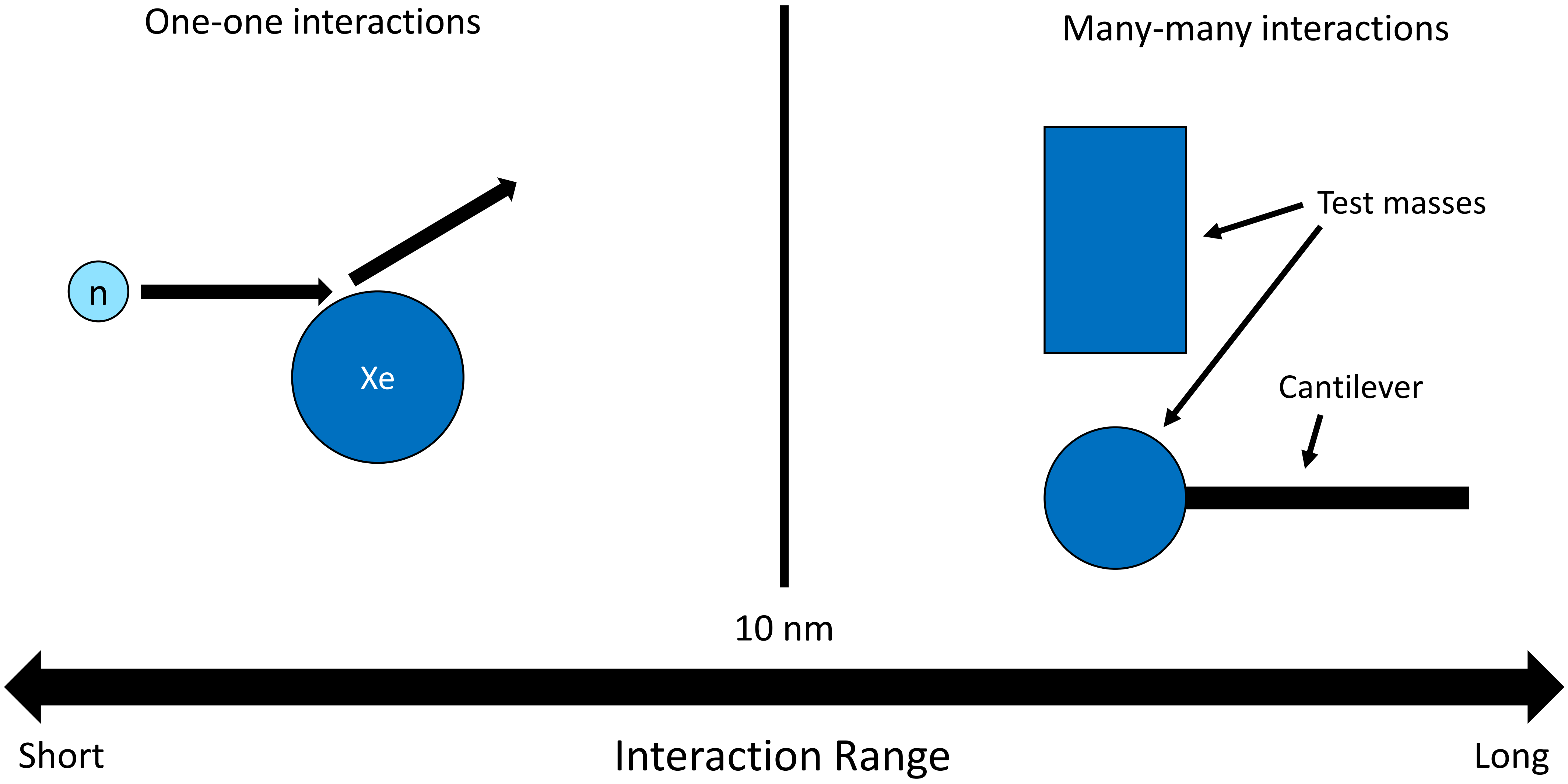


(Relatively) recent limits using neutron scattering from xenon

Potential reach of this work's proposal

Y. Kamiya *et al.*, PRL **114**, 161101 (2015)

Single-particle versus collective interactions



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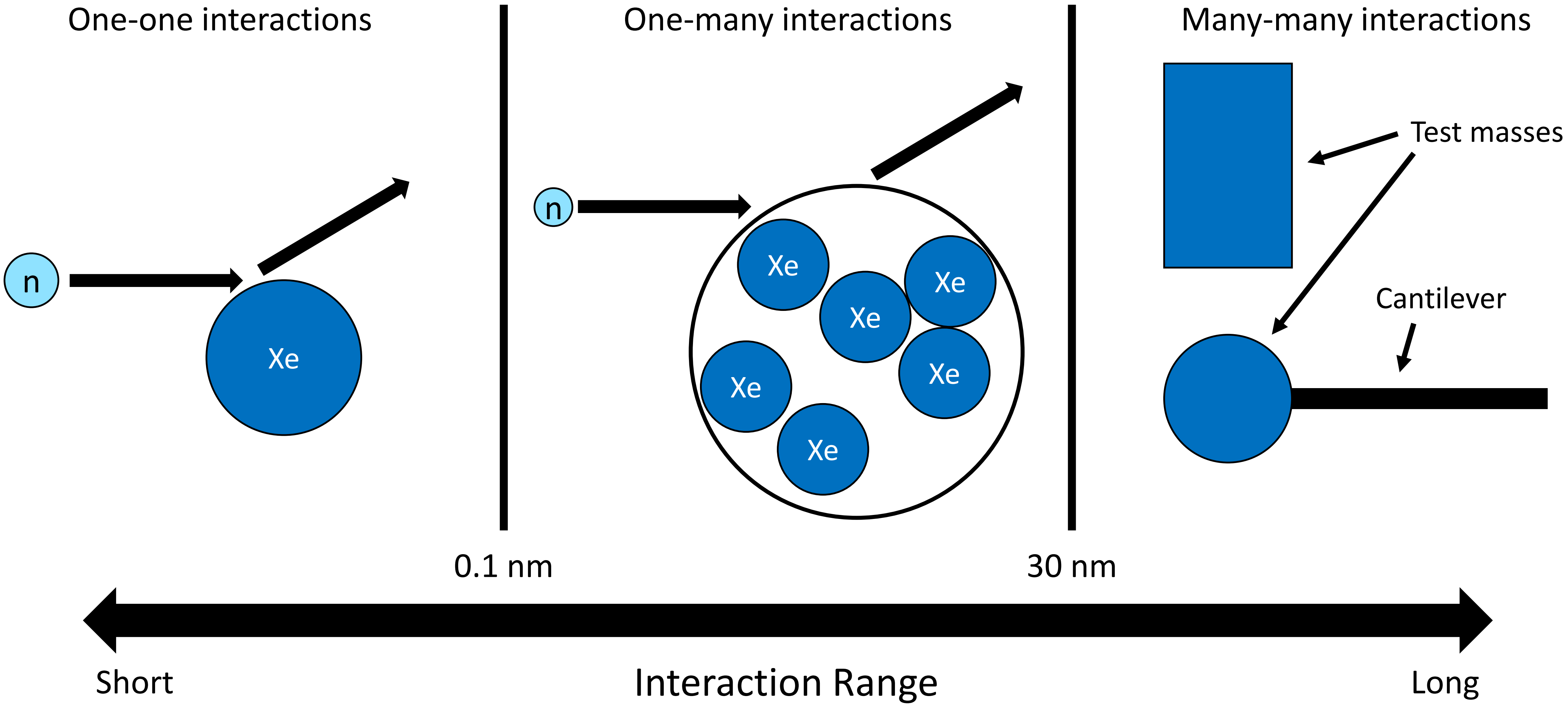


Outline

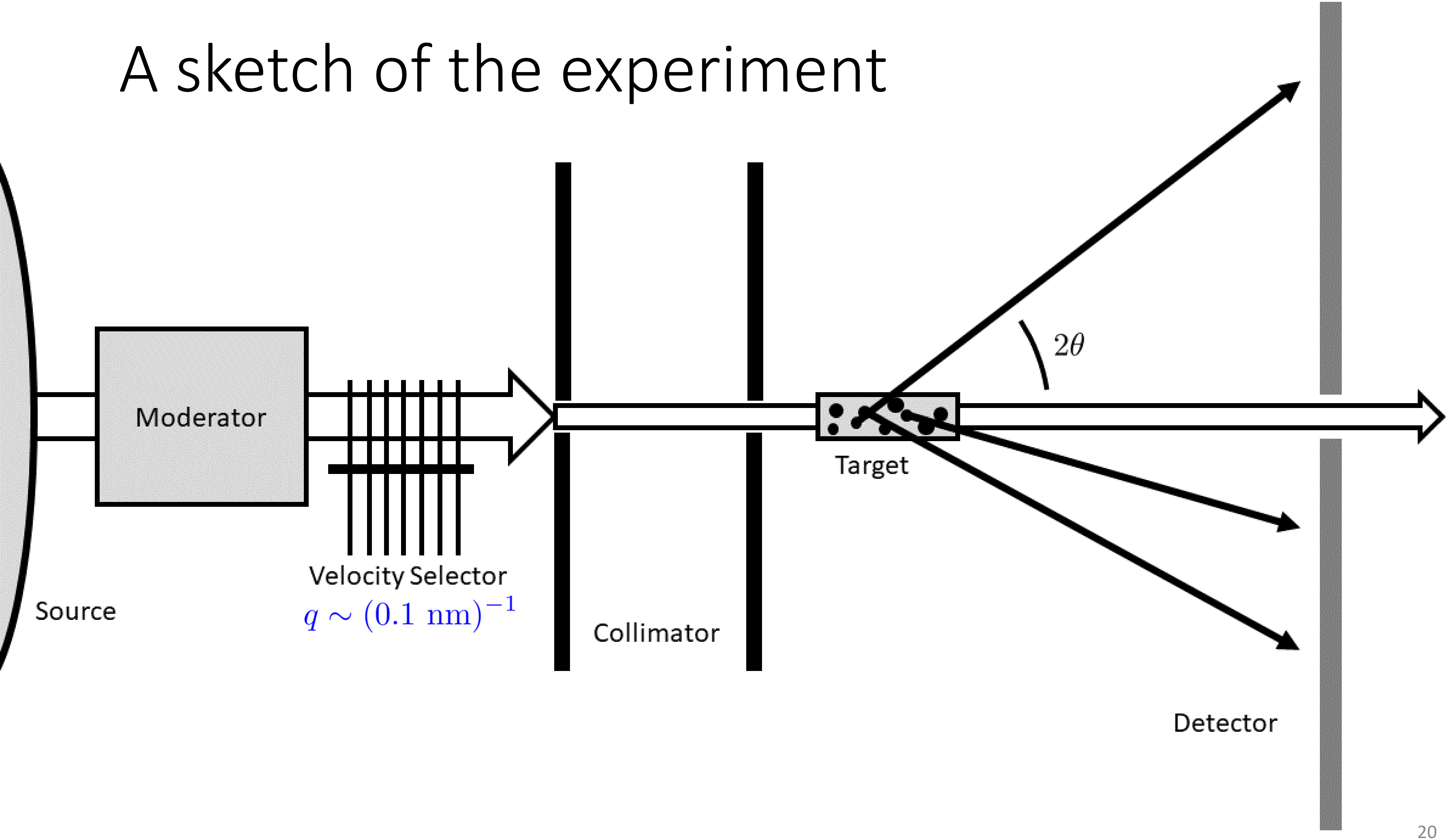
- New forces: motivation and previous experiments
- Single-material targets: how they work and possible implementation
- Two-material targets: challenges and possible implementation

