Explicit and multipole solutions of viscous fireball hydrodynamics

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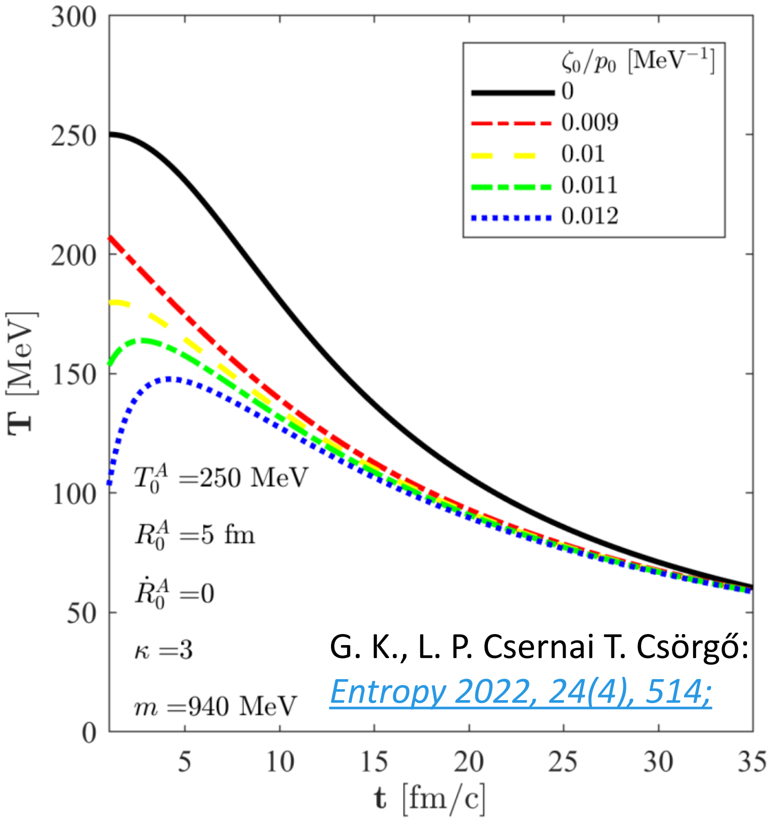
Introduction

Asymptotic perfection

- New exact solutions of viscous hydrodynamics have been found recently
- ***These solutions are asymptotically perfect***
- **Perfect fluid attractor**: how to extract the effect of bulk and shear viscosity in final state measurments?



G. K., L. P. Csernai T. Csörgő: [Entropy 2022, 24(4), 514;]

Introduction

List of variety of viscous, asymptotically perfect fluid solutions:

Self-similar solutions were found

- …in relativistic regime: T. Csörgő, G. K.: arXiv:2003.08859 M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498
- …in non-relativistic regime: G. K., L. P. Csernai and T. Csörgő: arXiv:2203.05859

In the relativistic regime, we have found solutions of…

- …Israel-Stewart theory: T. Csörgő, G. K.: arXiv:2003.08859
- …Navier-Stokes theory: M. Csanád, M. I. Nagy, Z. Jiang, T. Csörgő: arXiv:1909.02498
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Solutions above were found with different assumptions, with different symmetries:

- $\zeta \sim p$, $\eta \sim p$ → spherical (only bulk), spheroidal, ellipsoidal
- $\zeta \sim n$ → spherical
- $\zeta = \zeta(p(t))$ → spherical, spheroidal, ellipsoidal (in prep.)

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Our recent paper was awarded „*Feature paper*" by MDPI Entropy:

Entropy 2022, 24(4), 514;

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Spheroidal Hubble-profile

Common properties of these solutions: Hubble-flow

In this work we have **spheroidal symmetry**:

Longitudinal scale: $Y(t)$

Transverse scale: $R(t)$

$$v_H(\vec{r},t)=\left(\frac{\dot{R}}{R}r_x,\frac{\dot{Y}}{Y}r_y,\frac{\dot{R}}{R}r_z\right)$$

Self similar solutions:

$$(\partial_t+\vec{v}\nabla)s=0$$

$$s=\frac{r_x^2+r_z^2}{R^2}+\frac{r_y^2}{Y^2}$$

M. I. Nagy, T. Csörgő: arXiv:1309.4390

Equations of Non Relativistic Hydro

Non relativistic hydrodynamics

- The same effects can be understood in a much simpler formalism
- ***The basic equations of non relativistic, viscous hydro are fully clarified***

$$\partial_t n + \nabla(n\vec{v}) = 0$$

$$\partial_t \varepsilon + \nabla(\varepsilon\vec{v}) + p\nabla\vec{v} = \zeta(\nabla\vec{v})^2 + 2\eta\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right]$$

$$(\varepsilon + p)(\partial_t + \vec{v}\nabla)\vec{v} + \nabla p = \nabla(\zeta\nabla\vec{v}) + \eta\left[\Delta\vec{v} + \frac{1}{3}\nabla(\nabla\vec{v})\right]$$

