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Stabilizing complex Langevin for real-time YM theory

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Doktoratskolleg Particles and Interactions

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Outline

Schwinger-Keldysh formalism for Yang-Mills theory

Expectation values

- Gauge fields on SK-contour

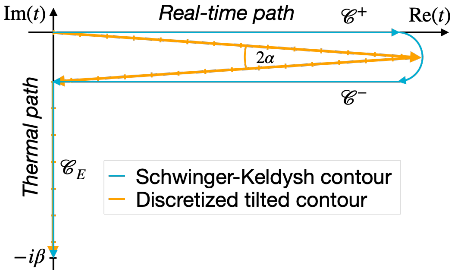
$$\langle \mathcal{O}[A] \rangle = \frac{1}{Z} \int \mathcal{D}A_E \, e^{-S_E[A_E]} \int \mathcal{D}A_+ \, \mathcal{D}A_- \, e^{iS[A_+,A_-]} \, \mathcal{O}(x)$$

- Periodic boundary conditions

$$A^a_\mu(t=0) = A^a_\mu(t=-i\beta)$$

- Yang-Mills action

$$S_{\rm YM} = -\frac{1}{4} \int_{\mathcal{C}} d^4x F^{\mu\nu}_a F^a_{\mu\nu}$$



Sign problem of real-time YM theory

Complex weight function leads to the sign problem

- Real-time contour leads to complex weight function
- Direct calculation of oscillatory integrals is not feasible

$$\langle \mathscr{O}[A] \rangle = \frac{1}{Z} \int \mathcal{D}A_E \, e^{-S_E[A_E]} \int \mathcal{D}A_+ \, \mathcal{D}A_- \, \mathbf{e}^{\mathbf{iS[A_+, A_-]}} \, \mathscr{O}(x)$$

- Numerical integration methods exponentially costly

Alternative integration methods needed

- Reweighting, Contour deformation, Analytic continuation, Taylor expansion, Lefschetz thimbles, **Complex Langevin method**, ...

Complex Langevin method for gauge fields

- Introduction of (auxiliary) Langevin time θ
- CL equation for gauge fields [1]

$$\partial_\theta A_\mu^a(\theta, x) = i \frac{\delta S_{\mathrm{YM}}}{\delta A_\mu^a(t,x)} + \eta_\mu^a(\theta, x), \quad A_\mu(\theta, x) \in \mathfrak{sl}(N_c, \mathbb{C})$$

- Gaussian distributed noise term

$$\langle \eta_\mu^a(\theta, t, \mathbf{x}) \rangle = 0,$$
$$\langle \eta_\mu^a(\theta, t, \mathbf{x}) \eta_\nu^b(\theta', t', \mathbf{x}') \rangle = 2\delta(\theta - \theta')\delta(t - t')\delta^{(d-1)}(\mathbf{x} - \mathbf{x}')\delta^{ab}\delta_{\mu\nu}$$

Goal: Overcoming the sign problem

- CL bypasses the sign problem by sampling at late θ

$$\langle \mathcal{O}[A] \rangle \approx \lim_{\theta_0 \to \infty} \frac{1}{T} \int_{\theta_0}^{\theta_0 + T} d\theta \, \mathcal{O}[A(\theta)]$$

Instabilities of the CL method

CL suffers from instabilities

- Runaway instabilities → numerical blow-up
- Wrong-convergence instabilities → distorted expectation values
- Severity of the instabilities increase with shrinking tilt-angles

- Discretized path integral is regularized for tilted contours
- Complex time contours leads to ambiguous noise correlator expression (δ-distribution for complex arguments)
  - Ambiguities are resolved by a parametrization dependent CL formulation

$$t : [a, b] \mapsto \mathbb{C}, \qquad t(a) = 0, \quad t(b) = -i\beta$$

  - Noise correlator in terms of λ

$$\delta(t(\lambda) - t(\lambda')) \rightsquigarrow \delta(\lambda - \lambda')$$

Outline

Lattice gauge fields

- Link variables and plaquette variables

$$U_{x,\mu} \simeq \exp\left[iga_\mu A_\mu\big(x+\hat{\mu}/2\big)\right] \in \mathrm{SU}(N_c) \rightsquigarrow \mathrm{SL}(N_c,\mathbb{C}),$$
$$U_{x,\mu\nu} = U_{x,\mu}U_{x+\hat{\mu},\nu}U^{-1}_{x+\hat{\nu},\mu}U^{-1}_{x,\nu}$$

- Wilson plaquette action

$$S_{\mathrm{W}}[U] = \frac{1}{2N_c}\sum_{k,\mathbf{x},\mu\neq\nu}\beta_{\mu\nu}\mathrm{Tr}\left[U_{x,\mu\nu}-1\right],$$

