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Algebraic methods for reconstruction of coordinates in strip detectors

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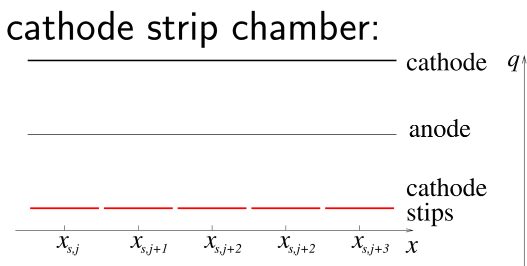
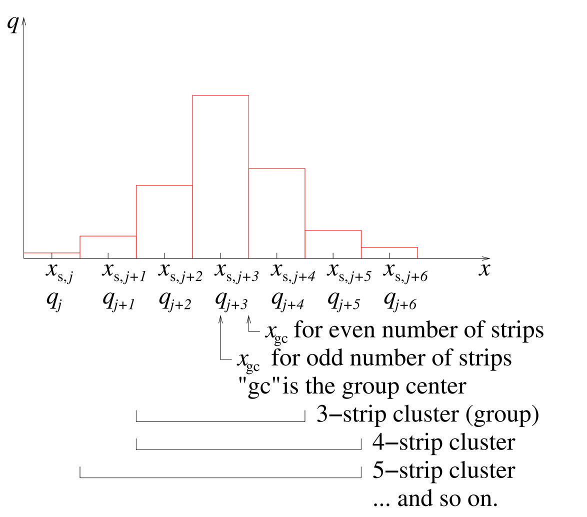
1. Overview

Many types of detectors, such as cathode strip chambers and some MPGDs, allow us to reconstruct track positions by induced strip charges. There are two main types of algebraic methods for this reconstruction: center of gravity methods and little known differential methods. In the differential methods the track coordinate is a ratio of linear combinations of strip charges with parameters constrained by considerations of symmetry and continuity. The resulting formulas are elegant and effective. They do not depend on the common pedestal. Only special cases of these formulas can be found in the literature, usually under different names. In this work general differential formulas are derived and tested. In order to compare them with alternative approaches, center of gravity methods are also considered and improved. One of the new center of gravity methods is almost free from systematic errors and has nearly perfect statistical resolution. The other studied methods need corrections of systematic shifts to obtain perfect results. Algebraic methods may always be useful, and they are the only choice for very high rate experiments, for which the Maximum Likelihood Estimate (MLE) of coordinates with the strip response function takes too much computer time.

2. Center Of Gravity (COG)

Example of strip detector:
cathode strip chamber:



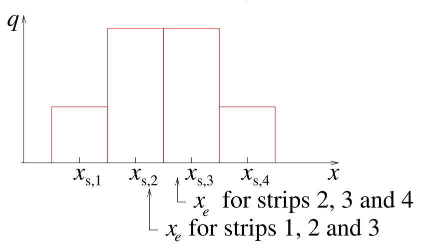


In the case of MPGD there are some analogue of anode (conducting strips or resistive surface), sensitive strips below it and resistive or dielectric layers between them.

Simple center of gravity [1]:

$$x_{\rm er} = \frac{\sum_{i=j}^{j+n-1} (x_{{\rm s},i} - x_{\rm gc}) q_i}{\sum_{i=j}^{j+n-1} q_i}$$

q_i is the charge on the ith strip, $x_{{\rm s},i}$ is the ith strip center (usually $i + 0.5$) $x_{\rm er}$ is the "Estimate Relative" to $x_{\rm gc}$. The final estimate is $x_{\rm e} = x_{\rm gc} + x_{\rm er}$.



- Systematic shifts (systematic errors).
- Discontinuous.
- Depends on the common pedestal.
- Parameters for optimization are absent.

3. Center Of Gravity with Bias level (COGB) and with Power (COGBP)

Center Of Gravity with subtraction of the Bias level (COGB):

$$x_{\rm er} = \sum_{i=j}^{j+n-1} (x_{{\rm s},i} - x_{\rm gc}) \max\{0, q_i - B\} \Big/ \sum_{i=j}^{j+n-1} \max\{0, q_i - B\},$$

the bias level $B = \alpha \sum_{i=1,n} q_i$, α is a small non-negative constant [2,3].