Background

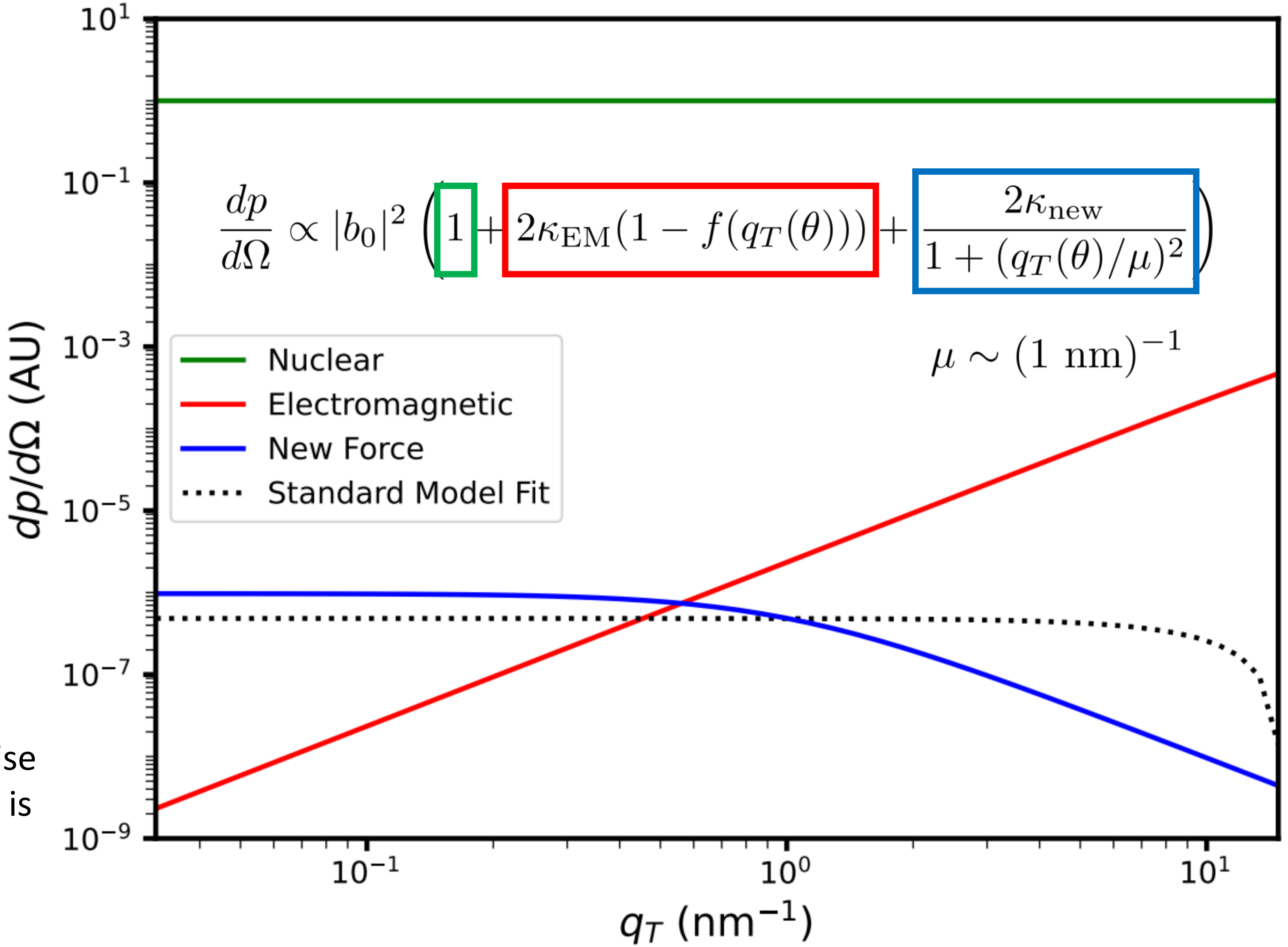
Single-Material

Two-Material

A sketch of the experiment



Three main sources of neutron scattering



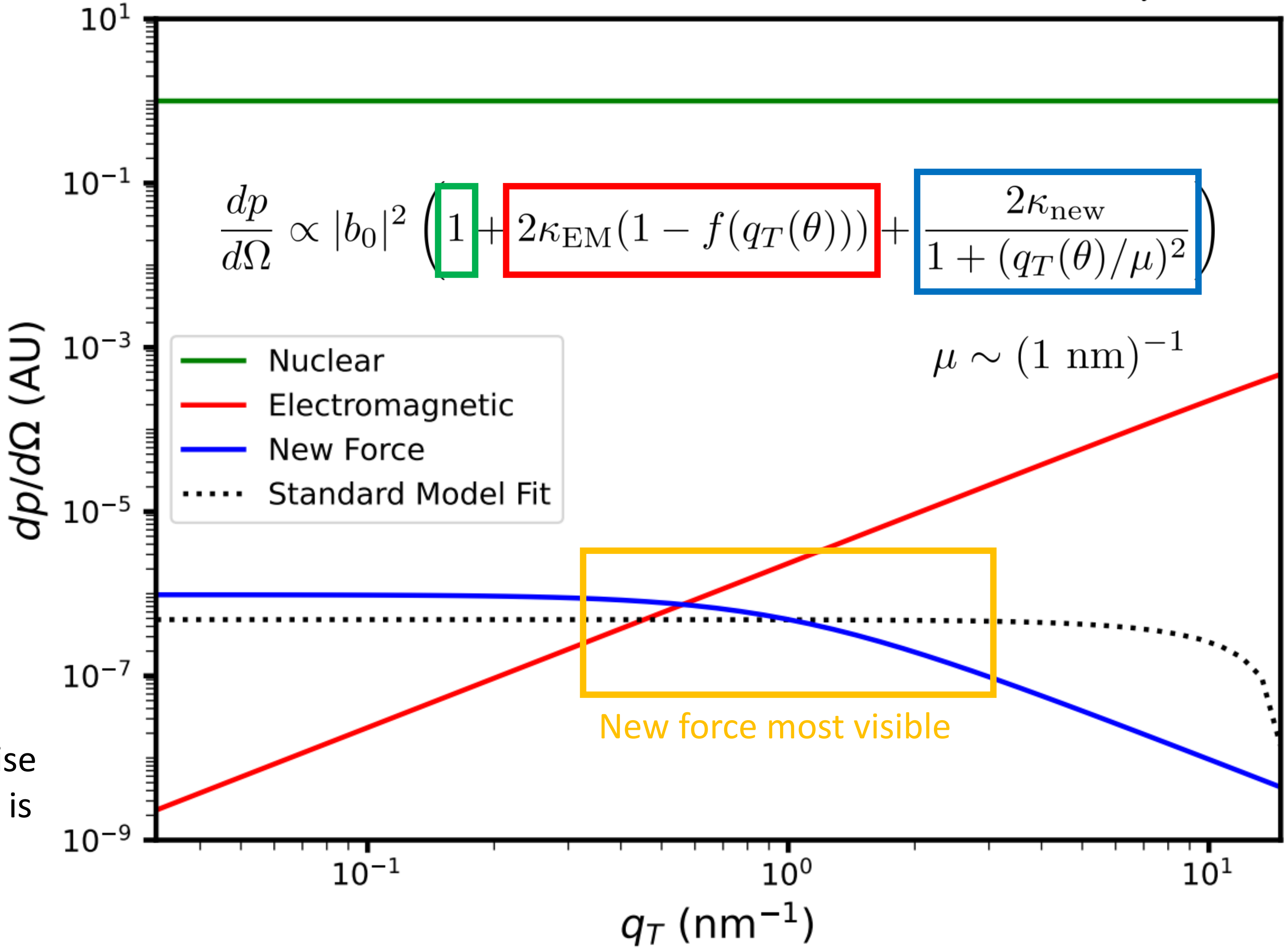
Plot is for noble elements: otherwise electromagnetism is much more complicated

Background

Single-Material

Two-Material

New forces are most visible at $q_T \sim \mu$



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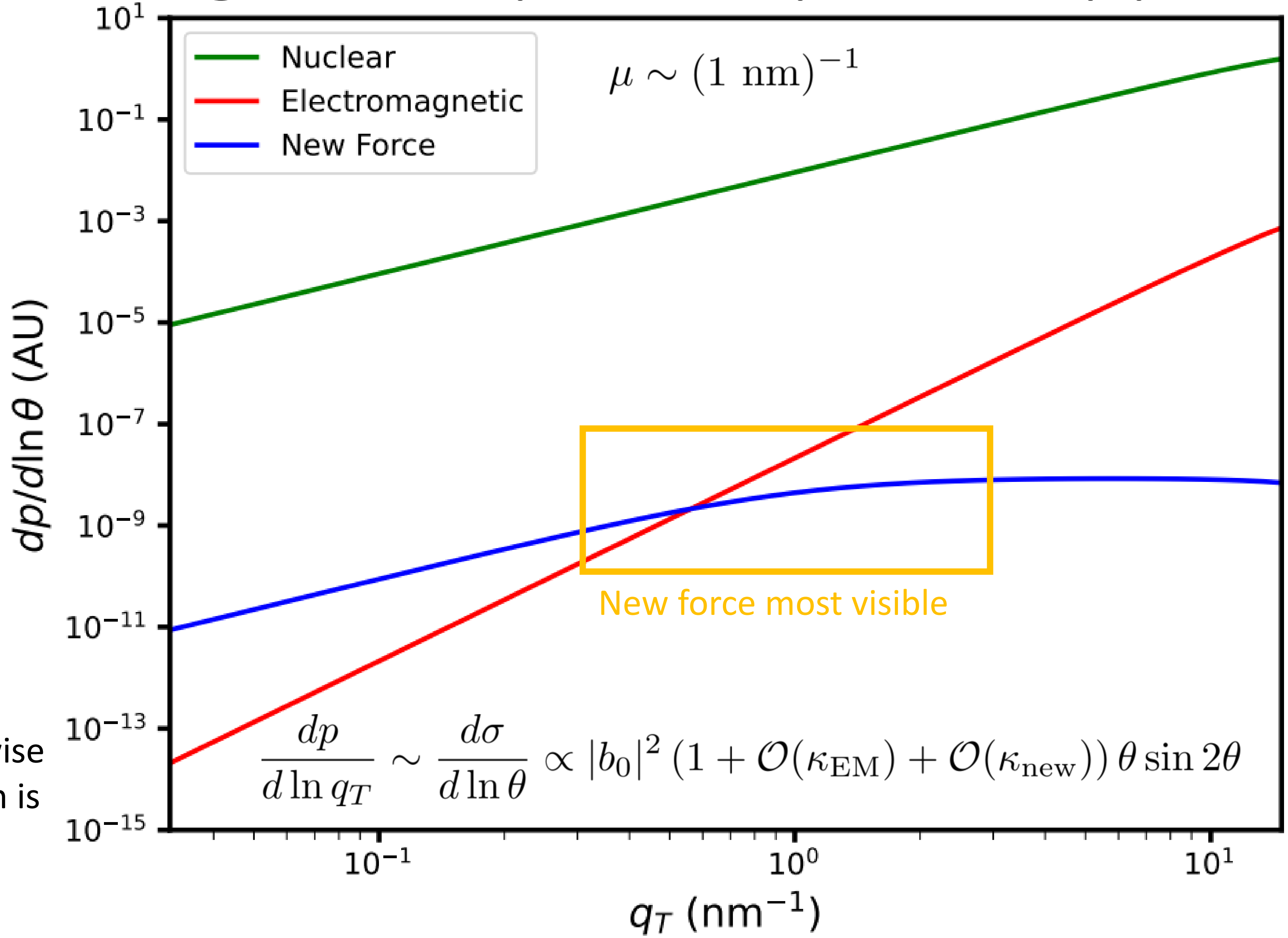
Optimal angles are phase space-suppressed



Plot is for noble elements: otherwise electromagnetism is much more complicated

A familiar modification: Bragg scattering

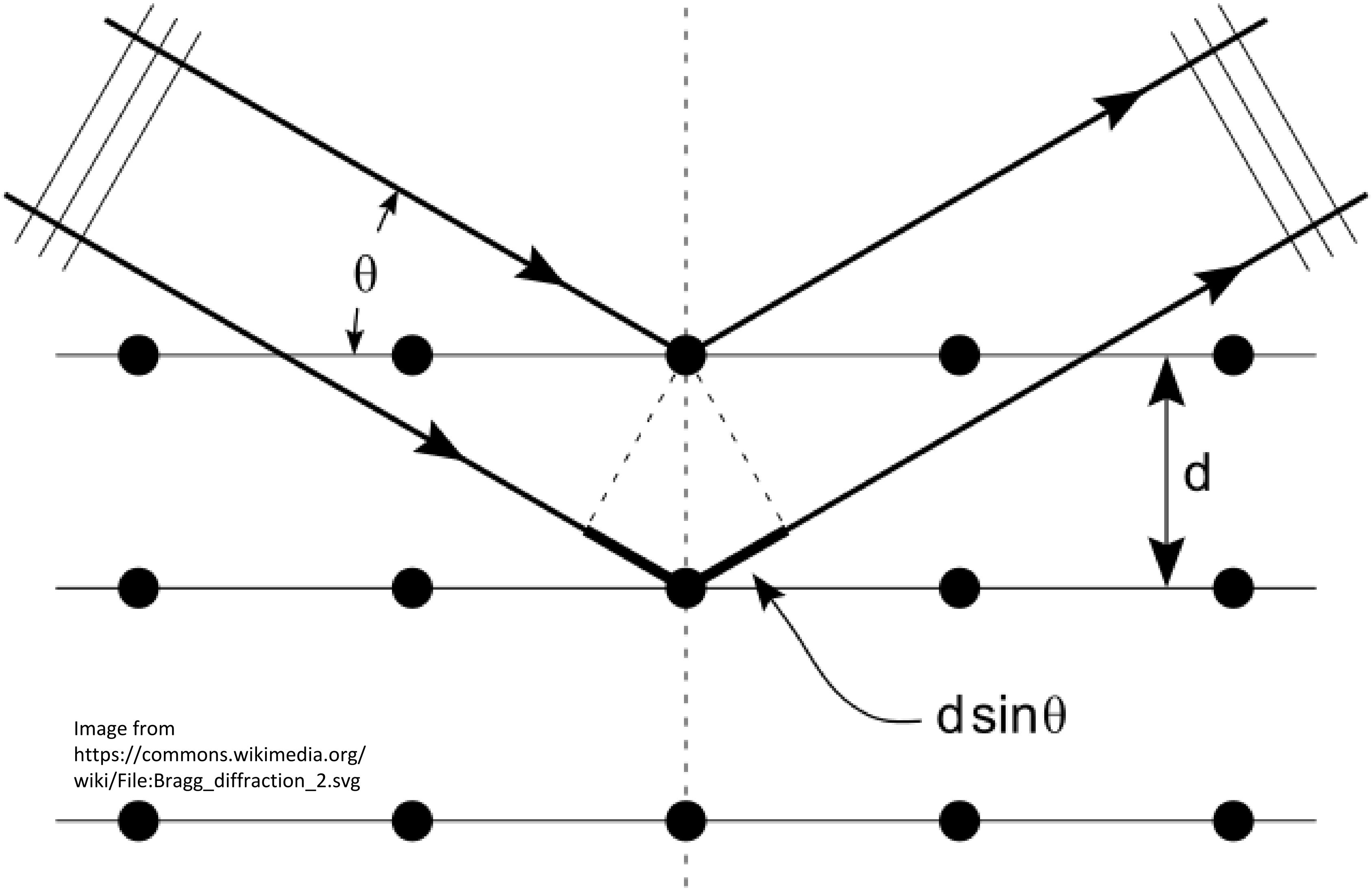


Image from
https://commons.wikimedia.org/
wiki/File:Bragg_diffraction_2.svg

Target structure can change scattering distributions

- Each scatterer in the target comes in with a factor of

$$e^{i\mathbf{q}_T\cdot\mathbf{r}}$$

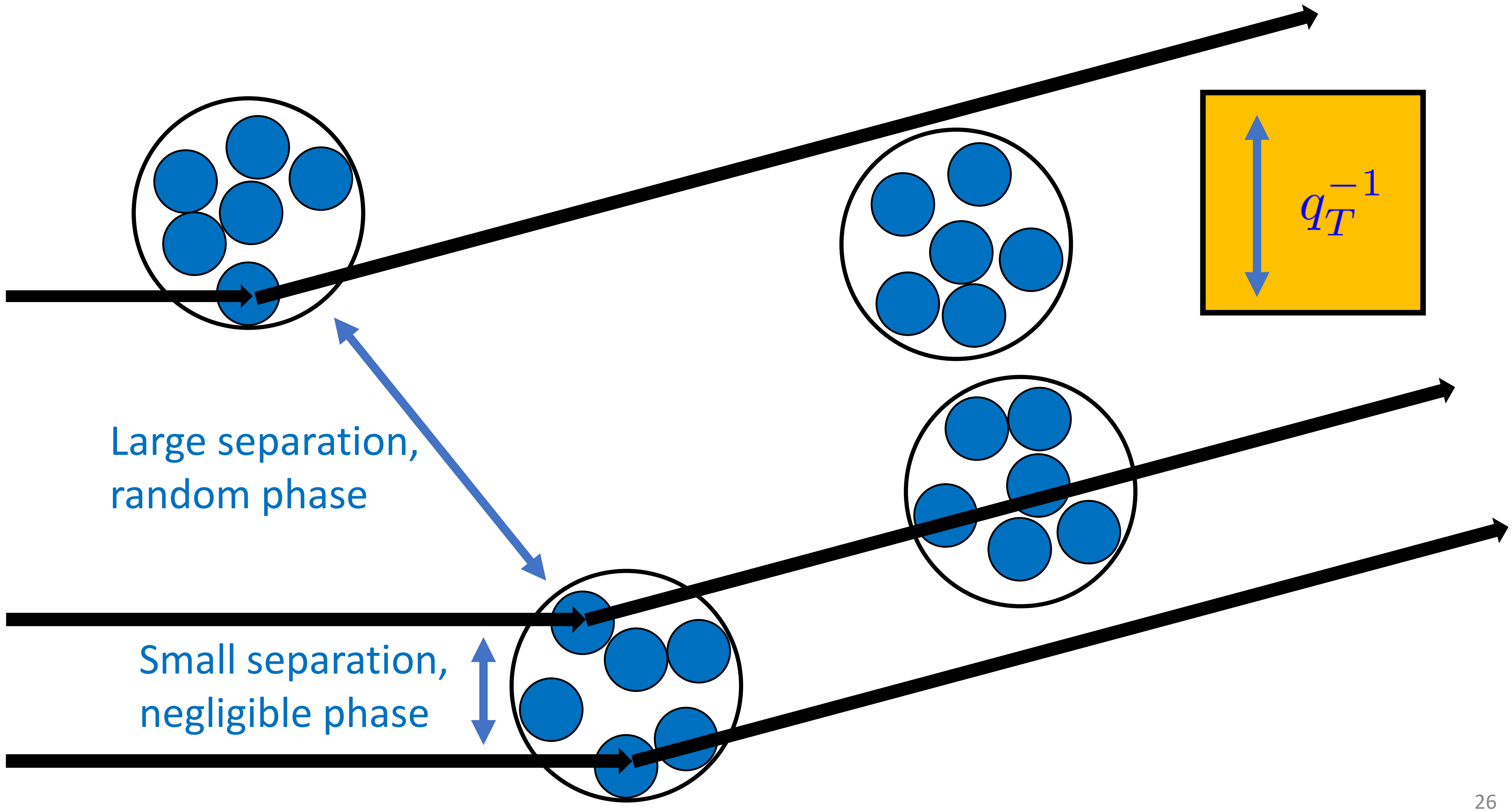
  due to different path lengths

- The total scattering cross-section from a collection of identical scatterers is then proportional to

$$N S(q_T) = \left| \sum_{j=1}^{N} e^{i\mathbf{q}_T\cdot\mathbf{r}_j} \right|^2$$

