New type of self-similar solutions of the regular diffusion equation

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Outline

- *Introduction* *various PDEs, basic solutions, self-similar and travelling wave and other Anzätze, our fomer activity*

- *The regular diffusion equation*
- *various self-similar solutions*

- *Summary & Outlook* *possible generalizations eg. time, or space dependent diffusion equations*

The very basics of the idea

Initially: a (nonlinear) partial differential equation (system) for u(x,t) is given

Ansatz: combination from space and time variable x,t to a new variable η(x,t) so u(η) called reduction technics

Result: a (nonlinear) ordinary differential equation (system) u' (η)

\+ with some tricks analytic solutions can be found which depend on some free physical parameter(s), general global properties can be studied like: non-continous solutions, compact supports, oscillations, asymptotic power-law behaviour etc. etc.

A way of interpretation

there are the two basic time-dependent linear PDEs which describe propagation in space and time

each has it's natural Ansatz with physical interpretation

$$u_t = a u_{xx}$$

$$u = t^{-\alpha} \cdot f\left(\frac{x}{t^\beta}\right)$$

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

$$u = f(x \pm ct)$$

we may attack any kind of non-linear PDE with self-similar or traveling wave Ansatz ☺ asking how diffusive/dissipative or how wave-like it is

(the connection between the two Ansätze is nontivial !! Sometimes can be seen

How these solutions look like

- Self-similar in 1D Sedov, Zeldovich, Barenblatt

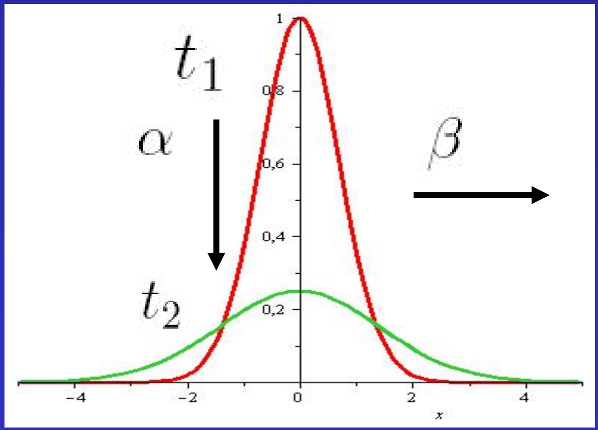
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$ *in Fourier heat-conduction the Gaussian*

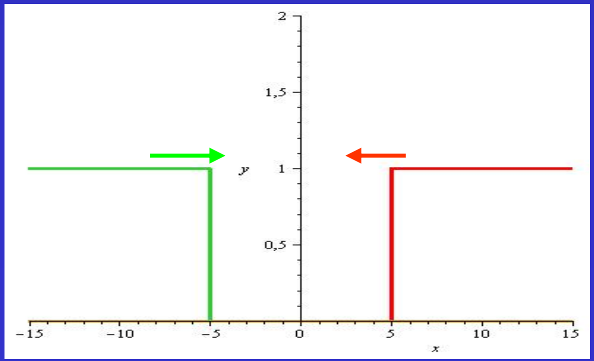


- Traveling waves:

arbitrary wave fronts

u(x,t) ~ g(x-ct), g(x+ct)

c is the speed of the sound



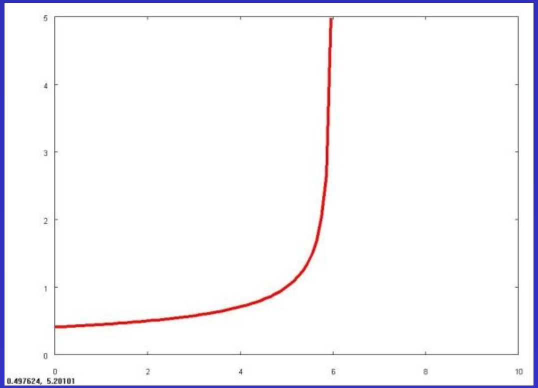
Additional relevant solutions for PDEs I