Local conservation laws

$$\partial_t \sigma + \nabla(\sigma\vec{v}) = \frac{\zeta}{T}(\nabla\vec{v})^2 + \frac{2\eta}{T}\left[\mathrm{Tr}\left(D^2\right) - \frac{1}{3}(\nabla\vec{v})^2\right] \geq 0$$

Balance eq. of entropy

Equation of State

To close the equation system:

EoS: $\varepsilon = \kappa(T) p$

$p = nT$

Relationship to the speed of sound

T. Csörgő, S. V. Akkelin, Y. Hama, B. Lukács, Yu. Sinyukov:
Phys.Rev.C **67** (2003) 034904, hep-ph/0108067

G. Kasza, L. P. Csernai and T. Csörgő, 2203.05859 [hep-th]
Published in: *Entropy* 2022, **24**(4), 514,
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Relationship to the speed of sound

κ is constant:

$$c_s^2 = \frac{1}{\kappa}$$

κ is temperature dependent:

$$c_s^2 = \frac{1}{(1+\kappa)}\left[\frac{d}{dT}\left(\frac{\kappa T}{1+\kappa}\right)\right]^{-1}$$

Only for μ=0

Equation of State

To close the equation system:

EoS: ε=κ(T)p

p=nT

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Relationship to the speed of sound (μ≠0)

κ is constant:

$$c_s^2(T) = \left(1+\kappa^{-1}\right)\frac{T}{m}$$

κ is temperature dependent:

$$c_s^2(T) = \underbrace{\left[1+\left(\kappa + T\frac{d\kappa}{dT}\right)^{-1}\right]}\frac{T}{m}$$

the adiabatic index can be temperature dependent (γ(T)) or constant (γ)

New, self-similar solutions

Self-similar solutions: product of a homogeneous term and arbitrary functions of *s*

$$n(t,s) = n_H(t)\mathcal{V}(s) \longrightarrow n(t,s) = n_0 \frac{V_0}{V}\mathcal{V}(s)$$

$$T(t,s) = T_H(t)\mathcal{T}(s) \longrightarrow$$

$$p(t,s) = n(t,s)T(t,s) = p_H(t)\mathcal{V}(s)\mathcal{T}(s)$$

For constant κ we assume:

$$T_H(t) = T_P(t)D_T(t)$$

ν(s) and τ(s) are not independent from each other, from Euler eqs.:

$$-\frac{C_E}{2} = \mathcal{T}'(s) + \frac{\mathcal{T}(s)}{\mathcal{V}(s)}\mathcal{V}'(s)$$

Homogeneous *p*: C_E=0 $\longrightarrow$

Multipole solutions

$$\mathcal{V}(s) = \frac{1}{\mathcal{T}(s)}$$

$$\downarrow$$

$$\mathcal{V}\left(s_x, s_y, s_z\right) = \frac{1}{\mathcal{T}\left(s_x, s_y, s_z\right)}$$

Conditions to find exact solutions

Assumption: $\zeta \equiv \zeta(p)\,,\quad \eta \equiv \eta(p)$

If the pressure is inhomogeneous:

- Conditions to find exact solutions: $\zeta \propto p\,,\ \eta \propto p$
- Parametric, rotating solutions, partial diff. eqs. are reduced to ordinary diff. eqs.

If the pressure is homogeneous:

- Arbitrary functions of *p*: $\zeta = \zeta(p(t))\,,\ \eta = \eta(p(t))$
- Parametric, ***multipole solutions***, partial diff. eqs. are reduced to ordinary diff. eqs.
- ***Explicit, multipole solution were found for*** $\zeta \propto p\,,\ \eta \propto p$

  Spherically symmetric: G. K., L. P.. Csernai, T. Csörgő: arXiv:2203.05859 (non-relativistic)
  T. Csörgő, G. K.: arXiv:2003.08859 (relativistic)

  Spheroidally symmetric: today's topic (without rotation)