The "structure factor"

Coherence from separated lengthscales



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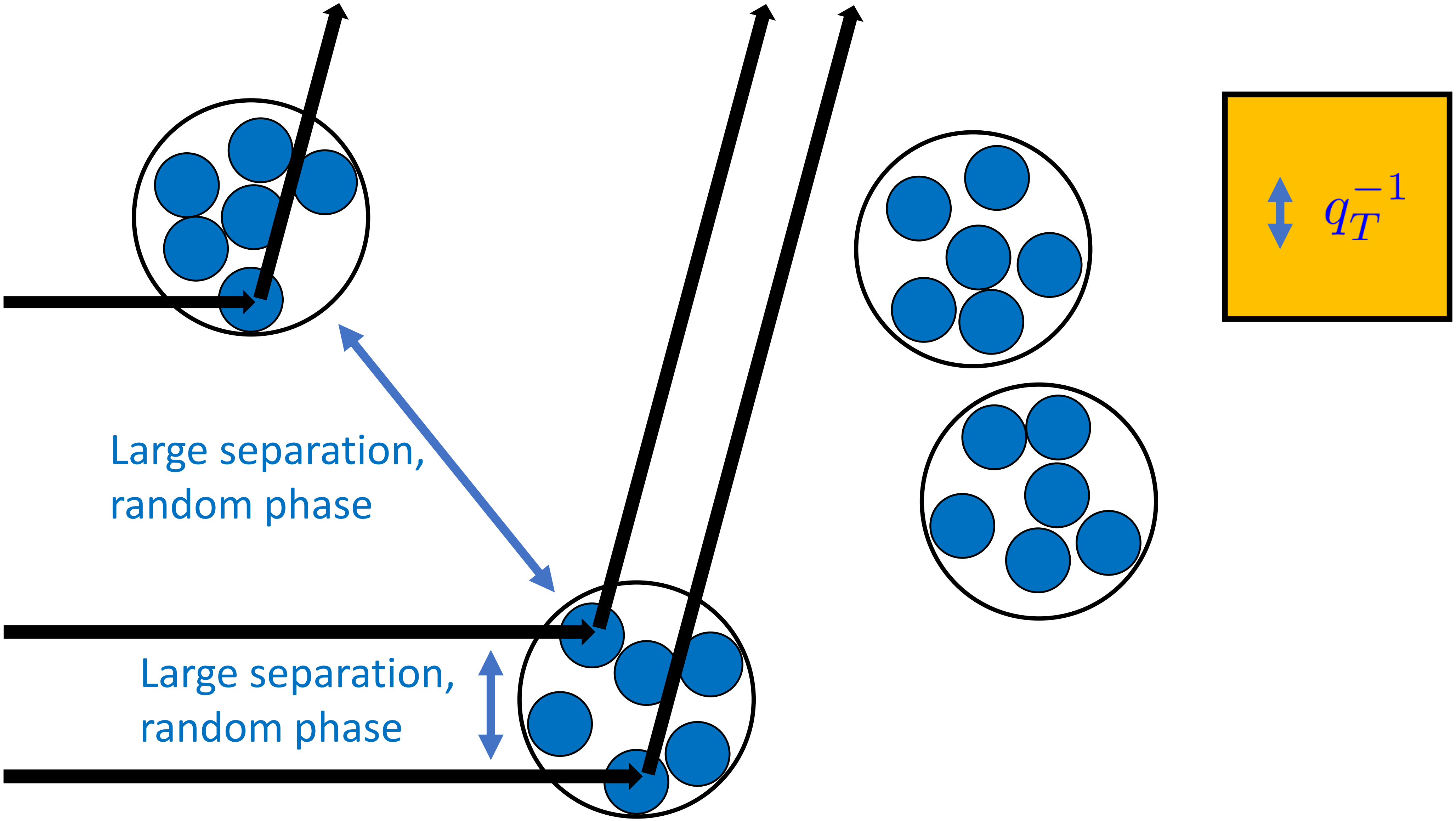


Structure factors for uniform spheres: exact form

- The structure factor of a sphere of radius R with number density n is

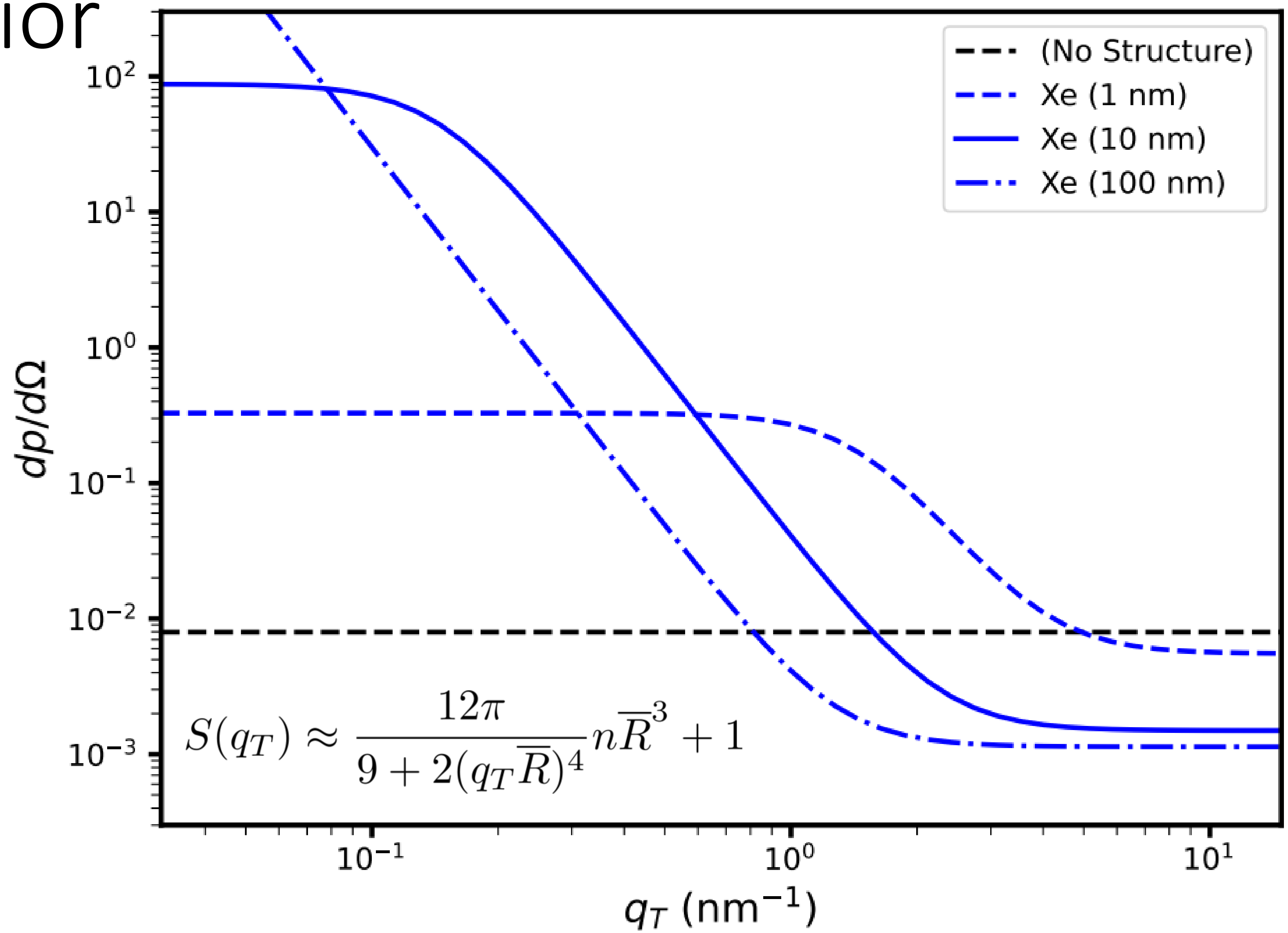
$$S(q_T) = \left(\frac{3(\sin(q_T R) - q_T R \cos(q_T R))}{(q_T R)^3} \right)^2 nR^3 + 1$$

- Averaging over a small spread in radii smooths this to

$$S(q_T) \approx \frac{12\pi}{9 + 2(q_T \overline{R})^4} n\overline{R}^3 + 1$$

Incoherent scattering

Structure factors for uniform spheres: scaling behavior



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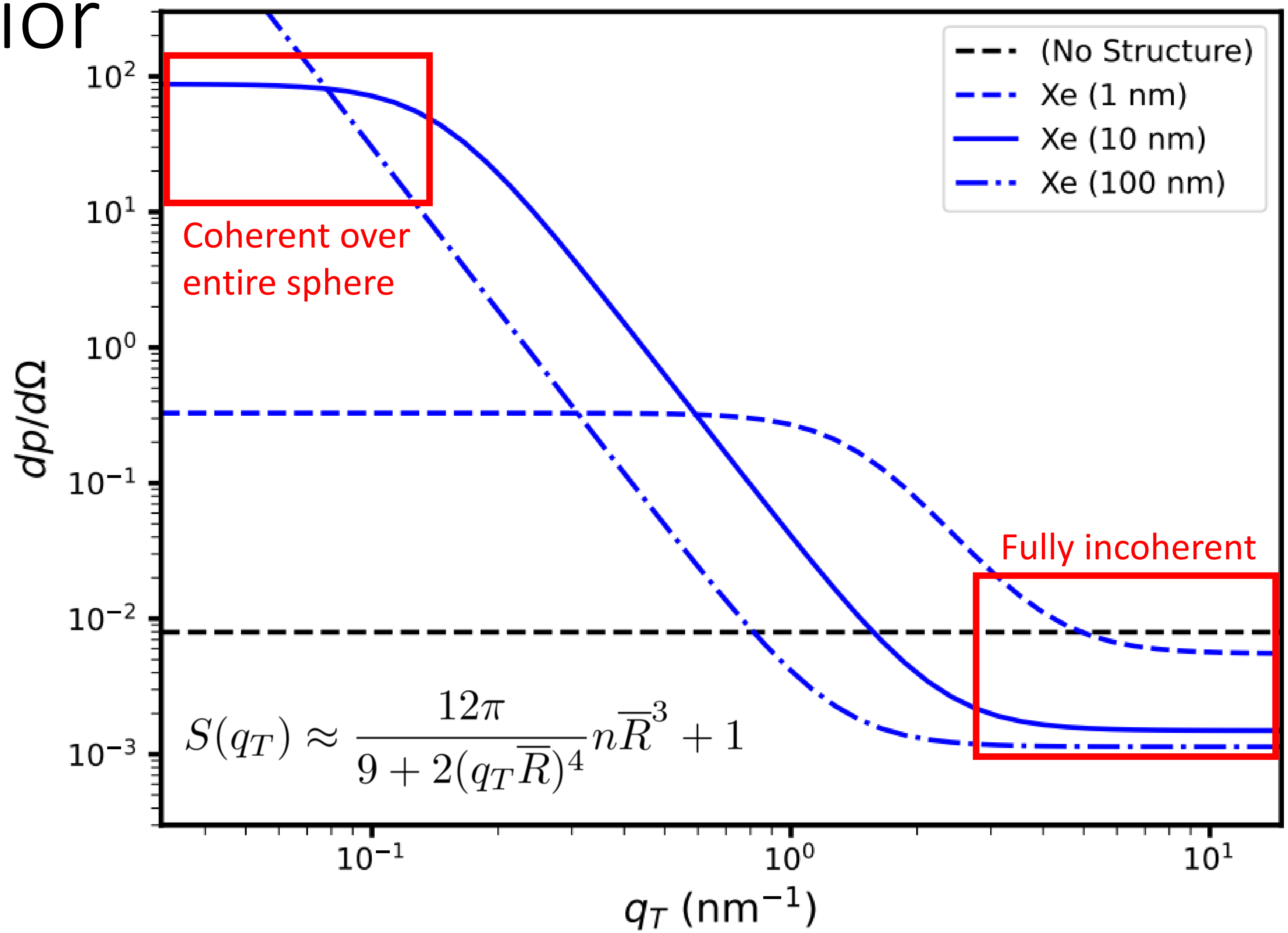
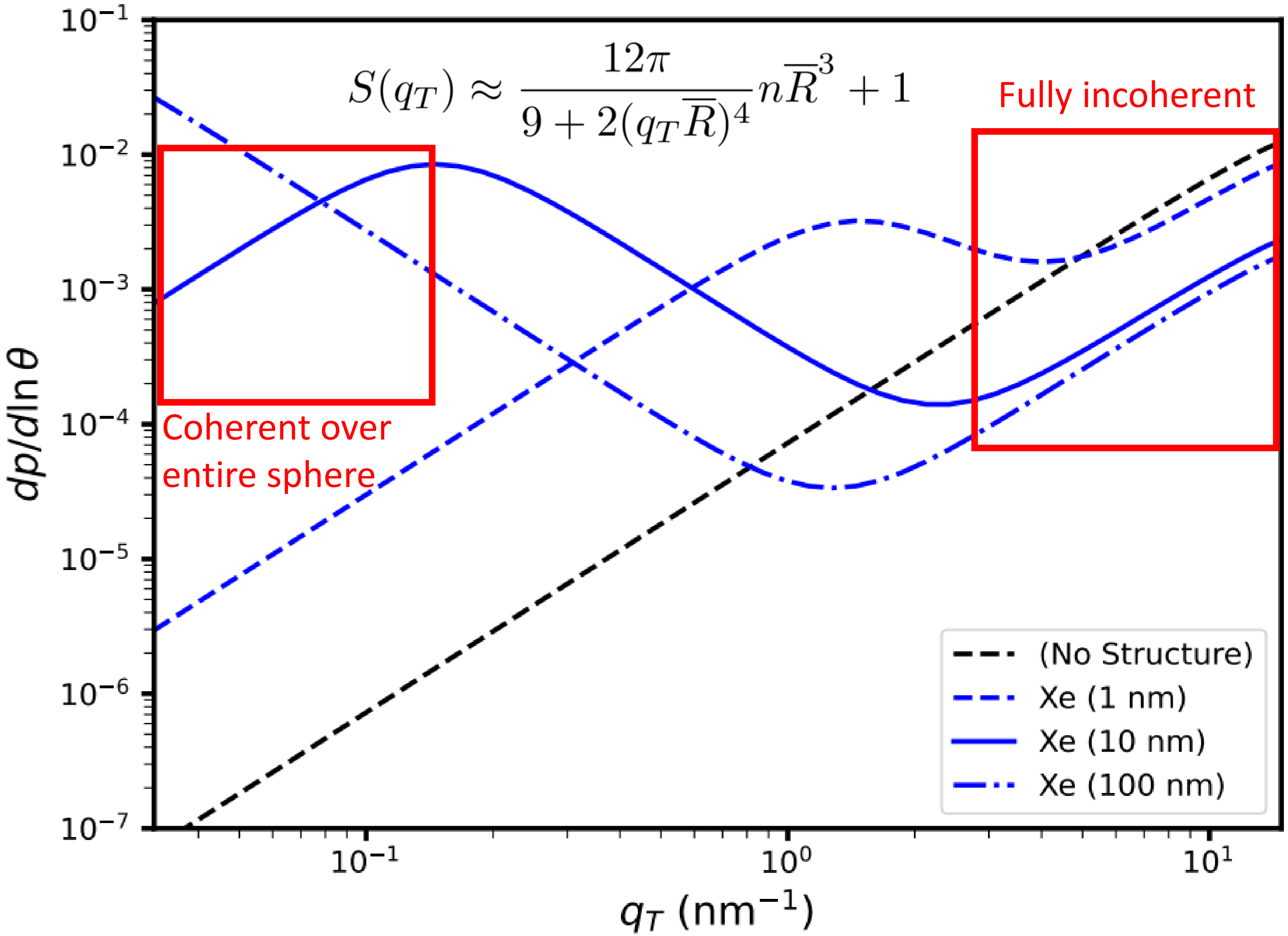