- generalised self-similar: $u(x,t) = l(t) f(x/g(t))$

– one case: blow-up solution: $u(x,t) = \frac{1}{\sqrt{T-t}} f\left(\frac{x}{\sqrt{T-t}}\right)$

goes to infinity in finite time

other forms are available too



- generalized travelling waves,

exp-function method

$$u(x,t) = U(\eta), \quad \eta = kx + wt,$$

He & Wu 2006

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-\mathrm{d}\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}.$$

Additional relevant solutions for PDEs II

- An interpolation in between „traveling profiles method”:

$$u(x,t) = c(t)\,\psi\left[\frac{x - b(t)}{a(t)}\right]$$

Behamioduce 2008 the unknown functions a(t), b(t), c(t) have to be deteminded

- additional Ansäzte:

additive separable

multiplicative separable

generalised separable eg .

$$u(x,t) = \varphi(t) + \phi(x)$$

$$u(x,t) = \varphi(t)\phi(x)$$

$$u(x,t) = \frac{x + C_1}{at + C_2} + \frac{2ab}{(at + C_1)^2}$$

- classical (on Lie algebra symmetries, we skip) and non – classical symmetry reductions

$$u(x,t) = \alpha(x,t) + \beta(x,t)\,w(z(x,t))$$

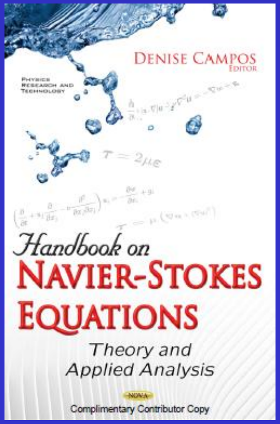
Clarson & Kruskal 1989

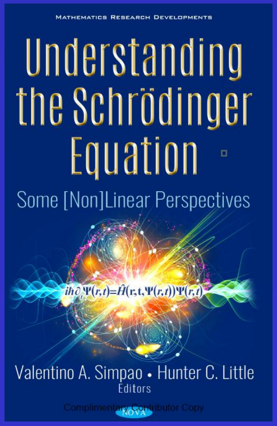
Additional personal remarks

- We apply the self-similar and the traveling-wave Ansätze almost a decade, which can easily be generalized for PDE systems for 2 or 3 dim. ☺

- Published more than 10 papers in transpor theory + 2 book chapters in hydrodynamics, and hydrodynamical formulation of quantum mechanics

- More on www.kfki.hu/~barnai





The regular diffusion/heat conduction equation

$$\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2}$$

Written in one Cartesian coordinate
The diffusion/heat conduction coefficient is a positive $D > 0$ real number, and independent of the concentration/temperature

(the non-linear problem is also under investigated and will be mentioned later)

The self-similar Ansatz

$$C(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} f(\eta)$$

Calculating the derivatives and pluging back to the diffusion equation

$$-\alpha t^{-\alpha-1} f(\eta) - \beta t^{-\alpha-1} \eta \frac{df(\eta)}{d\eta} = D t^{-\alpha-2\beta} \frac{d^2 f(\eta)}{d\eta^2}$$

This object is obtained

This must not depend on time if want to get an ODE so the $\alpha+1 = \alpha+2\beta$ constraint has to be fulfilled which means that α = arb. real number and $\beta = 1/2$