- Systematic shifts.
+ Continuous, if $q_{j-1} < d$ and $q_{j+n} < d$.
- Depends on the common pedestal.
+ One free parameter for optimization.
- Many "local" minima of precision with almost identical precision but with different systematic shifts. (Local minima are also inherent to all the other methods with adjustable parameters.)

Better performance, if to rise $\max\{0, q_i - B\}$ to a power p. This gives a second adjustable parameter (there is no privileged value of p). Let us call it the Center Of Gravity with Bias level and with Power, COGBP.

4. Center of Gravity with floating cluster, Weight function(COGW) and with Splines(COGWS)

The charge is multiplied by a modulating or weight function $w(x)$, such that $w(x) \geqslant 0$, $w(x) = w(-x)$, $w(x) = 0$ for $x > t$, where t halfwidth. The generalized center of gravity:

$$R(x) = \frac{\sum_{i=j}^{j+n-1} q_i \int_i^{i+1} w(z-x) z \, {\rm d}z}{\sum_{i=j}^{j+n-1} q_i \int_i^{i+1} w(z-x) \, {\rm d}z}$$

The estimate of the coordinate x_e is given by the equation:

$$x_{\rm e} = R(x_{\rm e}). \tag{1}$$

Iterative solution: at $(k+1)$th iteration $x_{\rm e}^{(k+1)} = R(x_{\rm e}^{(k)})$.
Non-iterative algebraic solution exists for $w(x)$ expressed by polynomials of the order not higher than two. Numerical tests show that $w(x)$ expressed by a constant in the interval $[-t, t]$, or by a linear function in the same interval, or by a single square polynomial function does not produce perfect results.
Good choice of w is a quadratic spline with continuous first derivative. Fitted parameters are coordinates of breaking points ("knots"). The resulting quartic equation is very complex, but can be solved "by radicals". Of the four solutions, one useful solution can be chosen. Let us call it the Center Of Gravity with Weight function with Splines, COGWS.

+ Systematic shifts can probably be reduced to a negligible level and the statistical resolution can be made the best possible by optimizing free parameters.
+ Continuous.
+ Solutions do not depend on the common pedestal.
- Large complexity and large calculation time (but $\ll$ the time of MLE).

5. Differential methods (ADF, SDF, ADFBP), 6 strips

Let $x_{\rm er} = \sum_{i=1}^{n} a_i q_i / \sum_{i=1}^{n} b_i q_i$ with any parameters a_i and b_i and $n = 6$ (for brevity, here we suppose that $j = 1$). Assume that the maximal charge is q_4 and $q_4 \geqslant q_3 \geqslant q_5$ (the track is most likely between $(x_{{\rm s},3} + x_{{\rm s},4})/2$ and $x_{{\rm s},4}$). Then, $x_{\rm er}$ should be zero, if $q_1 = q_6$, $q_2 = q_5$, $q_3 = q_4$ (condition of symmetry), and 0.5, if $q_2 = q_6$, $q_3 = q_5$ (condition of continuity). Then,

$$x_{\rm er} = \frac{a_1 d_{16} + a_2 d_{25} + a_3 d_{34}}{2a_1 d_{12} + 2a_2 d_{23} + 2a_3 d_{34} + b_5 d_{53} + b_6 d_{62}}, \; d_{ij} = q_i - q_j,$$
$$a_1 \leqslant 0, \; a_2 \leqslant 0, a_3 < 0, b_5 < -2a_2, b_6 \leqslant -2a_1$$