Background

Single-Material

Two-Material

Structure enhances low-angle scattering



Background

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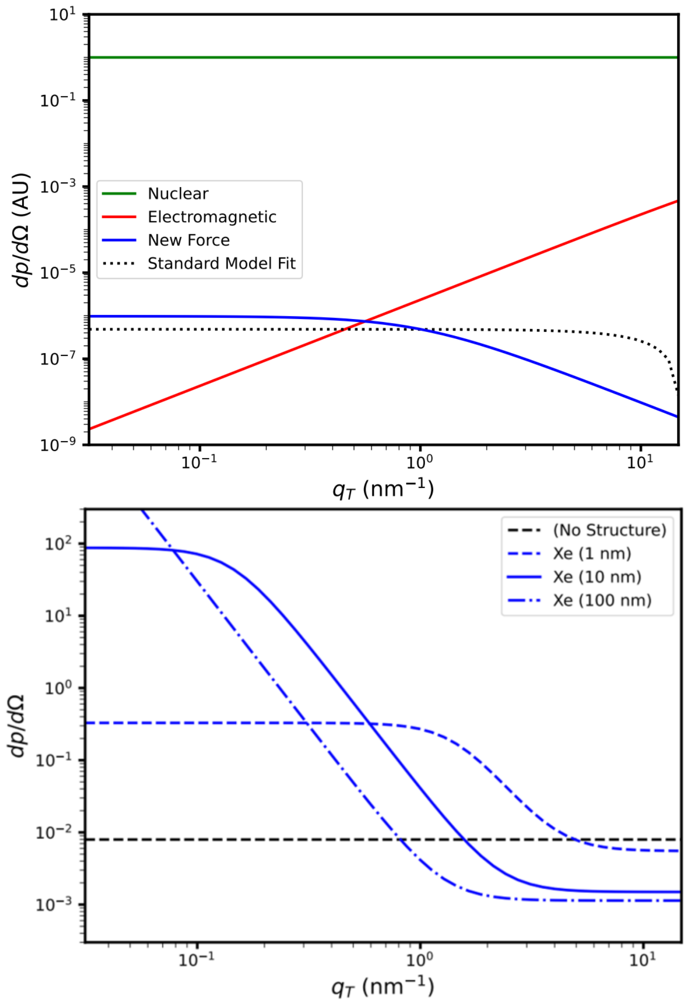
Two-Material

The problem: how do you distinguish a new force from a change in the structure factor?

- Both new forces and structure factors look like low-angle bumps
- No way to know the structure factor *a priori*
  - In fact, a typical use of neutron scattering is to measure structure factors



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The solution: X-ray scattering

- Can perform the same measurements with X-rays
- X-ray scattering distributions will be proportional to the same structure factor
  - Structure factors are a property of geometry alone
- Then look at

$$\underbrace{\frac{dp_{n,s}}{dq_T}\left(\frac{dp_{X,u}/dq_T}{dp_{X,s}/dq_T}\right)}_{\text{Measurable}} = \frac{dp_{n,u}}{dq_T}$$

Compare to prediction

Background

Single-Material

Two-Material

Single-material target candidates

- Noble “snow”
- Aerosols
- Boiling liquids

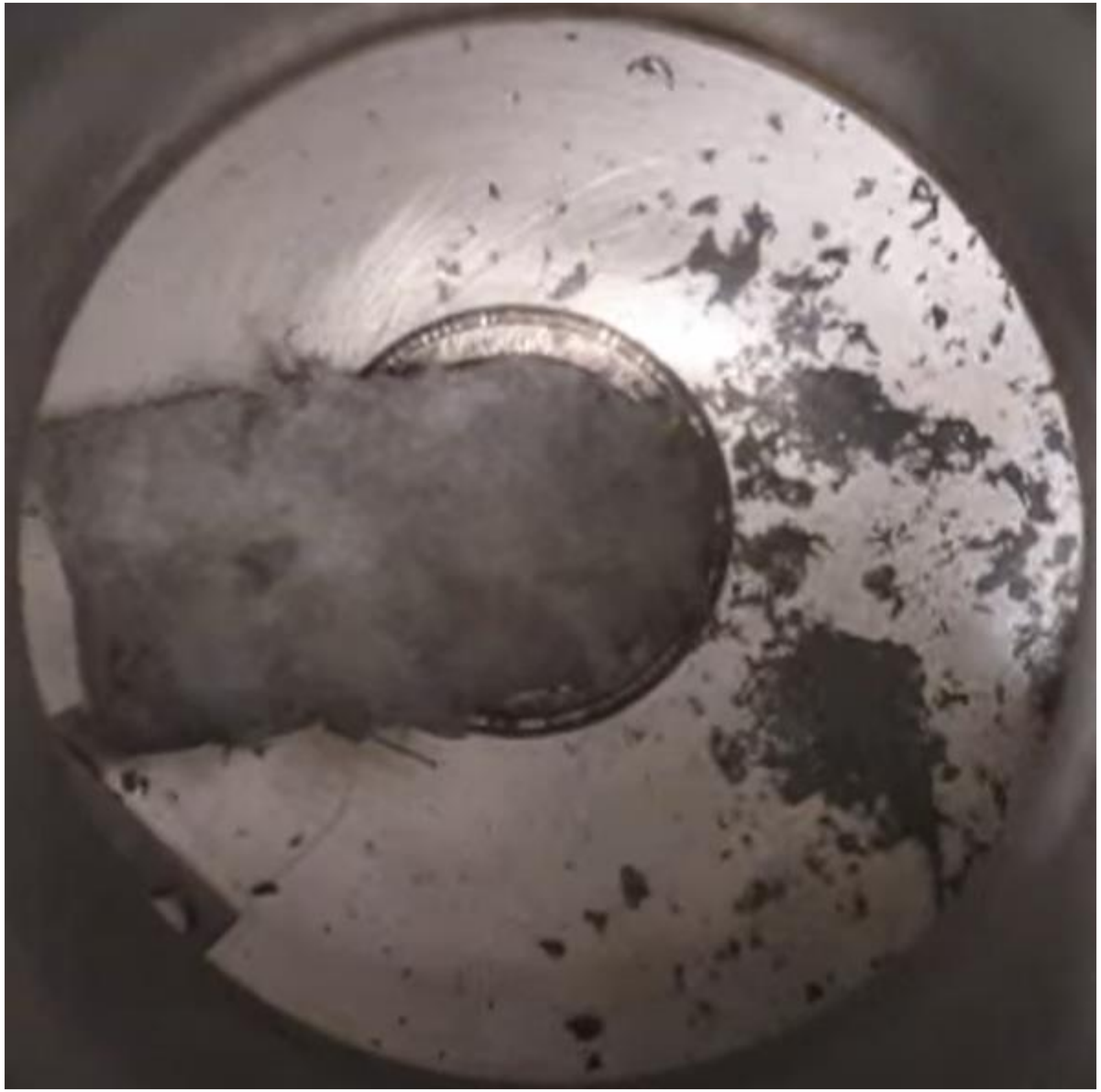


Image from https://www.youtube.com/watch?v=QtDPv637KHY&ab_channel=AttilaDobi , from Carter Hall’s group at UMD

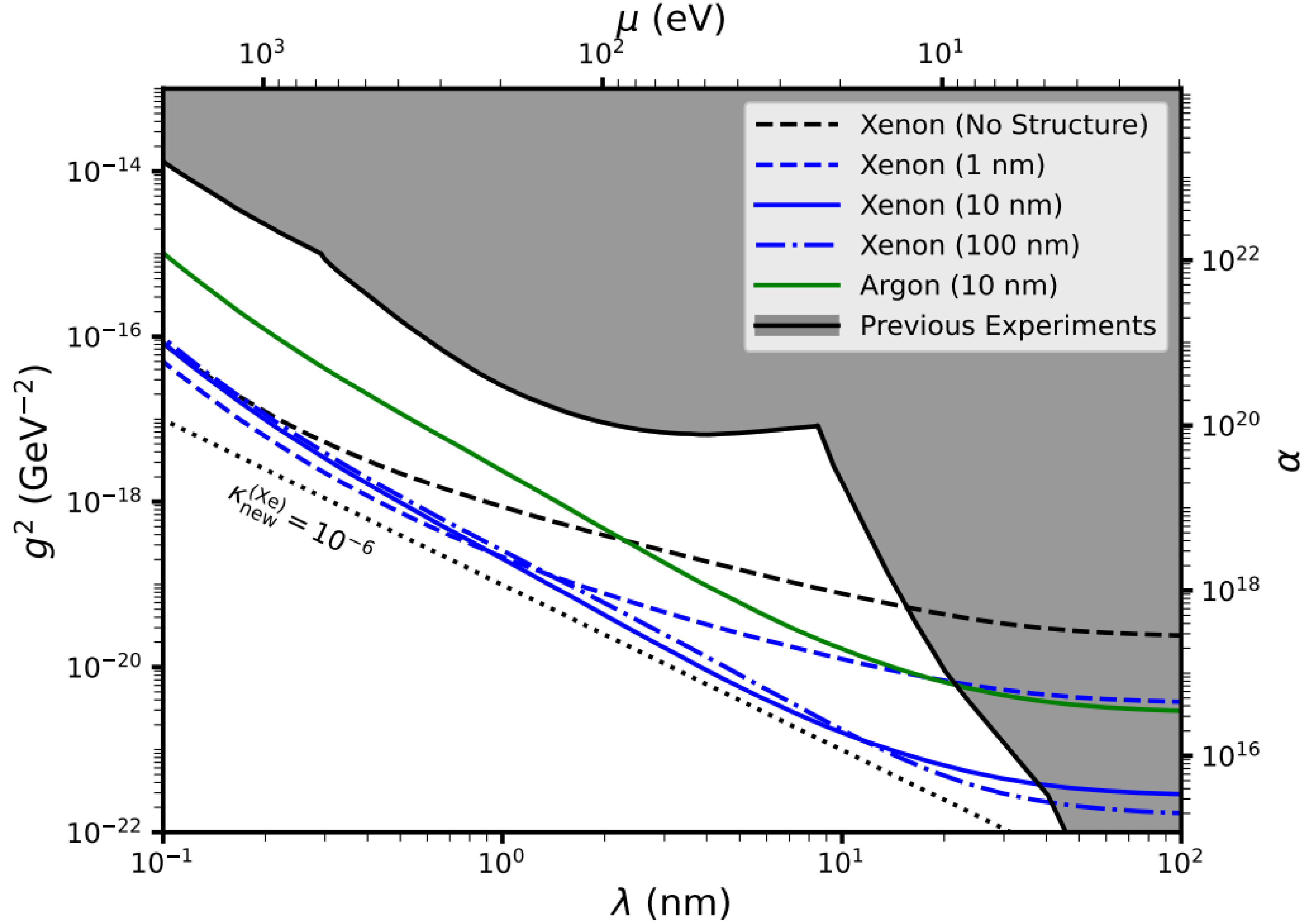
Signal fitting and statistics

- Given a signal, want to compare two fits:

$$\frac{d\sigma}{d\ln\theta} = \mathcal{N}^{(\mathrm{fit})}\left(1 + \frac{2\kappa_{\mathrm{EM}}^{(\mathrm{fit})}}{\sqrt{1 + \left(q_T(\theta)/q_0^{(\mathrm{fit})}\right)^2}} \left[+ \frac{2\kappa_{\mathrm{new}}^{(\mathrm{fit})}}{1 + \left(q_T(\theta)/\mu^{(\mathrm{fit})}\right)^2}\right]\right) S(q_T(\theta))\,\theta\sin 2\theta$$

- Formally done using an F-test of whether improvement in fit from including new force parameters is significant