$$-\alpha f - \frac{1}{2}\eta f' = D f''$$

if $\alpha = 1/2$ it is a total derivative and can be integrated once

$$-\frac{1}{2}\eta f + c_1 = D f'$$

c_1 integral constant

The derived solutions

$$-\frac{1}{2}\eta f + c_1 = Df'$$

If $c_1 = 0$ we get the usual Gauss solution

$$f(\eta) = c_2 e^{-\frac{\eta^2}{4D}}$$

$$C(x,t) = c_2 t^{-1/2} e^{-\frac{x^2}{4Dt}}$$

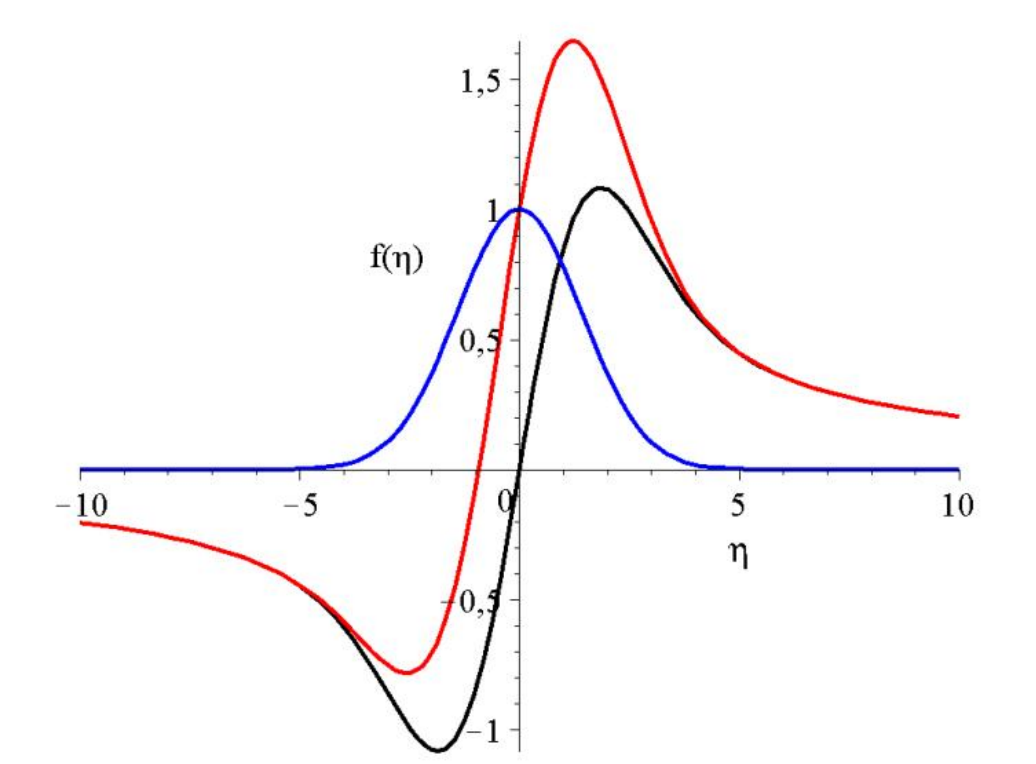
If $c_1 \neq 0$ we get the usual a more complicated solution

$$f(\eta) = \left(\frac{c_1 \sqrt{\pi} \cdot erf\left[\frac{1}{2}\sqrt{-\frac{1}{D}\eta}\right]}{D\sqrt{-\frac{1}{D}}} + c_2 \right) \cdot e^{-\frac{\eta^2}{4D}}$$

Note, that the two complex numbers $\sqrt{-D}$ together give a real result

The derived solutions

How the solutions look like?



for three different initial conditions the black,red and blue curves are for $c_1 = 1, c_2 = 0$, $c_1 = c_2 = 1$ and for $c_1 = 0, c_2 = 1$, respectively.

The general solution

What happens if $\alpha \neq 1/2$ the general case never,examined

$$-\alpha f - \frac{1}{2}\eta f' = D f''$$

The solutions contain the Kummer's M and U functions where α is a real number

$$f(\eta) = \eta \cdot e^{-\frac{\eta^2}{4D}} \left(c_1 M\left[1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right] + c_2 U\left[1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right] \right)$$

Other interesting point, that positive interger α gives

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \left(\tilde{c}_1 H_{2\alpha-1}\left[\frac{\eta}{2\sqrt{D}}\right] + \tilde{c}_2 \cdot {}_1F_1\left[\frac{1-2\alpha}{2}, \frac{1}{2}; \frac{\eta^2}{4D}\right] \right)$$

Hermite polynom and the a confluent hypergeometric function

The general solution

defintion of Kummer's M function via the hypergeometric series:

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \ldots + \frac{(a)_n z^n}{(b)_n n!},$$