Constraints are sufficient to have non-zero denominator at $q_1 \leqslant q_6 \leqslant q_2 \leqslant q_5 \leqslant q_3 \leqslant q_4 \wedge (q_3 \neq q_4 \vee q_3 \neq q_5)$.
If the maximal charge is q_3 and $q_3 \geqslant q_4 \geqslant q_2$ (the track is most likely between $x_{{\rm s},3}$ and $(x_{{\rm s},3} + x_{{\rm s},4})/2$), it needs to swap $q_1 \leftrightarrow q_6$, $q_2 \leftrightarrow q_5$, $q_3 \leftrightarrow q_4$, to calculate $x_{\rm er}$ and to change its sign. Let us call it Asymmetric Differential Formula, ADF.
Symmetric Differential Formula, SDF, does not require permutations:

$$x_{\rm er} = \frac{1}{2} \frac{a_1 d_{16} + a_2 d_{25} + (a_2 - a_1) d_{34}}{a_1 s_{16} + (a_2 - 2a_1) s_{25} + (a_1 - a_2) s_{34}}, \quad s_{ij} = q_i + q_j, \quad a_1 \leqslant 0, \; a_2 < a_1$$

Both ADF and SDF can be used with the bias and power (notation with suffix BP): $q_i \to (\max\{0, q_i - B\})^p$. α and p are additional adjustable parameters (there is no privileged value of p). At $a_1 = 0$ and $b_6 = 0$ these formulas are converted in 4-strip formulas. The symmetric 4-strip formula $x_{\rm er} = 0.5(-q_1 - q_2 + q_3 + q_4)/(-q_1 + q_2 + q_3 - q_4)$ is proposed in Ref. [4].
The asymmetric 4-strip formula (never proposed):
$x_{\rm er} = (a_1(q_1 - q_4) + a_2(q_2 - q_3))/(2a_1 q_1 + (2(a_2 - a_1) - b_4) q_2 - 2a_2 q_3 + b_4 q_4)$.

6. Differential methods (ADF, RADF, SDF, ADFBP), 7 strips

The asymmetric formula (q_4 is maximum and $q_4 \geqslant q_5 \geqslant q_3$):

$$x_{\rm er} = \frac{a_1 d_{17} + a_2 d_{26} + a_3 d_{35}}{2a_1 d_{12} + 2a_2 d_{23} + 2a_3 d_{34} + b_5 d_{54} + b_6 d_{63} + b_7 d_{72}},$$
$$a_1 \leqslant 0, \; a_2 \leqslant 0, \; a_3 < 0, \; b_5 < -2a_3, \; b_6 \leqslant -2a_2, \; b_7 \leqslant -2a_1$$

If $q_3 > q_5$, permute $q_1 \leftrightarrow q_7$, $q_2 \leftrightarrow q_6$, $q_3 \leftrightarrow q_5$, calculate $x_{\rm er}$ and change the sign. The symmetric formula:

$$x_{\rm er} = \frac{1}{2} \frac{a_1 d_{17} + a_2 d_{26} + a_3 d_{35}}{a_1 s_{17} + (a_2 - 2a_1) s_{26} + (2a_1 - 2a_2 + a_3) s_{35} + 2(-a_1 + a_2 - a_3) q_4},$$
$$a_1 \leqslant 0, \; a_2 \leqslant a_1, \; a_3 < a_2 - a_1.$$

If $a_1 = b_7 = 0$, these formulas are converted into 5-strip formulas.
If $a_1 = a_2 = b_6 = b_7 = 0$, then 3-strip formulas are obtained.
If $b_3 = 0$, the asymmetric 3-strip formula is "the ratio method":
$r = 0.5(Q_{\rm mid} - Q_{\rm min})/(Q_{\rm max} - Q_{\rm min})$. from Ref. [5]. The full asymmetric 3-strip formula (never proposed): $x_{\rm er} = a_1(q_1 - q_3)/(2a_1 q_1 - (2a_1 + b_3) q_2 + b_3 q_3)$.
The symmetric 3-strip formula $x_{\rm er} = 0.5(-q_1 + q_3)/(-q_1 + 2q_2 - q_3)$ is also an algebraic fit of the parabolic strip response function [6].
An intermediate 7-strip formula, Restricted Asymmetric Differential Formula, RADF, can also be useful, if to fix some b_i (one or two) at their values for the symmetric case, for example: $b_7 = 2a_1$, $b_6 = -4a_1 + 2a_2$. Then, one can fit b_5 together with a_2 and a_3 (a_3 can be assumed to be equal to -1).
General asymmetric formula:
$\forall n \geqslant 3$, m is the whole part of $n/2$, $l = m + 1$, $j = 1$:

$$x_{\rm er} = \frac{\sum_{i=1}^{m} a_i d_{i,n-i+1}}{2\sum_{i=1}^{m} a_i d_{i,i+1} + \sum_{i=m+2}^{n} b_i d_{i,n-i+2}},$$

- Systematic shifts.
+ Continuous.
+ Independent on the common pedestal (if used without bias and power).
+ Many parameters for optimization.