- Coupling constants

$$\beta_{0i} = -\frac{2N_c}{g^2}\frac{a_s}{a_{t,k}}, \qquad \beta_{ij} = +\frac{2N_c}{g^2}\frac{\bar{a}_{t,k}}{a_s}$$

- Time reversibility is retained by averaged spacing

$$\bar{a}_{t,k} = (a_{t,k}+a_{t,k+1})/2$$

Euler-Maruyama for numerical solution of the CLE

CL update step for link variables on a SK-contour

- Additional lattice spacing factors compared to common CL-step [2]

$$U_{x,t}(\theta+\epsilon) = \exp\left(it^a\left[i\epsilon\frac{a_{\lambda,k}}{a_s}\left.\frac{\delta S_{\mathrm{W}}}{\delta \tilde{A}^a_{x,t}}\right|_\theta + \sqrt{\epsilon}\sqrt{\frac{a_{\lambda,k}}{a_s}}\eta^a_{x,\lambda}(\theta)\right]\right)U_{x,t}(\theta)$$

$$U_{x,i}(\theta+\epsilon) = \exp\left(it^a\left[i\epsilon\frac{a_s}{\bar{a}_{\lambda,k}}\left.\frac{\delta S_{\mathrm{W}}}{\delta \tilde{A}^a_{x,i}}\right|_\theta + \sqrt{\epsilon}\sqrt{\frac{a_s}{\bar{a}_{\lambda,k}}}\eta^a_{x,i}(\theta)\right]\right)U_{x,i}(\theta)$$

- Langevin time step $\epsilon \to 0$
- Noise is approximated by a Gaussian function

Stabilization techniques

- **Gauge cooling [3]:** Exploit gauge freedom to minimize non-unitarity measured by a functional $F[U]$

$$U_{x,\mu} \mapsto U^V_{x,\mu} = V_x U_{x,\mu} V^{-1}_{x+\mu}, \qquad F[U] \geq F[U^V]$$

- **Adaptive stepsize [4]:** Regulate large drift terms which lead to instabilities

$$\epsilon \mapsto \tilde{\epsilon} = \epsilon \min\left(1, \frac{B}{\max\limits_{x,\mu,a} |K^a_{x,\mu}|}\right)$$

- **Dynamical stabiliziation [5]:** Penalize drift terms depending on the local non-unitary of the configuration

$$K^a_{x,\mu} \mapsto \tilde{K}^a_{x,\mu} = K^a_{x,\mu} + i\alpha_{\mathrm{DS}} M^a_x$$

We introduce an anisotropic kernel to improve stability!

Field-independent kernels

Field-independent Kernel Freedom [1]

- *Kerneled Langevin equation* yields same limiting density function

$$\partial_\theta A^a_\mu(\theta, x) = i \int dx' \, \Gamma^{ab}_{\mu\nu}(x, x') \frac{\delta S}{\delta A^b_\nu(x')} + \int dx' \, \tilde{\Gamma}^{ab}_{\mu\nu}(x, x') \eta^b_\nu(\theta, x')$$

- $\Gamma^{ab}_{\mu\nu}(x, x')$ is required to be factorizable

$$\Gamma^{ab}_{\mu\nu}(x, x') = \int dx'' \, \tilde{\Gamma}^{ac}_{\mu\sigma}(x, x'') \tilde{\Gamma}^{bc}_{\nu\sigma}(x', x'')$$

- Kerneled Langevin equations correspond to the same Fokker-Plank equation
- *In practice:* Kernels may aggravate or mitigate instabilities

Novel anisotropic kernel

Kerneled CL update step

- Rescaling of Langevin time for temporal and spatial d.o.f.

$$U_{x,t}(\theta+\epsilon) = \exp\left(it^a \left[i\epsilon \left(\frac{a_{\lambda,k}}{a_s}\right)^2 \left.\frac{\delta S_{\mathrm{W}}}{\delta \tilde{A}^a_{x,t}}\right|_\theta + \sqrt{\epsilon}\frac{a_{\lambda,k}}{a_s}\eta^a_{x,\lambda}(\theta) \right] \right) U_{x,t}(\theta)$$

$$U_{x,i}(\theta+\epsilon) = \exp\left(it^a \left[i\epsilon \left.\frac{\delta S_{\mathrm{W}}}{\delta \tilde{A}^a_{x,i}}\right|_\theta + \sqrt{\epsilon}\eta^a_{x,i}(\theta) \right] \right) U_{x,i}(\theta)$$

- Noise of spatial update blows up for $a_{\lambda,k} \to 0$ $\quad\rightsquigarrow\quad \theta \mapsto \frac{\bar{a}_{\lambda,k}}{a_s}\theta$
- Temporal plaquette shows only slow dynamics $\quad\rightsquigarrow\quad \theta \mapsto \frac{a_{\lambda,k}}{a_s}\theta$