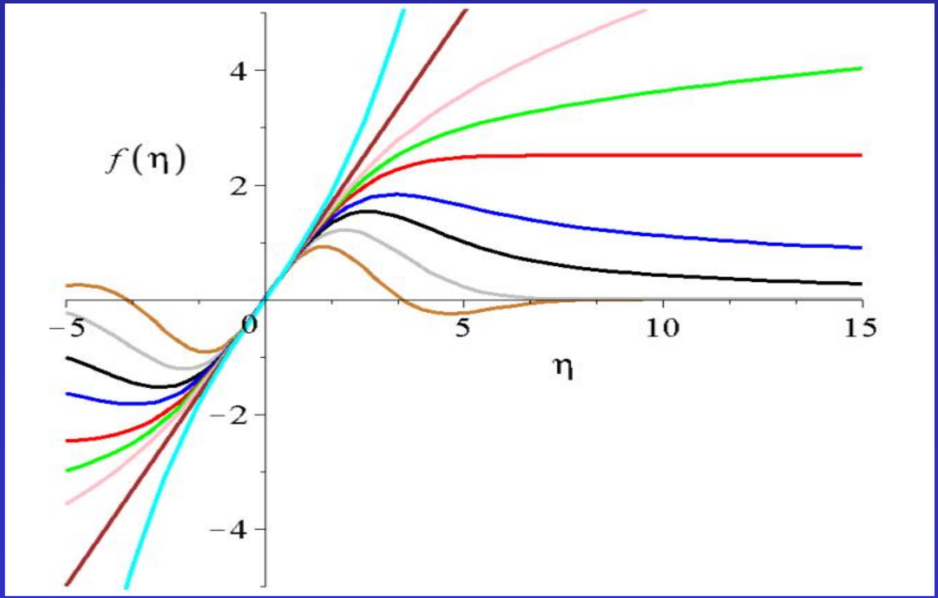
Where: $(a)_n = a(a+1)(a+2)\ldots(a+n-1), (a)_0 = 1$

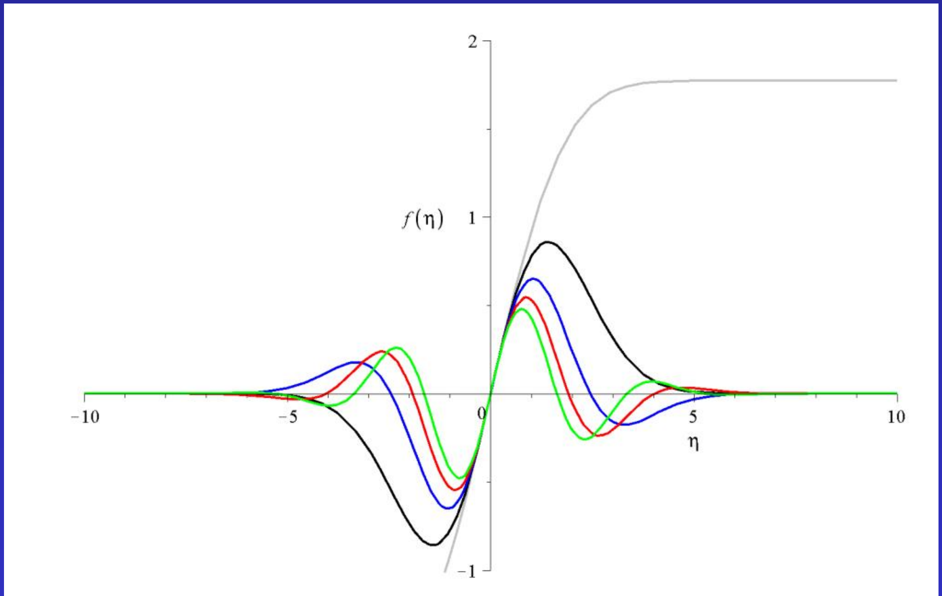
Rising-factorial or Pochammer symbol if „a" is a negative integer it is reduced to a finite series

Kummer's U can be expressed with Kummer's M:

$$U(a,b,z) = \frac{\pi}{sin(\pi b)} \left(\frac{M[a,b,z]}{\Gamma[1+a-b]\Gamma[b]} - z^{1-b} \frac{M[1+a-b, 2-b, z]}{\Gamma[a]\Gamma[2-b]} \right)$$

The general solution





Numerous shape functions $f(\eta)$ Eq. (24) for various αs all are for $\beta = 1/2$ and for $c_1 = 1, c_2 = 0, D = 2$. The gold, gray, black, blue, red, green, pink, brown and cyan curves are for $\alpha = 2, 1, 1/2, 1/4, 0, -1/8, -1/4, -1/2$ and for -1, respectively.