If n is odd, $a_{l-1} < 0$, $b_{l+1} < -2a_{l-1}$, $\forall i \in [2; n-l]$, $a_{l-i} \leqslant 0$, $b_{l+i} \leqslant -2a_{l-i}$; for $q_l \geqslant q_{l+1} \geqslant q_{l-1} \geqslant q_{l+2} \ldots \geqslant q_1$ and $q_l > q_{l-1}$. If n is even, $a_{l-1} < 0$, $b_{l+1} < -2a_{l-2}$, $\forall i \in [2; n-l]$, $a_{l-i} \leqslant 0$, $b_{l+i} \leqslant -2a_{l-i-1}$, $a_1 \leqslant 0$; for $q_l \geqslant q_{l-1} \geqslant q_{l+1} \geqslant q_{l-2} \ldots \geqslant q_1$ and $q_l > q_{l+1}$.

7. Fitting parameters

Suppose, a "true" coordinate $x_{\rm t}$ is known (in simulations or in a test experiment).
Minimization of the "standard" sample variance: $S(x_{\rm e}) = \frac{1}{N_{\rm e}} \sum_{i=1}^{N_{\rm e}} (x_{{\rm e},i} - x_{{\rm t},i})^2$ for $N_{\rm e}$ events can ignore "local" minima with almost the best total resolution, but with much better systematic shifts. Good minimum is found by the minimization of a generalized $S_\beta(x_{\rm e}) = \frac{1}{N_{\rm e}} \sum_{i=1}^{N_{\rm e}} (x_{{\rm e},i} - x_{{\rm t},i})^2 + \beta Y$, β is a small constant, and Y is an estimate of systematic shifts or an estimate of deflection of the spatial distribution (fluctuations of the occupancy). The latter is used in tests. If $n_{{\rm h},i}$ is the number of events in i-th bin of the histogram of $x_{\rm er}$ with $N_{\rm h}$ bins, and $n_{{\rm h},i}^{({\rm s})} = (n_{{\rm h},i} + n_{{\rm h},N_{\rm h}+1-i})/2$, then

$$Y = \frac{N_{\rm h}}{N_{\rm e}} \frac{\left(\max\{0, \sigma(n_{\rm h}^{({\rm s})}) - E[\sigma(n_{\rm h}^{({\rm s})})]\}\right)^2}{E^2[\sigma(n_{\rm h}^{({\rm s})})]}, \quad \sigma(n_{\rm h}^{({\rm s})}) = \sqrt{\frac{1}{N_{\rm h}} \sum_{i=1}^{N_{\rm h}} \left(n_{{\rm h},i}^{(s)} - \frac{N_{\rm e}}{N_{\rm h}}\right)^2}.$$

8. Correction of systematic shifts, smoothing

Fourier series [7] or polynomial [8] correction is possible. An alternative approach is smoothing the measured spatial distributions. Let us suppose a true coordinate distribution is uniform. Let $x_{er} \in [x_{\rm er,min}, x_{\rm er,min} + 1[$, where $x_{\rm er,min}$ is either -0.5 or 0. Recovery of uniformity: $x_{\rm er,corr} = \int_{x_{\rm er,min}}^{x_{\rm er,min}+1} {\rm p}(x_{\rm er}) {\rm d}x_{\rm er} - x_{\rm er,min}$, where ${\rm p}(x_{\rm er})$ is probability density of $x_{\rm er}$ [9,10]. Let us call it integrated smoothing or just smoothing. It is applied with minimization of $S(x_{\rm e,corr})$ (without βY).