Projected sensitivity: single-material targets



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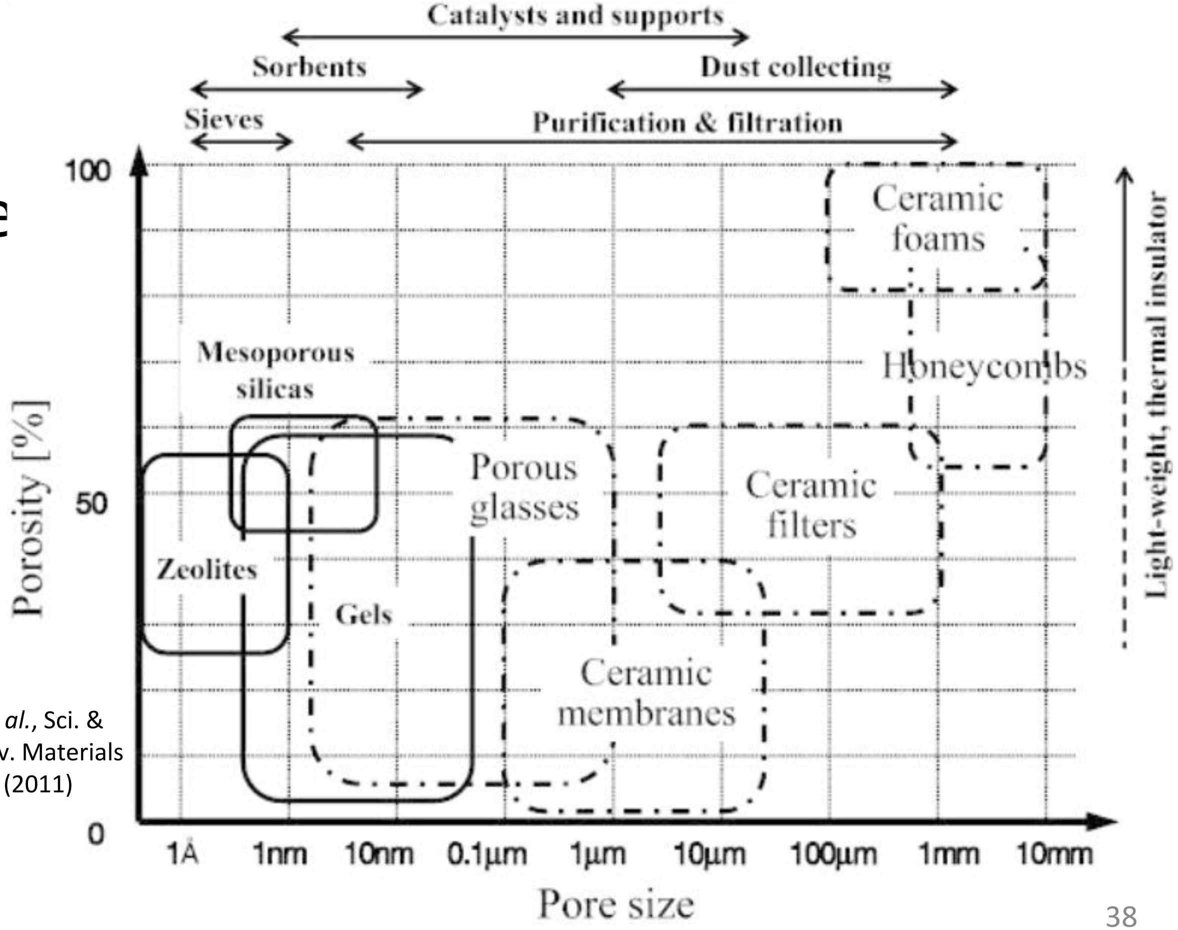
- New forces: motivation and previous experiments
- Single-material targets: how they work and possible implementation
- Two-material targets: challenges and possible implementation

Two-material targets: less effective but more certain

- Noble elements have simpler electromagnetic scattering, but giving them structure is hard

- However, many other solids have structures of the right size; can then add noble gas to them

- Two broad categories:
  - Porous
  - Granular



K. Okada *et al.*, Sci. & Tech. of Adv. Materials **12**, 064701 (2011)

Structure factors for two-material targets: contrast dependence

- Coherent scattering depends only on the "scattering length density," or "SLD" of a material:

$$\mathcal{S}(q_T) := \sum_j n_j b_j(q_T)$$

- Structure factors thus depend only on the contrast between the SLD of different regions:

$$S(q_T) \approx \frac{12\pi\overline{R}^3}{9 + 2(q_T\overline{R})^4} \left(\frac{f\,|\Delta\mathcal{S}|^2}{f n_g |b_g(\mathbf{q}_T)|^2 + (1-f)\sum_j n_{s,j} |b_{s,j}(\mathbf{q}_T)|^2} \right) + 1$$

Background

Single-Material

Two-Material

Two-material target candidates

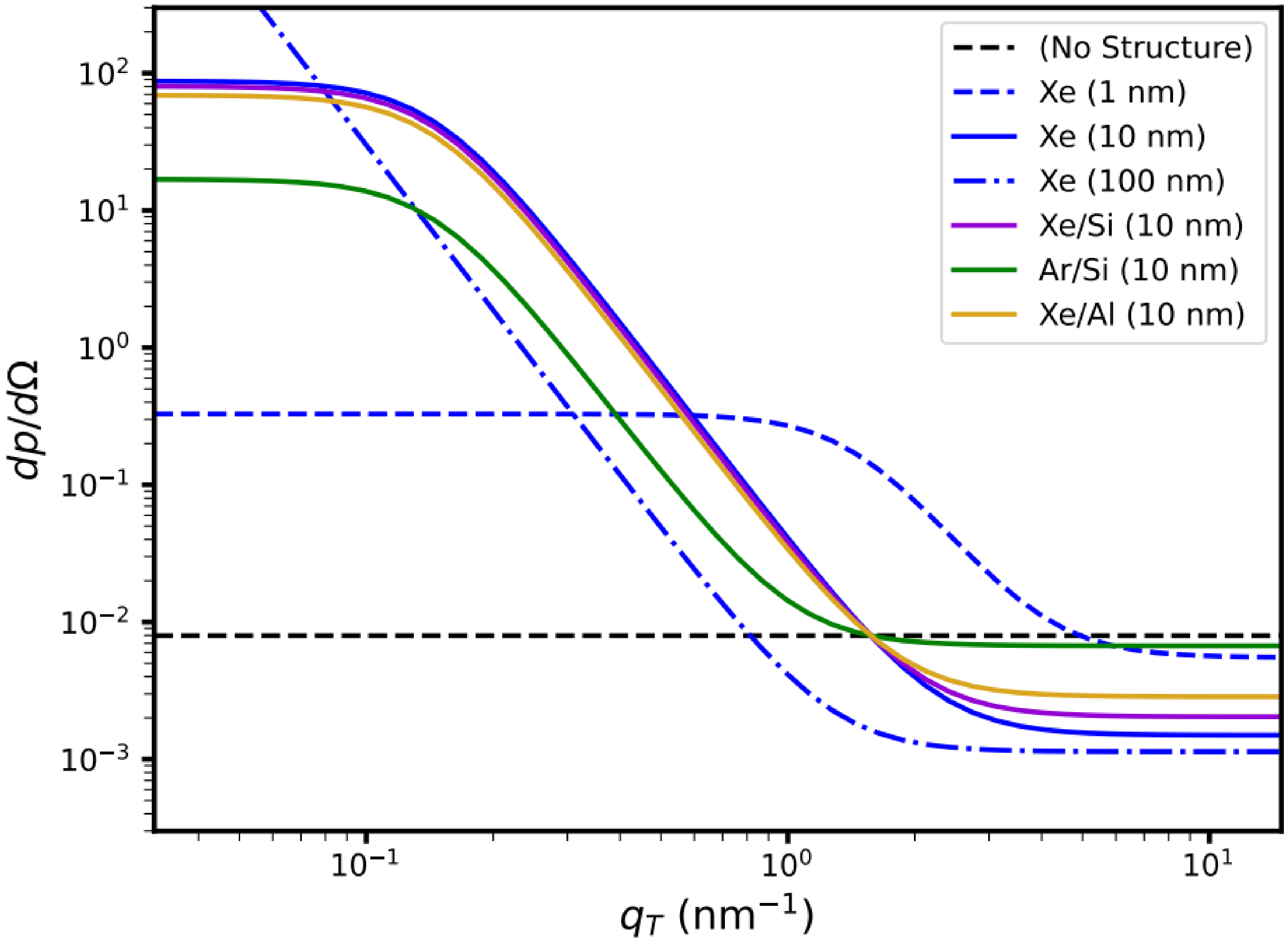
Material	b_c (fm)	n_{liquid} (nm^{-3})	$\mathrm{SLD}_{\mathrm{liquid}}$ ($\mathrm{fm\ nm}^{-3}$)
He-4	3.3	22	72
Ne-20	4.6	37	170
Ar-36	25	21	530
Kr-86	8.1	18	140
Xe-136	9.0	14	120
Material	**b_c^{unit} (fm)**	**n^{unit} (nm^{-3})**	**$\mathrm{SLD}_{\mathrm{max}}$ ($\mathrm{fm\ nm}^{-3}$)**
SiO_2	16	27	420
Al_2O_3	24	24	580
$Al_2Ti_3O_9$	49	5.6	275
$BaTiO_3$	19	15	290
CeO_2	16	25	410
CNTs	6.7	100	670

Background

Single-Material

Two-Material

Structure factors for two-material targets



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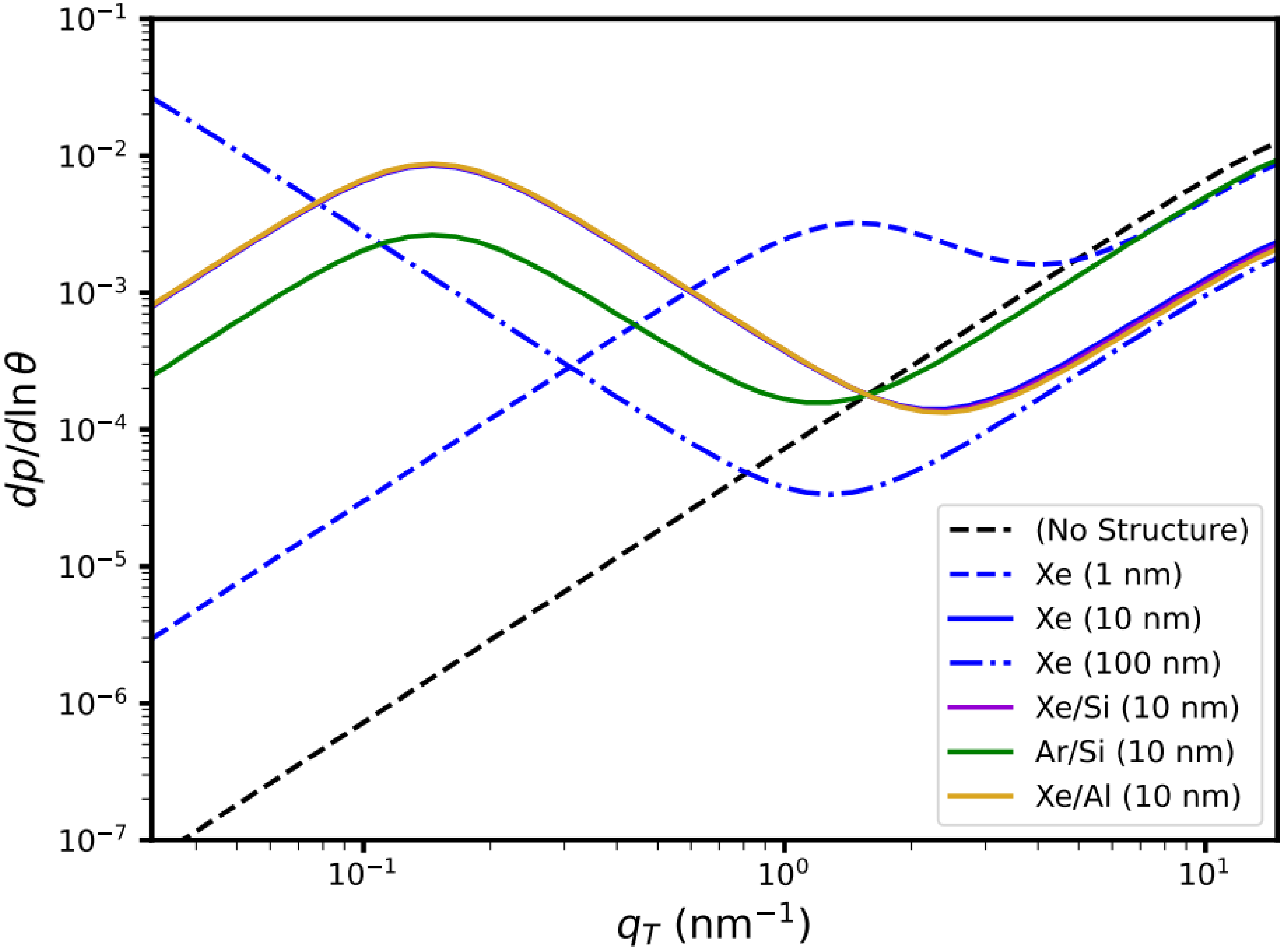

Distinguishing new forces from two-material structure factors is difficult…

- Tempting to simply subtract solid-only scattering from combined target scattering

- This doesn’t work: can only measure scattering *probabilities* of different targets, but there’s interference between *amplitudes*

$$\frac{d\sigma}{d\Omega} = \left|\sum_g b_g(\theta)e^{i\mathbf{q}\cdot\mathbf{r}} + \sum_s b_s(\theta)e^{i\mathbf{q}\cdot\mathbf{r}}\right|^2$$

$$\neq \left|\sum_g b_g(\theta)e^{i\mathbf{q}\cdot\mathbf{r}}\right|^2 + \left|\sum_s b_s(\theta)e^{i\mathbf{q}\cdot\mathbf{r}}\right|^2$$

Distinguishing new forces from two-material structure factors is difficult… but possible

- Can still obtain noble element scattering distribution through a combination of measurements using two different noble gases

	Neutrons	X-Rays
Xenon Alone	✗	✓
Argon Alone	✗	✓
Solid Alone	✓	✓
Solid + Xenon	✓	✓
Solid + Argon	✓	✓