Outline

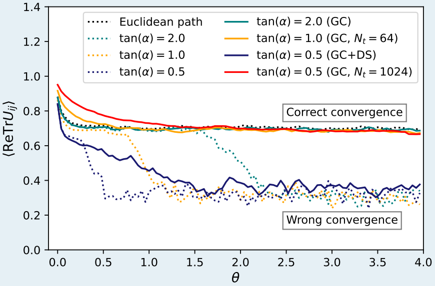
Numerical setup

- We simulate $SU(2)$ gauge theory on a $N_t \times 4^3$ lattice
- Coupling constant $g = 1$
- Inverse temperature of the system $\beta = 4.0$
- Simulations are initialized with identity matrices
- Same seed for random number generator for noise

Correct expectation values of one-point functions

Trace of average spatial plaquette

- **Kernels successfully stabilizes even small tilt angles**
- Existing methods not enough for stabilizing simulations



- Avg. spatial plaquette (considered in earlyier studies [2]):

$$\mathcal{O}[U] = \frac{1}{N_t N_s^3} \sum_x \frac{1}{3} \sum_{i<j} \frac{1}{N_c} \mathrm{ReTr}\, U_{x,ij}$$

- Time translation inv. allows comparison to Euclidean results

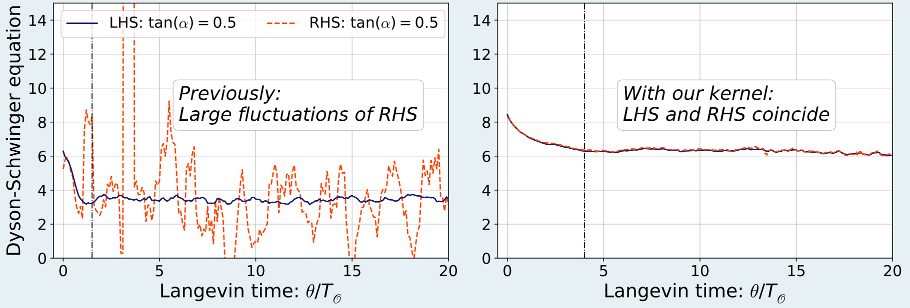
Expectation values from CL

Results for average spatial plaquette $\mathscr{O}$ via CL:

$\tan(\alpha)$	Stabilization techniques	N_t	$\langle \mathscr{O} \rangle$
Euclidean	None	16	0.704 ± 0.002
2.0	AS, GC	16	0.701 ± 0.002
1.0	AS, GC, DS	16	0.678 ± 0.002
0.5	AS, GC, DS	16	0.318 ± 0.007
2.0	AS, GC, Γ	16	0.701 ± 0.003
1.0	AS, GC, Γ	64	0.703 ± 0.003
0.5	AS, GC, Γ	1024	0.705 ± 0.004

Dyson-Schwinger equations

DSE for spatial plaquettes



- Self-consistency check of link configuration
- Derivation from first principles $\rightsquigarrow$ sampling process is "physical"

$$\frac{2(N_c^2-1)}{N_c}\left\langle \mathrm{ReTr}(U_{x,ij})\right\rangle = \frac{i}{2N_c}\sum_{|\rho|\neq i}\beta_{i\rho}\left\langle \mathrm{ReTr}\left[(U_{x,i\rho}+U_{x,i\rho}^{-1})U_{x,ij}\right]\right\rangle$$

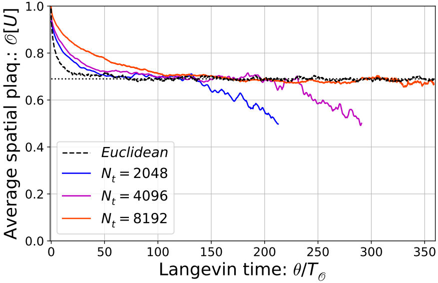
Systematic improvement of CL instabilities

Anisotropic kernel systematically stabilizes CL

- Our kernel allows simulations even for $\max \mathrm{Re}(t) > \beta$
- Stability is systematically improved for partial continuum limit

$\Rightarrow$ **Calculation of real-time observables may be feasible!**

- Stability region grows faster then auto-correlation time w.r.t. N_t
- Computational cost grows linearly with N_t



Tilt angle: $\tan(\alpha) = 0.4$

Outline

Conclusion & Outlook

Conclusion

- Stabilization techniques extend the applicability of CL
- We found an approach to systematically improve stability
- Anisotropic kernel enables simulations of $\max \mathrm{Re}(t) > \beta$

  $\rightsquigarrow$ Extrapolation to Schwinger-Keldysh contour might be possible

  $\rightsquigarrow$ Application to real-time observables

Outlook

- Calculation of unequal time correlation functions
- Development of further stabilization techniques
- Criterion of correctness and boundary terms
- Non-thermal quantum systems (future prospect)

References

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