9. Numerical testing, charge distribution function

A well known function describing the change distribution in cathode strip chambers [11,12,13] was used for numerical tests. The plots presented below are computed for $K_3 = 0.5$ and for strip width equal to the anode-cathode gap.
Some white noise (0.008 of the full cathode charge) was added to the charge of each strip in order to make the resolution similar to the typical experimental resolution of such chambers.

10. Numerical testing, resolution and shifts

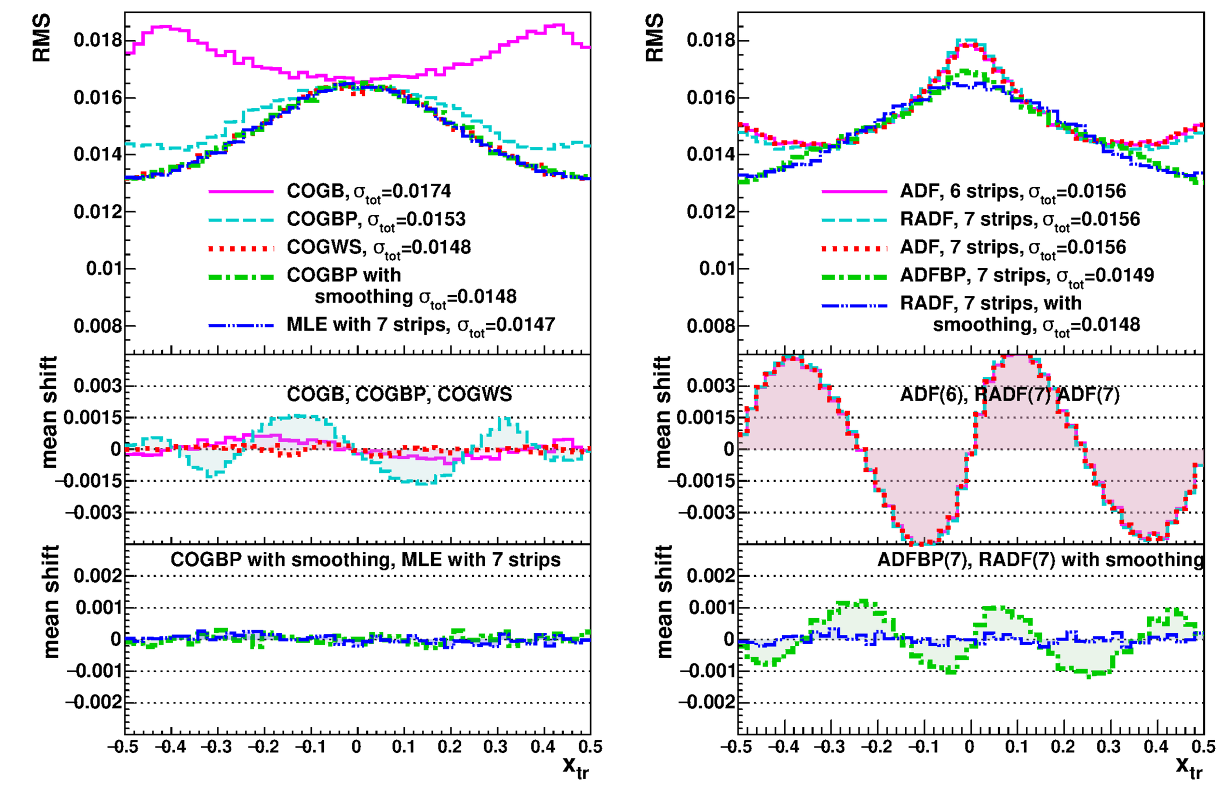


Figure: The statistical resolution (RMS) and systematic shifts (the "mean shifts", that is averages $(x_{\rm e} - x_{\rm t})$), as a functions of the true relative (to the center of its strip) coordinate $x_{\rm tr}$.

COGB: Center of Gravity with Bias;
COGBP: Center of Gravity with Bias and Power;
COGWS: Center of Gravity with Weights and quadratic Splines;
ADF: Asymmetric Differential Formula;
RADF: Restricted Asymmetric Differential Formula;
ADFBP: Asymmetric Differential Formula with Bias and Power;
MLE with 7 strips: Maximum Likelihood Estimation with charge distribution.