<table>
<tr><th></th><th>Neutrons</th><th>X-Rays</th></tr>
<tr><td>Xenon Alone</td><td colspan="2" rowspan="2">2x Atomic Form Factor</td></tr>
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2x Phase</td></tr>
<tr><td>Solid + Argon</td></tr>
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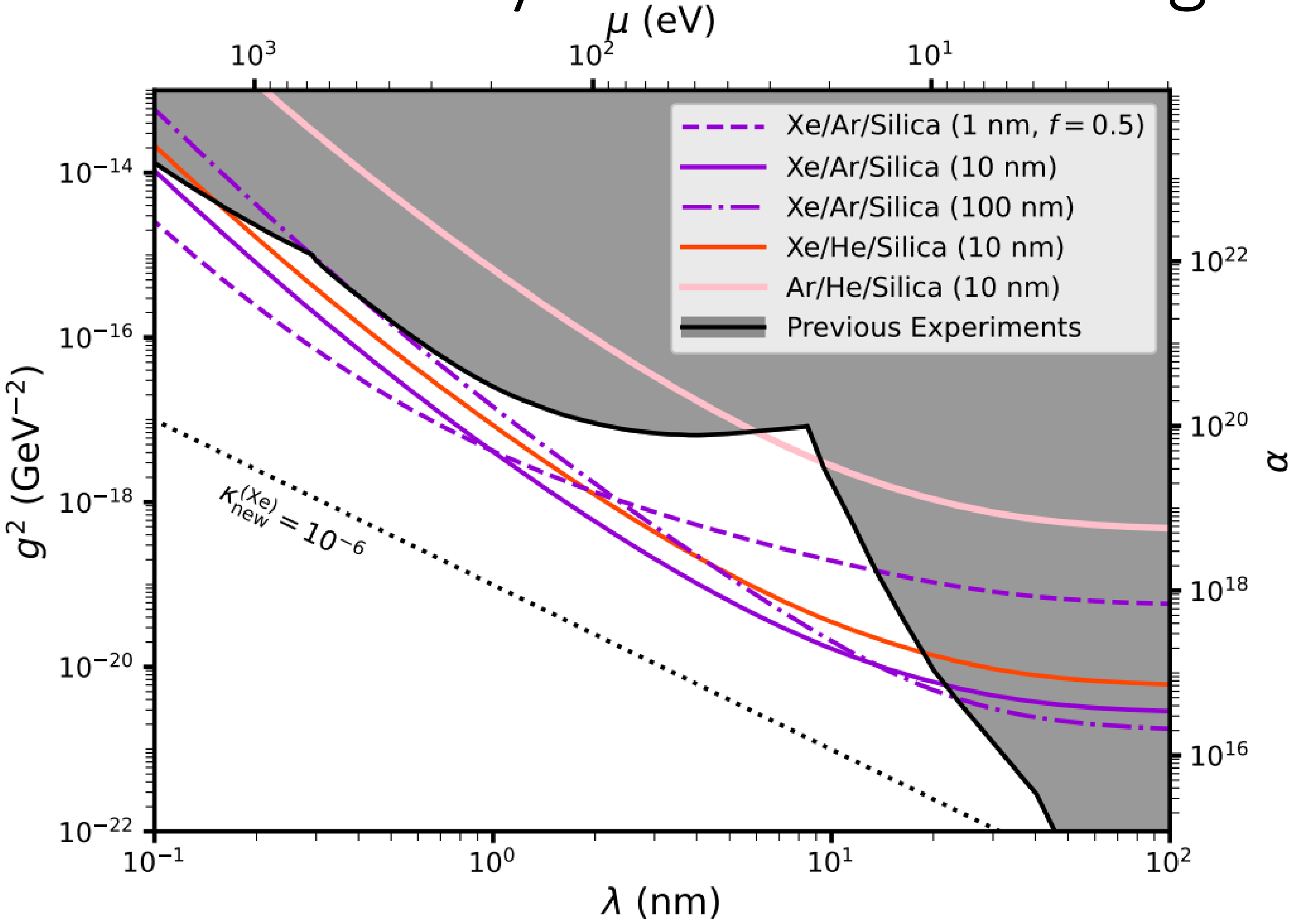
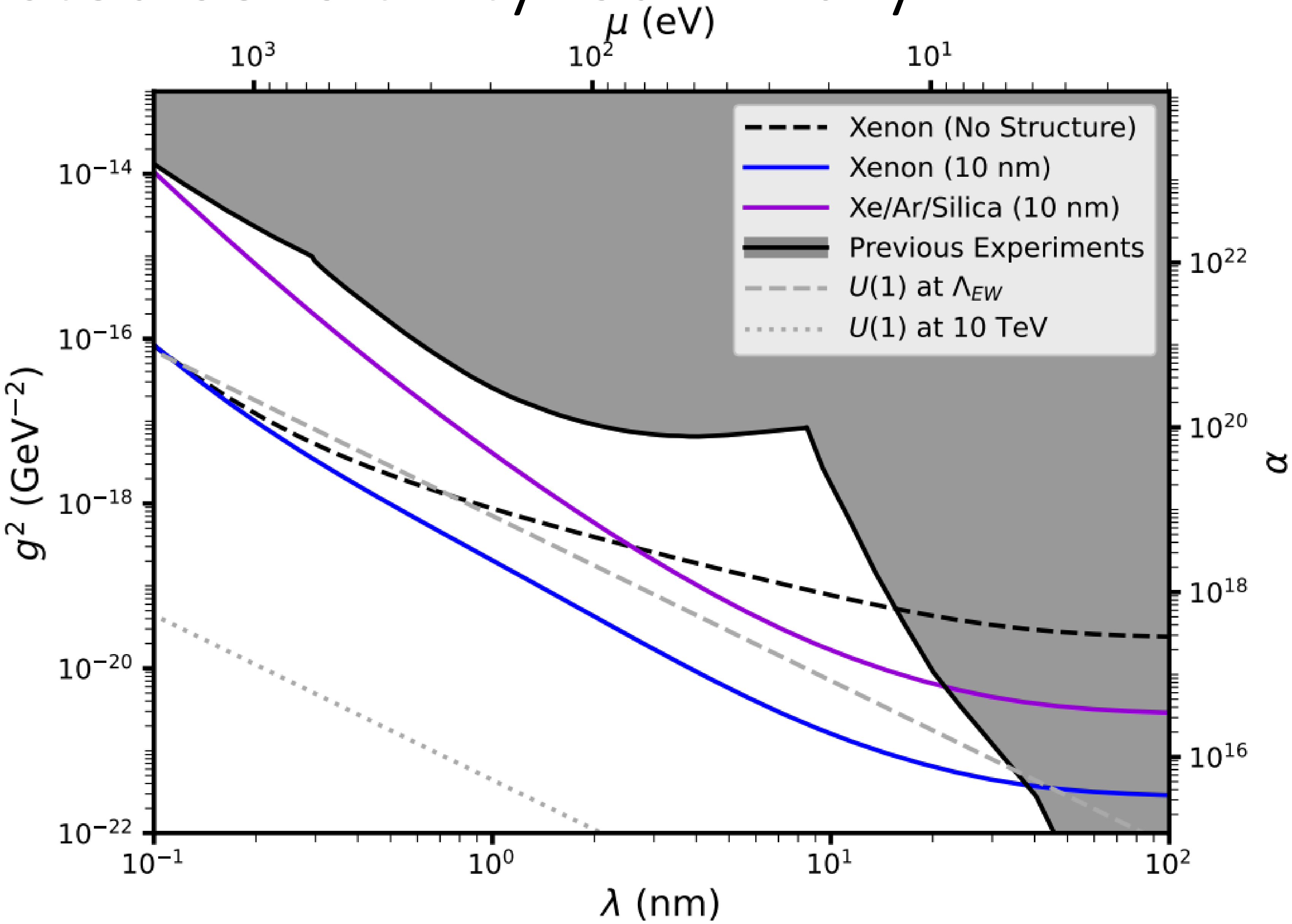
Projected sensitivity: two-material targets



Astrophysical constraints on new forces

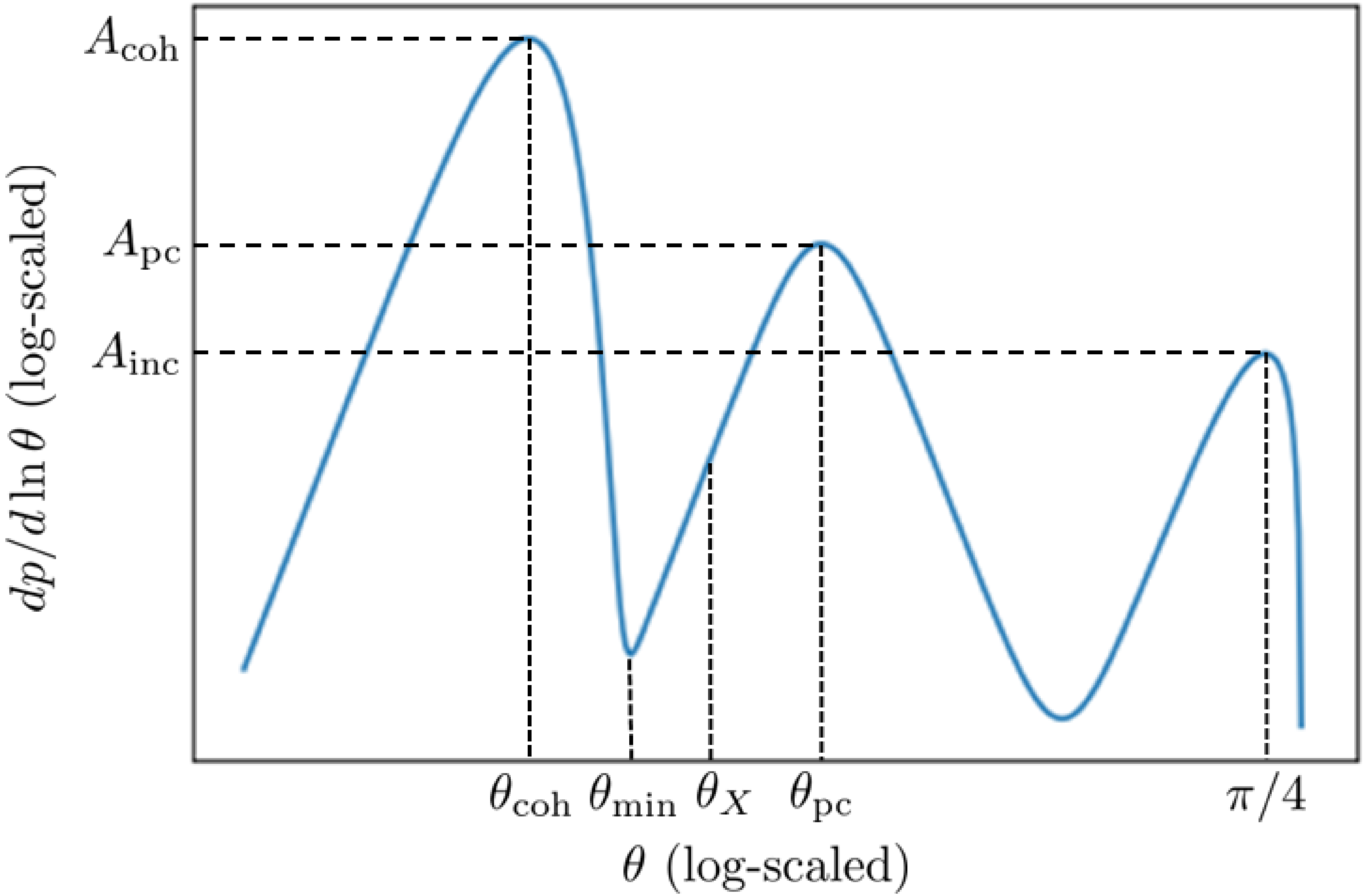
- New scalars could radiate from stars, increasing their cooling rates
- Relevant for masses $\mu \lesssim T_{\mathrm{core}} \sim 10^4\ \mathrm{eV}$
- Model-dependent:
  - B-coupled scalars $g^2 \lesssim 10^{-24}$
  - (B-L)-coupled scalars $g^2 \lesssim 10^{-30}$
  - Extra dimensions not constrained
  - Etc.

Projected sensitivity: summary



Backup^2 Slides

Multiple scattering: the fully coherent peak



Multiple scattering: upper bound

- Can divide scattering from invisibly small angles into two regimes:
  - Almost-visible angles that can be predicted from X-ray measurements
  - Very small angles where scattering is very common but rarely adds up to a significant angle

- Approximate probability of the latter:

$$p \sim \exp\left(-53\left(\frac{n}{10\ \mathrm{nm}^{-3}}\right)^2\left(\frac{b}{10\ \mathrm{fm}}\right)^2\left(\frac{\Delta r_\perp}{10\ \mu\mathrm{m}}\right)^2\left(\frac{\lambda_0}{10\ \mathring{\mathrm{A}}}\right)^2\left(\frac{0.1}{A_{\mathrm{pc}}}\right)^2\right)$$

Interactions at the solid-gas surface

- Magnetic moments of solid surface atoms can distort noble atoms' electron orbitals

- A magnetic field gradient should thus change the scattering length by

$$|\Delta b| \sim \frac{|g_n|e^2}{8\pi m_e}\frac{\mu_B|\Delta\mathbf{B}|}{\Delta E}$$

- The average change in the scattering length is then

$$|\overline{\Delta b}| \sim \frac{|g_n|e^2\mu_B^2}{8\pi m_e R_{\text{atom}}^3 \Delta E}\left(\frac{R_{\text{atom}}}{R_{\text{grain}}}\right)\frac{1}{\sqrt{(R_{\text{grain}}/\xi)^2}} \lesssim \left(\frac{R_{\text{atom}}\xi}{R_{\text{grain}}^2}\right) 3\times 10^{-3}\ \text{fm}$$

Fiducial neutron beam parameters

- Flux: 10^8 cm^{-2} s^{-1}
    - Target area: 10 cm^2
    - Integrated over 28 hours: 10^{14} incident neutrons
    - 10% of neutrons scattered
- Neutron wavelength: 0.6 nm
- Minimum observed angle: 3 mrad
    - Minimum observed momentum transfer: (30 nm)$^{-1}$

Systematic errors

- Nuclear force momentum-dependence: $\mathcal{O}((q_T b_{\rm nuc})^2) \lesssim 10^{-8}$
- Electric polarizability: $\mathcal{O}(q_T\sqrt{\langle r_n^2\rangle} b_P/b_{\rm nuc}) \sim 10^{-9}$
- Also: multiple scattering, atomic interactions, thermal effects, structure degradation, temperature/pressure drift, etc.