Three different shape functions $f(\eta)$, the gray, black, blue, red and green curves are for $\alpha = 0, 1, 2, 3$ and 4, respectively. Additional parameters κ_1 and D are set to unity.

$\alpha < 0$ explodes

$\alpha = 0$ asymptot

$0 < \alpha < 1$ decays

$1 < \alpha$ decays & oscillates

The form of the solutions

The shape functions for some integer numerical values

$\alpha = 0, 1, 2, 3, 4$

$$f(\eta) = erf\left(\frac{\eta}{2\sqrt{D}}\right),$$

$$f(\eta) = \kappa_0 \cdot \eta \cdot e^{-\frac{\eta^2}{4D}},$$

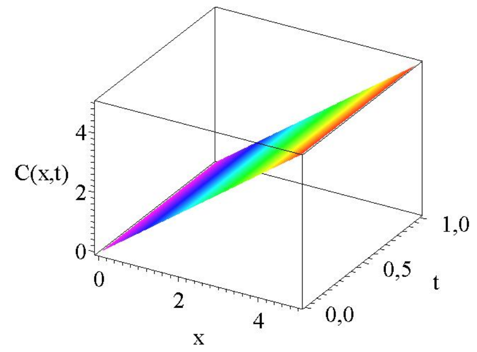
$$f(\eta) = \kappa_0 \cdot \eta \cdot e^{-\frac{\eta^2}{4D}} \left(1 - \frac{1}{6D}\eta^2\right),$$

$$f(\eta) = \kappa_0 \cdot \eta e^{-\frac{\eta^2}{4D}} \cdot \left(1 - \frac{1}{3D}\eta^2 + \frac{1}{60}\frac{1}{D^2}\eta^4\right),$$

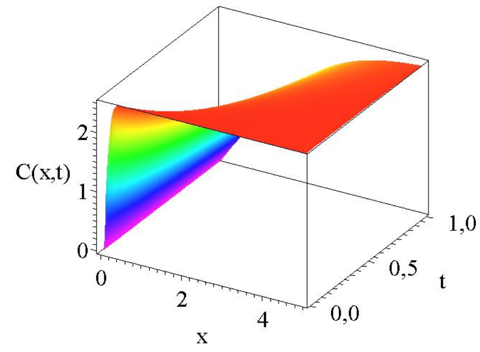
$$f(\eta) = \kappa_0 \cdot \eta e^{-\frac{\eta^2}{4D}} \cdot \left(1 - \frac{1}{2D}\eta^2 + \frac{1}{20}\frac{1}{D^2}\eta^4 - \frac{1}{840}\frac{1}{D^3}\eta^6\right)$$

Our interpretation: the ground and the higher harmonics on the infinite horizon because in finite interval we have:
Gauss(t)*sinus(x) or Gauss(t)*cosinus(x) solutions