11. Numerical testing, spatial distribution

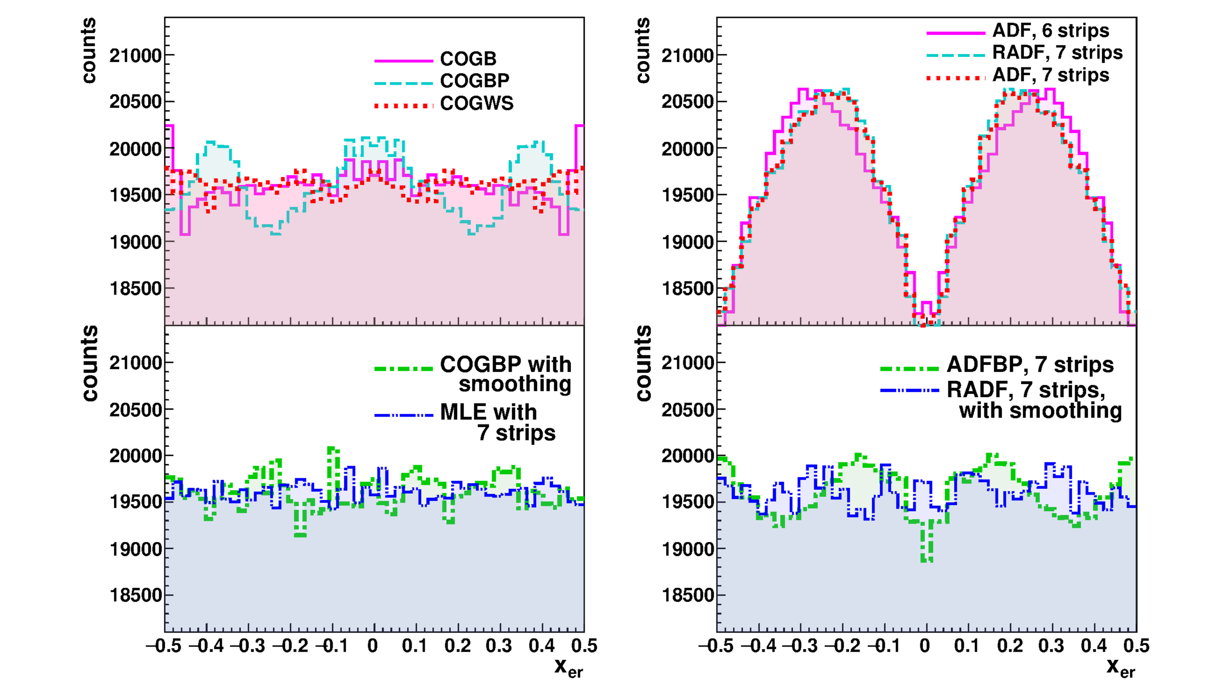


Figure: Measured spatial distributions (occupancy histograms) for the uniform irradiation for all methods given in the previous figure.

12. Features of methods

notation of method	simplicity	continuity	independence on common pedestal	accuracy
COG	yes	no	no	bad
COGB	yes	yes	no	moderate
COGBP	yes	yes	no	good
COGWS	no	yes	yes	very good
COGBP with smoothing	yes	yes	no	very good
MLE	no	yes	yes	very good
ADF(RADF, SDF)	yes	yes	yes	moderate
ADFBP	yes	yes	no	good
ADF(RADF, SDF) with smoothing	yes	yes	yes	very good

13. Conclusions

A range of new differential methods and center of gravity methods is developed (in particular, denoted by SDF, ADF, RADF, ADFBP, COGBP, COGWS).
Several methods of both classes, when applied with the integrated smoothing, and the COGWS method (the center of gravity method for which the smoothing is unnecessary) provide almost zero systematic shifts and the resolution which is very close to the best resolution attainable by MLE.
Several methods of both classes (SDF, ADF, RADF, COGWS) ensure the independence of the results on the common pedestal.

14. References

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