Pores and grains are largely equivalent

- Coherent scattering from a collection of grains is equal to coherent scattering from everything except those grains

- Negligible coherent scattering from isotropic targets, so

$$\langle b_{\text{tot}}(\mathbf{q}_T)\rangle = \int\limits_{R_{\text{gas}}} \mathcal{S}_{\text{gas}}(\mathbf{q}_T) e^{i\mathbf{q}_T\cdot\mathbf{r}} d^3\mathbf{r} + \int\limits_{R_{\text{solid}}} \mathcal{S}_{\text{solid}}(\mathbf{q}_T) e^{i\mathbf{q}_T\cdot\mathbf{r}} d^3\mathbf{r}$$

$$= \int\limits_{R_{\text{gas}}} (\mathcal{S}_{\text{gas}}(\mathbf{q}_T) - \mathcal{S}_{\text{solid}}(\mathbf{q}_T)) e^{i\mathbf{q}_T\cdot\mathbf{r}} d^3\mathbf{r} + \cancel{\int\limits_{R_{\text{total}}} \mathcal{S}_{\text{solid}}(\mathbf{q}_T) e^{i\mathbf{q}_T\cdot\mathbf{r}} d^3\mathbf{r}}$$

Two-material separation of contributions: a rough estimate

- Can write full two-material scattering distribution as

$$\left.\frac{dp}{d\Omega}\right|_{2-\mathrm{material}} = \left.\frac{dp}{d\Omega}\right|_{\mathrm{s,inc}} + \left.\frac{dp}{d\Omega}\right|_{\mathrm{g,inc}} + \left|\langle \mathcal{B}(\mathbf{q}_T)\rangle + b_g(\mathbf{q}_T)\,\langle W(\mathbf{q}_T)\rangle\right|^2$$

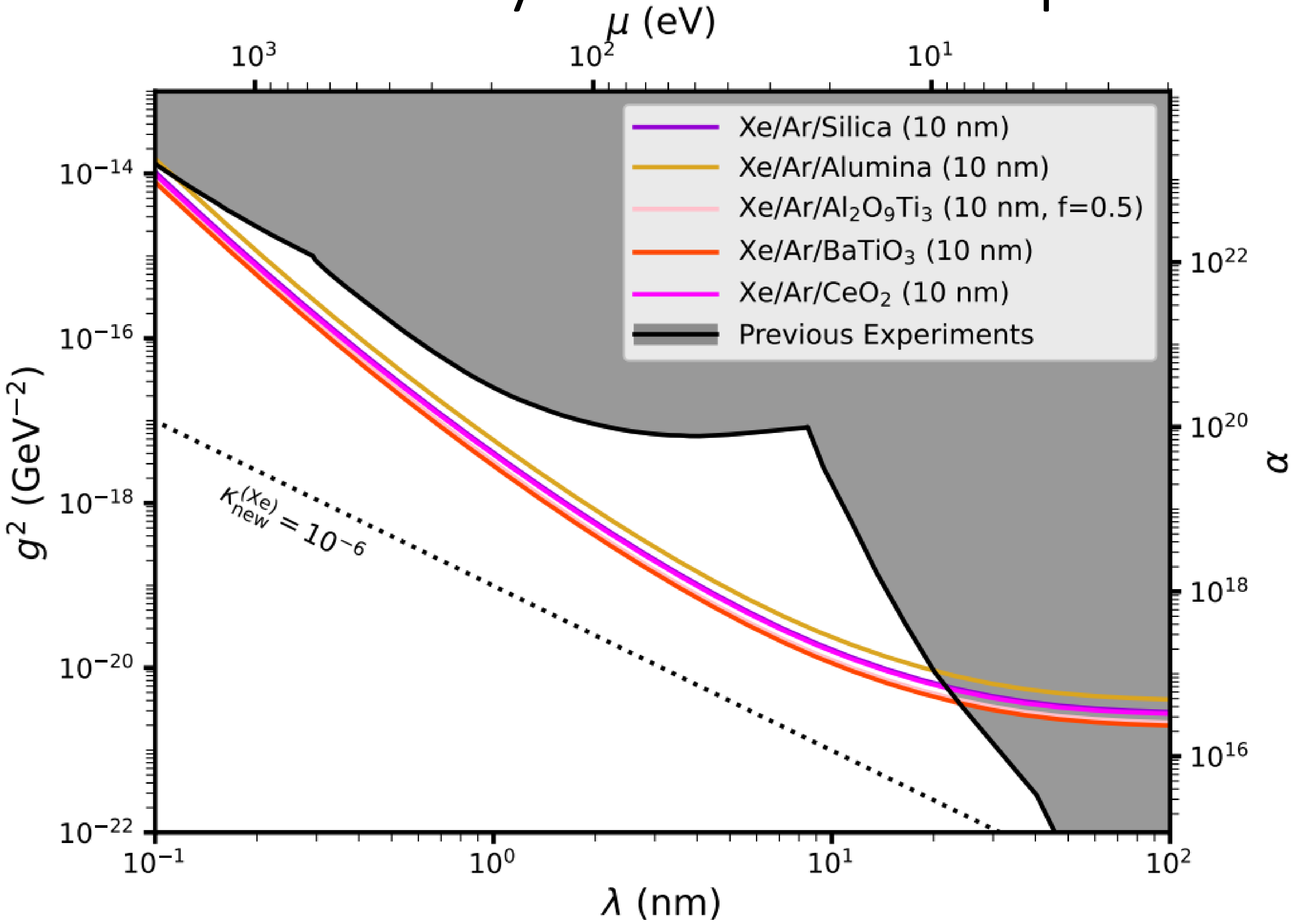
- As a conservative approximation, assume we predict the gas-only part; this leaves us with

$$\left.\frac{dp}{d\Omega}\right|_{\mathrm{cross}} = \left.\frac{dp}{d\Omega}\right|_{2-\mathrm{m}} - \left.\frac{dp}{d\Omega}\right|_{\mathrm{s}} - \left.\frac{dp}{d\Omega}\right|_{\mathrm{g}} - \left|b_g(\theta)\,\langle W(\theta)\rangle\right|^2_{\mathrm{predict}} = 2\ \mathrm{Re}\left(\langle \mathcal{B}^*(\theta)\rangle\,\langle W(\theta)\rangle\right) b_g(\theta)$$

- Then we have

$$\frac{dp/d\Omega|_{\mathrm{cross},1}}{dp/d\Omega|_{\mathrm{cross},2}} = (\mathrm{const.})\left(1 + \kappa_{\mathrm{EM},1}(1 - f_1(q_T)) - \kappa_{\mathrm{EM},2}(1 - f_2(q_T)) + \frac{\Delta\kappa_{\mathrm{new}}}{1 + (q_T/\mu)^2} + \mathcal{O}(\chi^2)\right)$$

Projected sensitivity: other solid options



New scalars can mediate three macroscopic forces

- Two possible fermion vertices:

$$\begin{array}{ll} \text{Scalar:} & \phi\bar{\psi}\psi \\ \text{Pseudoscalar:} & \phi\bar{\psi}i\gamma^5\psi \end{array}$$

- Three forces, depending on whether zero, one, or two scalar/pseudoscalar vertices are included:

$$V_{ss}(r) = -\frac{g_{s,1}g_{s,2}}{4\pi}\left(\frac{1}{r}\right)e^{-\mu r} \longleftarrow \text{Discussed in this talk}$$

$$V_{sp}(r) = \frac{g_{s,1}g_{p,2}}{8\pi m_2}(\hat{\sigma_2}\cdot\hat{r})\left(\frac{\mu}{r}+\frac{1}{r^2}\right)e^{-\mu r}$$

$$V_{pp}(r) = \frac{g_{p,1}g_{p,2}}{16\pi m_1 m_2}\left[(\hat{\sigma_1}\cdot\hat{\sigma_2})\left(\frac{\mu}{r^2}+\frac{1}{r^3}+\frac{4\pi}{3}\delta^{(3)}(r)\right)-(\hat{\sigma_1}\cdot\hat{r})(\hat{\sigma_2}\cdot\hat{r})\left(\frac{\mu^2}{r}+\frac{3\mu}{r^2}+\frac{3}{r^3}\right)\right]e^{-\mu r}$$

The solution: X-ray scattering

- Can perform the same measurements with X-rays
- X-ray scattering distributions will be proportional to the same structure factor
  - Structure factors are a property of geometry alone
- Then look at

Measurable (?) Predictable

$$\frac{dp_{n,s}/dq_T}{dp_{X,s}/dq_T} = \frac{dp_{n,u}/dq_T}{dp_{X,u}/dq_T} \quad \text{or} \quad \frac{dp_{n,s}}{dq_T}\left(\frac{dp_{X,u}/dq_T}{dp_{X,s}/dq_T}\right) = \frac{dp_{n,u}}{dq_T}$$

Distinguishing new forces from two-material structure factors is difficult… but possible

- Can still obtain noble element scattering distribution through a combination of measurements using two different noble gases

- Can show this by comparing number of possible measurements with number of degrees of freedom
  - Important to separate degrees of freedom *per bin* (e.g. structure factors) from degrees of freedom that are universal (e.g. a new force coupling)

Possible measurements with two materials

- 4 measurements from mixed targets:
  - Neutrons and X-rays, with each of two noble gases
- 2 measurements from solids alone:
  - Neutrons and X-rays
- 2 measurements from gases alone:
  - Only X-rays; avoiding this measurement with neutrons is the original goal

8 total constraints

Degrees of freedom per bin for two materials

- 1 phase sum for the gas region:

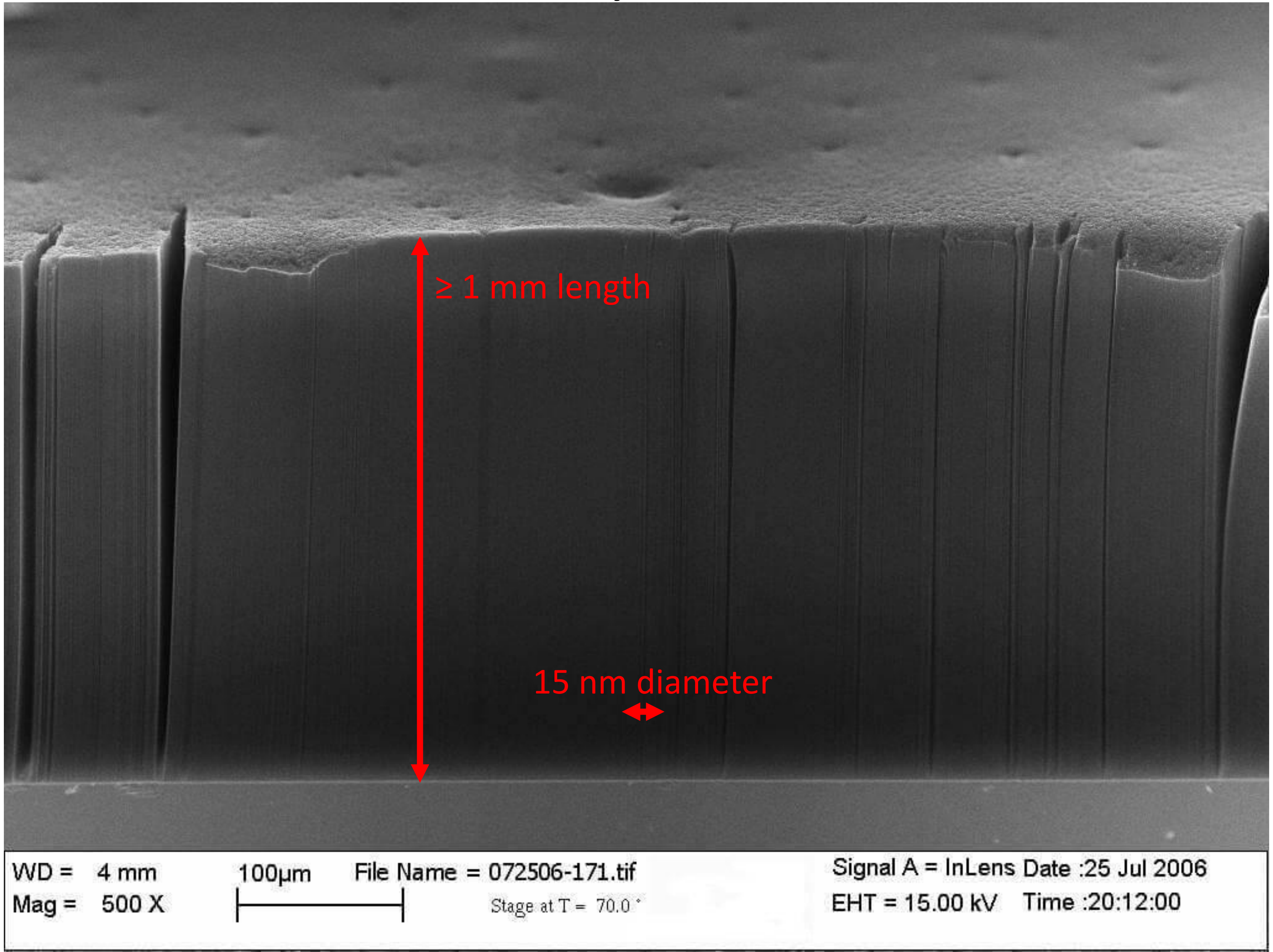
$$W(q_T) = \int_{V_{\mathrm{gas}}} d^3\mathbf{r}\ e^{i\mathbf{q}_T\cdot\mathbf{r}}$$

- 2 atomic form factors for the noble elements
- 4 components of the solid scattering lengths
  - Real and imaginary parts, for neutrons and X-rays

7 d.o.f./bin

- All other parameters (scattering lengths, new force mass, etc.) are not per-bin so they can be extracted from the “extra” measurement

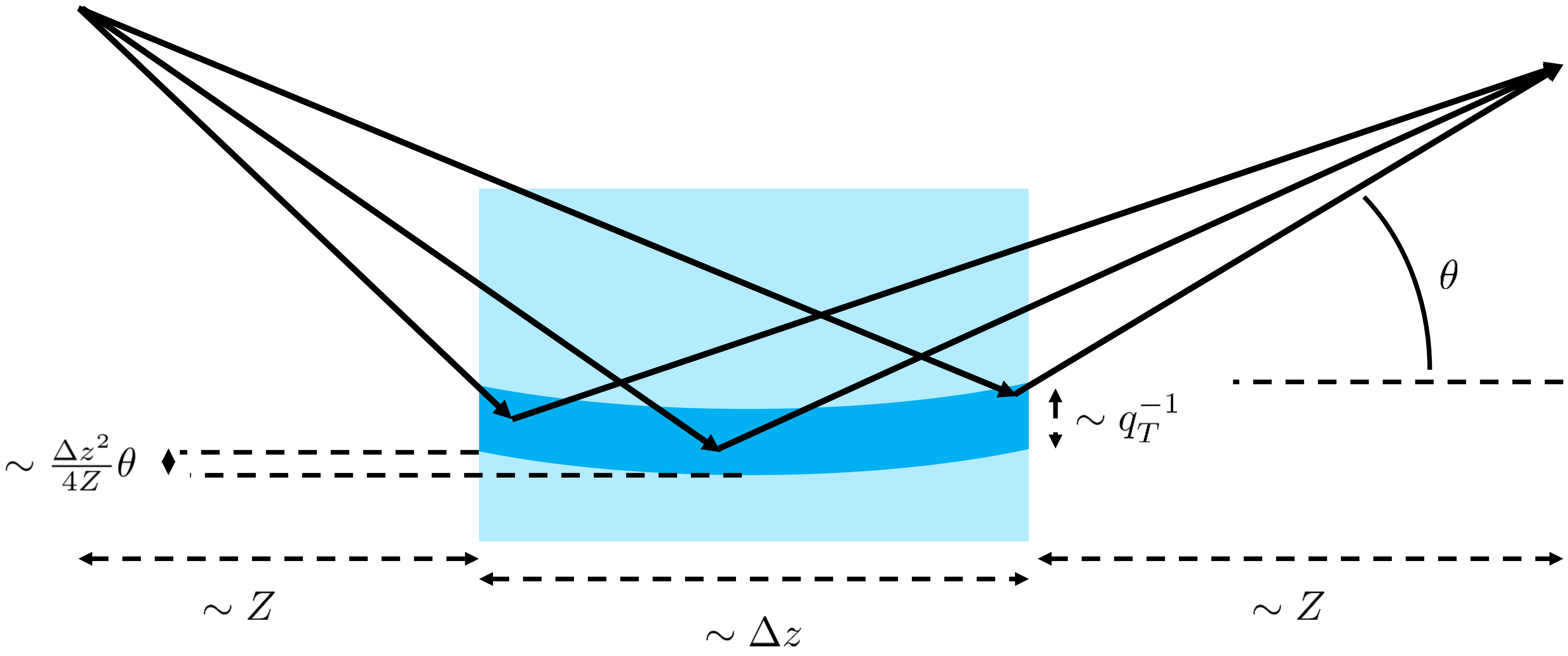
Carbon nanotube arrays/forests



Finite target depth effects

- A given coherent slice is effectively flat (to within σ) over depths of no more than

$$\Delta z \sim \sqrt{\frac{4Z\sigma}{\theta}}$$



$$\left\langle \frac{\partial\sigma}{\partial\Omega} \right\rangle = N_{\mathrm{atoms}} \left(\frac{2}{9} \left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2b_{i,S} b_{i,L} \langle \hat{\mathbf{S}} \cdot \hat{\mathbf{L}} \rangle_{\mathrm{grain}} \right) + \frac{4\pi(nR^3)}{9+2(q_T R)^4} \left(3b_c^2 + b_{i,n}^2 \right) \right)$$