  Ellipsoidally symmetric: in preparation

Spheroidally symmetric, explicit solution

If pressure is homogeneous

If no rotation in the system

$$R(t) = R_0 + \dot{R}_0(t - t_0),$$
$$Y(t) = Y_0 + \dot{Y}_0(t - t_0),$$

$\kappa=\kappa_0$ is temperature independent, constant

The solution for *n(t,s)* is known from perfect fluid solutions

We can use:

$$T_H(t) = T_P(t) D_T(t)$$

(„P" for perfect fluid and „D" for dissipation)

The energy conservation becomes:

$$\frac{\dot{D}_T}{D_T} = \frac{\xi_R}{\kappa_0}\frac{\dot{R}_0}{R(t)} + \frac{\xi_Y}{\kappa_0}\frac{\dot{Y}_0}{Y(t)} + \frac{\xi_{RY}}{\kappa_0}\frac{\dot{R}_0\dot{Y}_0}{R(t)Y(t)}$$

Assumptions for viscosities:

$$\frac{\zeta(t,s)}{\zeta_0} = \frac{\eta(t,s)}{\eta_0} = \frac{p(t,s)}{p_0}$$

The ***viscosities*** are carried by:

$$\xi_R = \frac{4\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$
$$\xi_Y = \frac{\zeta_0}{p_0} + \frac{4\eta_0}{3p_0}$$
$$\xi_{RY} = \frac{4\zeta_0}{p_0} - \frac{8\eta_0}{3p_0}$$

Spheroidally symmetric, explicit solution

The solution of the dissipative correction is:

$$D_T(t) = \left(\frac{R_0}{Y_0}\frac{Y(t)}{R(t)}\right)^{\frac{\xi_{RY}}{\kappa_0}\frac{H_R H_Y}{H_R - H_Y}} \exp\left[\frac{\xi_R H_R}{\kappa_0}\left(1 - \frac{R_0}{R(t)}\right) + \frac{\xi_Y H_Y}{\kappa_0}\left(1 - \frac{Y_0}{Y(t)}\right)\right]$$

For asymptotically late times D_T is constant:

$$D_T(t \to \infty) = \left(\frac{H_Y}{H_R}\right)^{\frac{\xi_{RY}}{\kappa_0}\frac{H_R H_Y}{H_R - H_Y}} \exp\left(\frac{\xi_R H_R}{\kappa_0} + \frac{\xi_Y H_Y}{\kappa_0}\right)$$

Hubble-parameters:

$$H_R = \frac{\dot{R}_0}{R_0}$$

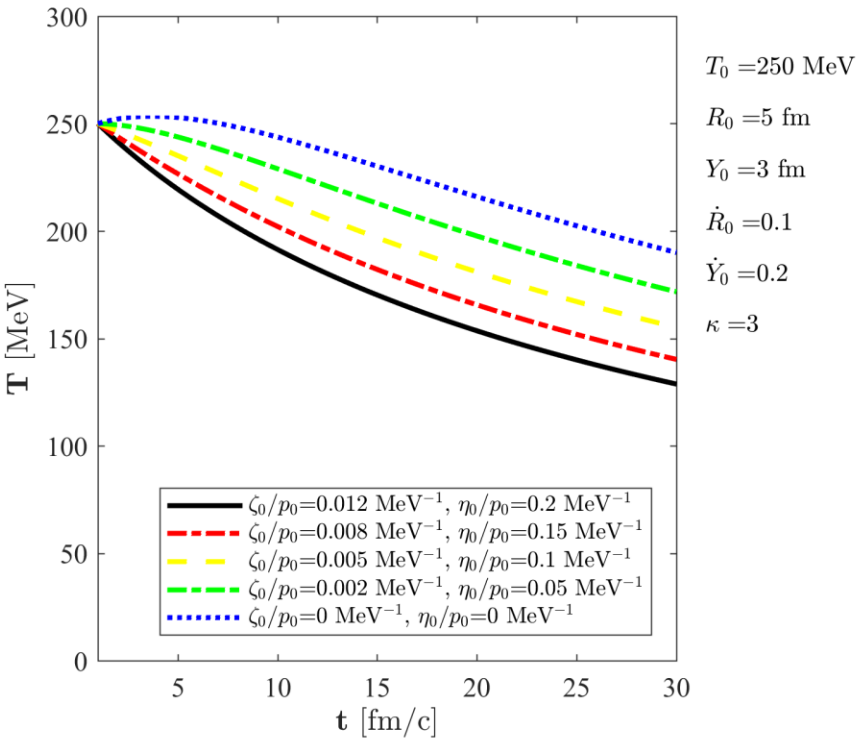
$$H_Y = \frac{\dot{Y}_0}{Y_0}$$

For late time, D_T does not affect the time evolution

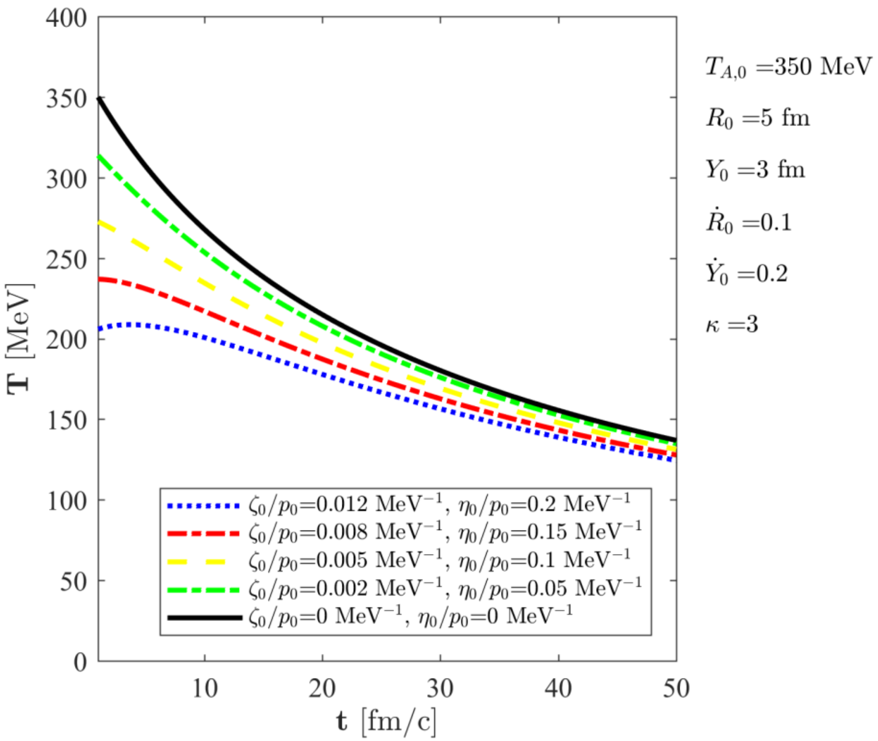
This solution tends to a perfect fluid solution:

$$T_A(t,s) = T_{A,0}(\kappa_0)\left(\frac{R_0^2 Y_0}{R^2 Y}\right)^{\frac{1}{\kappa_0}} \mathcal{T}(s)$$

Spheroidally symmetric, explicit solution



Fixed initial conditions



Fixed attractor

Spheroidally symmetric, explicit solution

Explicit solution with T dependent κ

Temperature equation:

$$\left[\frac{d}{dT}(\kappa T)\right]\partial_t \ln(T_H) + \frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y} = \frac{\zeta_0}{p_0}\left(\frac{2\dot{R}}{R} + \frac{\dot{Y}}{Y}\right)^2 + \frac{4\eta_0}{3p_0}\left(\frac{\dot{R}}{R} - \frac{\dot{Y}}{Y}\right)^2$$

We have seen that it can be solved analytically, if κ is constant

There is another possibility:

$$\frac{d}{dT}\left[\kappa(T)T\right] = \kappa_T = \text{constant}$$

→ Condition of finding exact, analytic solutions, if κ=κ(T)

$$\frac{\dot{D}_T}{D_T} = \frac{\xi_R}{\kappa_T}\frac{\dot{R}_0}{R(t)} + \frac{\xi_Y}{\kappa_T}\frac{\dot{Y}_0}{Y(t)} + \frac{\xi_{RY}}{\kappa_T}\frac{\dot{R}_0\dot{Y}_0}{R(t)Y(t)}$$

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→ ***The same temperature eq. as before, but with different constant coefficient:***

κ_0 → $\Delta\kappa_T$

Spheroidally symmetric, explicit solution

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$$D_T(t \to \infty) = \left(\frac{H_Y}{H_R}\right)^{\frac{\xi_{RY}}{\kappa_T}\frac{H_R H_Y}{H_R - H_Y}} \exp\left(\frac{\xi_R H_R}{\kappa_T} + \frac{\xi_Y H_Y}{\kappa_T}\right)$$

The same solution and asymptotic behaviour as before, but with different constant coefficient:

$\kappa_0 \rightarrow \Delta\kappa_T$

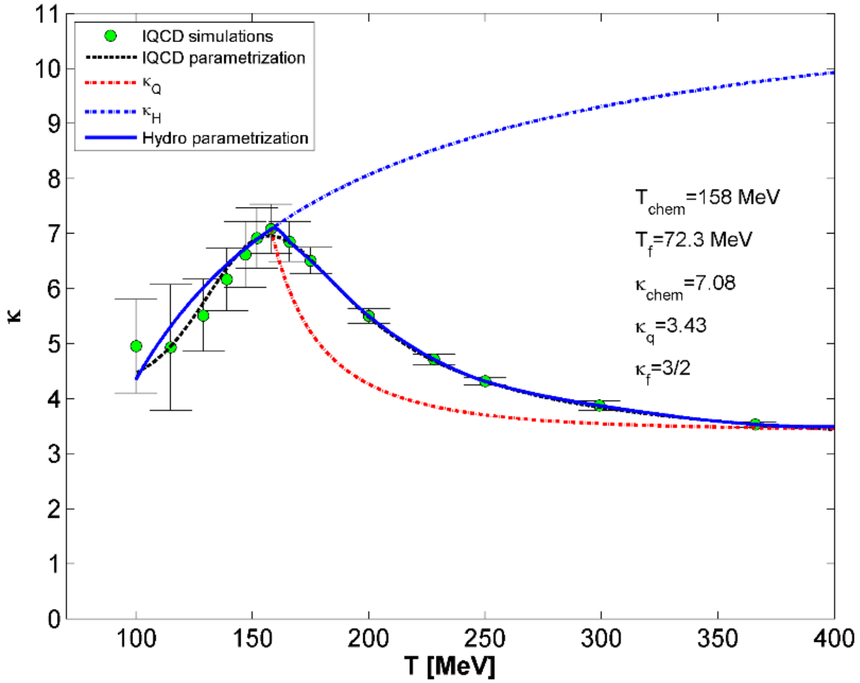
Generating explicit solutions with κ(T)

We constrain the κ(T) function by the condition to find analytic solutions of hydrodynamics

The power of this method:

1) *Provides hydro based parametrization of lattice QCD EoS:*
2) *Works in relativistic and non relativistic regimes*

T. Csörgő, G. K.: arXiv:1610.02197 and my MSc thesis



Lattice QCD EoS can be utilized in explicit, relativistic solutions

Summary

New non-relativistic solutions: **mostly academic results**, focus on understanding and clarity

New family of solutions of hydro: **asymptotically perfect fluid solutions**

Seems to be broad, open question: how broad is it?

The attractors are also known, perfect fluid solutions:

Relativistic, NS/IS solutions, perfect fluid attractor: T. Csörgő. L. P. Csernai, Y. Hama, T. Kodama: Acta Phys.Hung.A 21 (2004) 73-84
Non relativistic, spherical NS fireball, perfect fluid attractor: T. Csörgő: Central Eur. J. Phys. 2 (2004) 556-565
Non relativistic, spheroidal NS fireball, perfect fluid attractor: T. Csörgő, M. Nagy: Phys.Rev.C 89 (2014) 4, 044901
Non relativistic, ellipsoidal NS fireball, perfect fluid attractor: M. Nagy, T. Csörgő: Phys.Rev.C 94 (2016) 6, 064906

Common property of this family: **Hubble-flow**

Explicit, viscous, **asymptotically perfect solutions with lattice QCD** EoS have been found

Thank you for your attention!

Spheroidally symmetric, explicit solution

One more interesting property…

For fixed attractor and kinematic shear/bulk viscosity:

→ *the kinematic bulk/shear viscosity is a non monotonic function of the initial temperature*

→ ***higher limit for kinematic bulk and shear viscosity:***

$$\frac{\kappa_0 T_{A,0}}{e} > \frac{\zeta_0}{n_0}\left[4H_R + H_Y + \frac{4H_R H_Y}{H_Y - H_R}\ln\left(\frac{H_Y}{H_R}\right)\right] + \frac{4\eta_0}{3n_0}\left[H_R + H_Y - \frac{2H_R H_Y}{H_Y - H_R}\ln\left(\frac{H_Y}{H_R}\right)\right]$$

Lower limit for η was found by Danielewicz and Gyulassy: η/s > ħ/4π

P. Danielewicz, M. Gyulassy: Phys.Rev. D31, 53,1985

Spheroidally symmetric, explicit solution

Non monotonic shear and bulk viscosity as functions of the initial temperature