Want to measure

$$b_c = b_{nuc,c}(A,Z) - \frac{m_n Z}{3a_0 m_e} \langle r_n^2 \rangle (1 - f(\mathbf{q}_T)) + \mathcal{O}\left(\frac{m_e}{m_p^2}\right)$$
$$- (1 + \mathcal{O}(q_T \cdot 5\ \mathrm{fm})) \frac{2m\alpha Z^2 e^2}{\pi^2} \int_0^\infty |f_N(A,Z,k)|^2 dk + \boxed{\frac{m_n g^2 A}{2\pi\mu^2} \frac{1}{1+(q_T\lambda_\mu)^2}}$$

$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e} \left(\gamma_e f_S(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{S} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e} \left(f_L(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{L} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = b_{nuc,i}(A,Z)\sqrt{I(I+1)}\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{I}} - \frac{\gamma_n e^2}{2m_e} \left(\frac{m_e}{m_p} \gamma(A,Z) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{I} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \frac{\gamma_n e^2}{2m_e} \left(\frac{m_e}{m_p} i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$\left\langle \frac{\partial\sigma}{\partial\Omega} \right\rangle = N_{\text{atoms}} \left(\frac{2}{9} \left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2 b_{i,S} b_{i,L} \langle \hat{\mathbf{S}} \cdot \hat{\mathbf{L}} \rangle_{\text{grain}} \right) + \frac{4\pi(nR^3)}{9+2(q_T R)^4} \left(3b_c^2 + b_{i,n}^2 \right) \right)$$

Measurable using X-rays

$$b_c = b_{nuc,c}(A,Z) - \frac{m_n Z}{3a_0 m_e} \langle r_n^2 \rangle \left(1 - \boxed{f(\mathbf{q}_T)}\right) + \mathcal{O}\left(\frac{m_e}{m_p^2}\right)$$

$$- \left(1 + \mathcal{O}(q_T \cdot 5\ \text{fm})\right) \frac{2m\alpha Z^2 e^2}{\pi^2} \int_0^\infty |f_N(A,Z,k)|^2 dk + \boxed{\frac{m_n g^2 A}{2\pi\mu^2} \frac{1}{1+(q_T\lambda_\mu)^2}}$$

$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e} \left(\gamma_e f_S(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{S} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e} \left(f_L(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{L} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = b_{nuc,i}(A,Z)\sqrt{I(I+1)}\hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{I}} - \frac{\gamma_n e^2}{2m_e} \left(\frac{m_e}{m_p} \gamma(A,Z) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{I} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \frac{\gamma_n e^2}{2m_e} \left(\frac{m_e}{m_p} i\boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$\left\langle \frac{\partial\sigma}{\partial\Omega} \right\rangle = N_{\mathrm{atoms}} \left(\frac{2}{9} \left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2 b_{i,S} b_{i,L} \langle \hat{\mathbf{S}} \cdot \hat{\mathbf{L}} \rangle_{\mathrm{grain}} \right) + \frac{4\pi (nR^3)}{9+2(q_T R)^4} \left(3b_c^2 + b_{i,n}^2 \right) \right)$$

Angle-independent; combine to give one fit parameter

$$b_c = \boxed{b_{nuc,c}(A,Z)} - \frac{m_n Z}{3 a_0 m_e} \langle r_n^2 \rangle \left(\boxed{1} - \boxed{f(\mathbf{q}_T)} \right) + \mathcal{O}\left(\frac{m_e}{m_p^2} \right)$$

$$- \boxed{(1 + \mathcal{O}(q_T \cdot 5 \text{ fm})) \frac{2 m \alpha Z^2 e^2}{\pi^2} \int_0^\infty |f_N(A,Z,k)|^2 dk} + \boxed{\frac{m_n g^2 A}{2\pi \mu^2} \frac{1}{1 + (q_T \lambda_\mu)^2}}$$

$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2 m_e} \left(\gamma_e f_S(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{S} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2 m_e} \left(f_L(A,Z,q_T) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{L} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = \boxed{b_{nuc,i}(A,Z) \sqrt{I(I+1)} \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{I}}} - \frac{\gamma_n e^2}{2 m_e} \left(\boxed{\frac{m_e}{m_p} \gamma(A,Z) \boldsymbol{\sigma} \cdot (\mathbf{1} - \hat{\mathbf{q}}_T \hat{\mathbf{q}}_T^T) \cdot \mathbf{I}} + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \frac{\gamma_n e^2}{2 m_e} \left(\frac{m_e}{m_p} i \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} \cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p} \right)^2 \right) \right)$$

$$\left\langle \frac{\partial\sigma}{\partial\Omega}\right\rangle = N_{\text{atoms}}\left(\frac{2}{9}\left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2b_{i,S}b_{i,L}\langle \hat{\mathbf{S}}\cdot\hat{\mathbf{L}}\rangle_{\text{grain}}\right) + \frac{4\pi(nR^3)}{9+2(q_T R)^4}\left(3b_c^2 + b_{i,n}^2\right)\right)$$

Too small to matter

$$b_c = \boxed{b_{nuc,c}(A,Z)} - \frac{m_n Z}{3a_0 m_e}\langle r_n^2\rangle\left(\boxed{1} - \boxed{f(\mathbf{q}_T)}\right) + \mathcal{O}\left(\frac{m_e}{m_p^2}\right)$$
$$-\boxed{(1+\mathcal{O}(q_T\cdot 5\ \text{fm}))\,\frac{2m\alpha Z^2 e^2}{\pi^2}\int_0^\infty |f_N(A,Z,k)|^2 dk} + \boxed{\frac{m_n g^2 A}{2\pi\mu^2}\,\frac{1}{1+(q_T\lambda_\mu)^2}}$$

$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(\gamma_e f_S(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{S} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(f_L(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{L} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = \boxed{b_{nuc,i}(A,Z)\sqrt{I(I+1)}\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{I}}} - \frac{\gamma_n e^2}{2m_e}\left(\boxed{\frac{m_e}{m_p}\gamma(A,Z)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{I}} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \cancel{\frac{\gamma_n e^2}{2m_e}\left(\frac{m_e}{m_p}i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)}$$

$$\left\langle \frac{\partial\sigma}{\partial\Omega}\right\rangle = N_{\text{atoms}}\left(\frac{2}{9}\left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2 b_{i,S} b_{i,L}\langle \hat{\mathbf{S}}\cdot\hat{\mathbf{L}}\rangle_{\text{grain}}\right) + \frac{4\pi(nR^3)}{9+2(q_T R)^4}\left(3b_c^2 + b_{i,n}^2\right)\right)$$

Zero if all electrons paired

$$b_c = \boxed{b_{nuc,c}(A,Z)} - \frac{m_n Z}{3a_0 m_e}\langle r_n^2\rangle \left(\boxed{1} - \boxed{f(\mathbf{q}_T)}\right) + \mathcal{O}\left(\frac{m_e}{m_p^2}\right)$$

$$- \boxed{(1+\mathcal{O}(q_T\cdot 5\ \text{fm}))\,\frac{2m\alpha Z^2 e^2}{\pi^2}\int_0^\infty |f_N(A,Z,k)|^2 dk} + \boxed{\frac{m_n g^2 A}{2\pi\mu^2}\,\frac{1}{1+(q_T\lambda_\mu)^2}}$$

$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(\boxed{\gamma_e f_S(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{S}} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(\boxed{f_L(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{L}} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = \boxed{b_{nuc,i}(A,Z)\sqrt{I(I+1)}\,\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{I}}} - \frac{\gamma_n e^2}{2m_e}\left(\boxed{\frac{m_e}{m_p}\gamma(A,Z)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{I}} + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \cancel{\frac{\gamma_n e^2}{2m_e}\left(\frac{m_e}{m_p} i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)}$$

$$\left\langle \frac{\partial\sigma}{\partial\Omega}\right\rangle = N_{\text{atoms}}\left(\frac{2}{9}\left(b_{i,S}^2 + b_{i,L}^2 + \frac{3a^2-2a+1}{2} b_{i,I}^2 + 2 b_{i,S} b_{i,L}\langle \hat{\mathbf{S}}\cdot\hat{\mathbf{L}}\rangle_{\text{grain}}\right) + \frac{4\pi(nR^3)}{9+2(q_T R)^4}\left(3b_c^2 + b_{i,n}^2\right)\right)$$

Likely not an issue; can be computed if necessary

$$b_c = \boxed{b_{nuc,c}(A,Z)} - \frac{m_n Z}{3a_0 m_e}\langle r_n^2\rangle \left(\boxed{1} - \boxed{f(\mathbf{q}_T)}\right) + \boxed{\mathcal{O}\left(\frac{m_e}{m_p^2}\right)}$$
$$- \boxed{\left(1+\mathcal{O}(q_T\cdot 5\ \text{fm})\right)\frac{2m\alpha Z^2 e^2}{\pi^2}\int_0^\infty |f_N(A,Z,k)|^2 dk} + \boxed{\frac{m_n g^2 A}{2\pi\mu^2}\frac{1}{1+(q_T\lambda_\mu)^2}}$$

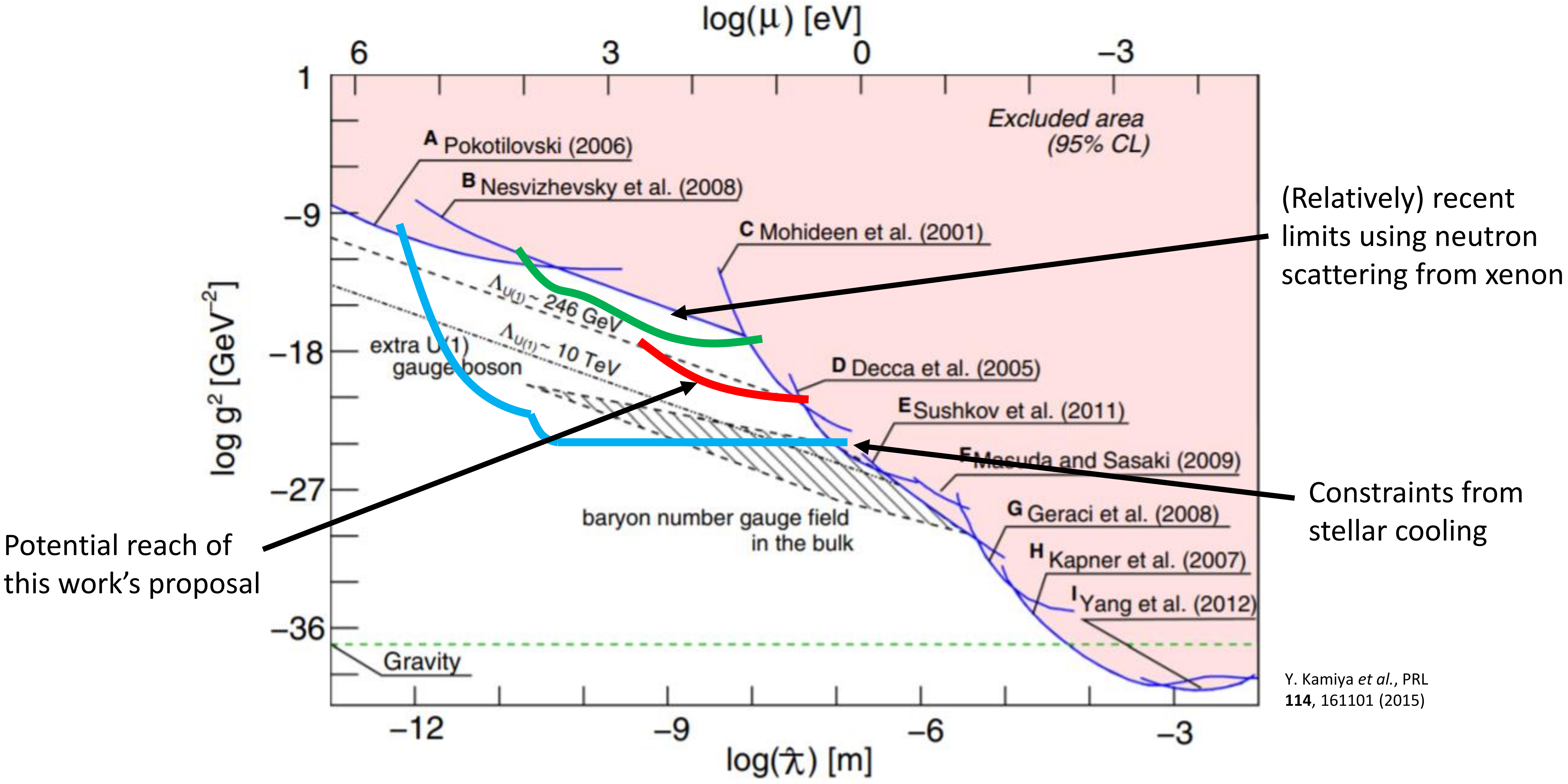
$$b_{i,S}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(\boxed{\gamma_e f_S(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{S}} + \boxed{\mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)}\right)$$

$$b_{i,L}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{S}) = \frac{\gamma_n e^2}{2m_e}\left(\boxed{f_L(A,Z,q_T)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{L}} + \boxed{\mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)}\right)$$

$$b_{i,I}(A,Z,q_T,\boldsymbol{\sigma},\mathbf{I}) = \boxed{b_{nuc,i}(A,Z)\sqrt{I(I+1)}\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{I}}} - \frac{\gamma_n e^2}{2m_e}\left(\boxed{\frac{m_e}{m_p}\gamma(A,Z)\boldsymbol{\sigma}\cdot(\mathbf{1}-\hat{\mathbf{q}}_T\hat{\mathbf{q}}_T^T)\cdot\mathbf{I}} + \boxed{\mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)}\right)$$

$$b_{i,n}(A,Z,q_T,\boldsymbol{\sigma}) = \cancel{\frac{\gamma_n e^2}{2m_e}\left(\frac{m_e}{m_p}i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\cot\theta + \mathcal{O}\left(\left(\frac{m_e}{m_p}\right)^2\right)\right)}$$

Existing limits on new forces



Y. Kamiya *et al.*, PRL **114**, 161101 (2015)