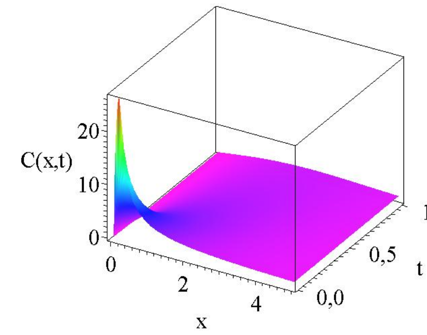
The C(x,t) solutions



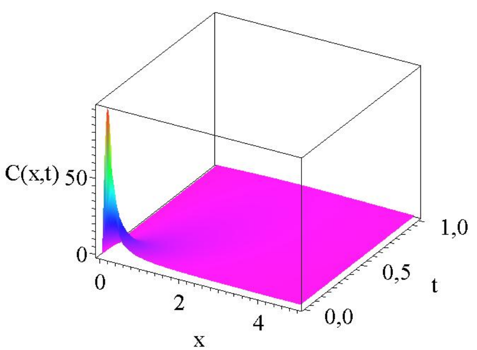
α = −1/2



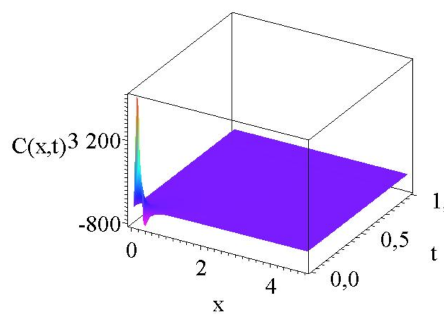
α = 0



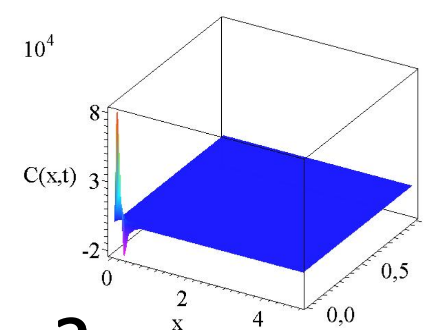
α = 1/2



α = 3/4



α = 3/2



α = 2

The question of the initial condition

The presented results just fulfills one special initial condition
e.g. for fixed $t_0 = 0$ we have C(0,x)

But how can it be generalized? Answer: With the corresponding Green's func.

Usually it looks like: where $w(x_0)$ is the initial condition

$$C(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} w(x_0) G(x - x_0) dx_0$$

$$G(x - x_0) = exp\left[-\frac{(x - x_0)^2}{4tD}\right]$$

Now it is changed to a more general form of:

$$C(x,t) = \frac{\kappa_n}{t^\alpha} \int_{-\infty}^{+\infty} w(x_0) \cdot \frac{(x-x_0)}{t^{1/2}} \cdot e^{-\frac{(x-x_0)^2}{4Dt}} \left(c_1 M\left[1-\alpha, \tfrac{3}{2}, \tfrac{(x-x_0)^2}{4Dt}\right] + c_2 U\left[1-\alpha, \tfrac{3}{2}, \tfrac{(x-x_0)^2}{4Dt}\right]\right) dx_0$$

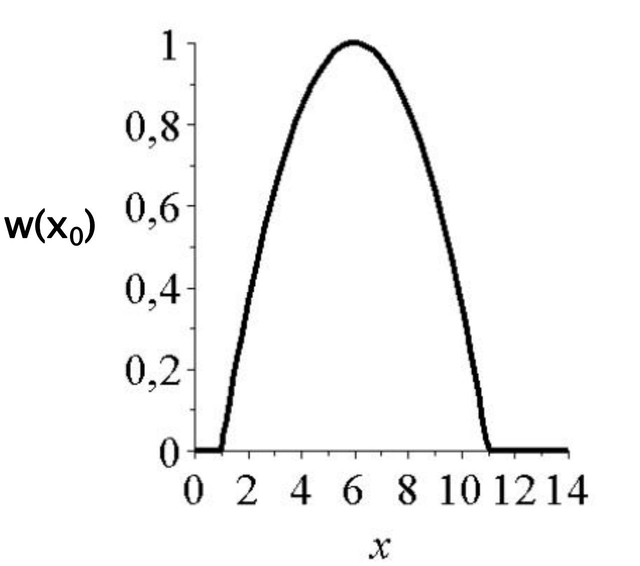
$w(x_0)$ is still the arbitraty initial condition

The question of the initial condition

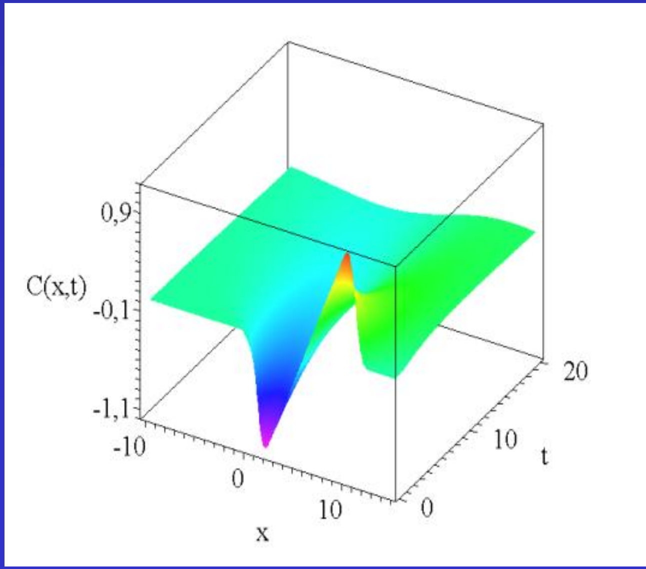
A real example as an initial condition:

$$w(x_0) = \frac{\text{Heaviside}(x_0 - 1) \cdot \text{Heaviside}(11 - x_0) \cdot (-[x_0 - 6]^2 + 25)}{25}$$

How it looks like:



how the solution propagates for $\alpha = 1$:



for completeness the final analytic formula:

$$C(x,t) = \frac{1}{t} \cdot \left(2tx\sqrt{\pi} \cdot erf\left[\frac{5+x}{2\sqrt{t}}\right] + 4te^{-\frac{(5+x)^2}{4t}} + -2tx\sqrt{\pi} \cdot erf\left[\frac{-5+x}{2\sqrt{t}}\right] - 4te^{-\frac{(-5+x)^2}{4t}} \right).$$

Other applied Ansätze

$$C(x,t) = f(t) + g(x)$$

$$C(x,t) = h(x) \cdot (t)$$

Additive separable and multiplicative separable

$$C(x,t) = f(x \mp ct)$$

traveling wave

$$C(x,t) = a(t) \cdot h\left(\frac{x - b(t)}{c(t)}\right) = a(t) \cdot h(\omega)$$

Traveling profile Ansatz interpolates between self-similar and traveling wave

$$C(x,t) = t^{-\alpha} f\left(\frac{x-ct}{t^{\beta}}\right)$$

It is also a kind of interpolating Ansatz

$$C(x,t) = a(t) \cdot h\left(\frac{x}{b(t)}\right) = a(t) \cdot h(\omega)$$

The „generalized self-similar Ansatz”

$$C(x,t) = at^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) + bt^{-\alpha} f\left(\frac{x}{t^{\beta}}\right)^2 = at^{-\alpha} f(\eta) + bt^{-\alpha} f(\eta)^2$$

Generalization with a finite series, an interesting idea

$$C(x,t) = \beta(x,t) \cdot W(z[x,t])$$

Clarkson-Kruskál 1989 a non-classical symmetry, also a kind of generalization

The obtained solutions

$$C(x,t) = f(t) + g(x)$$

$$C(x,t) = h(x) \cdot (t)$$

$$C(x,t) = f(x \mp ct)$$

just give trivial solution:

$$\left\{ t + \frac{Dx^2}{2}, \quad exp(t \mp \sqrt{D}x), \quad exp(-t) \cdot (cos[\sqrt{D}x] + sin[\sqrt{D}x]) \right\}$$

$$C(x,t) = a(t) \cdot h\left(\frac{x - b(t)}{c(t)}\right) = a(t) \cdot h(\omega)$$

Gives the Kummer's function with slightly different arguments

$$C(x,t) = t^{-\alpha} f\left(\frac{x - ct}{t^{\beta}}\right)$$

This does not work, gives contradiction

$$C(x,t) = a(t) \cdot h\left(\frac{x}{b(t)}\right) = a(t) \cdot h(\omega)$$

gives back the Gaussian and the Error function

$$C(x,t) = \beta(x,t) \cdot W(z[x,t])$$

if a solution is known it can generate new ones

$$C(x,t) = at^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) + bt^{-\alpha} f\left(\frac{x}{t^{\beta}}\right)^2 = at^{-\alpha} f(\eta) + bt^{-\alpha} f(\eta)^2$$

works fine, gives a non-linear ODE with the same exponents

α = arbitrary real number, $\beta = 1/2$.

The obtained solutions

The derived ODE reads:

$$-a\alpha f - \frac{a}{2}\eta f' - b(\alpha f^2 - \eta f f') = D(af'' + 2b[f'^2 + ff'']),$$

Has a general solutions for arbitrary a,b,α but a bit complicated:

$$f(\eta) = -\tfrac{1}{2}\Big(2aM_{1-\alpha}U_{-\alpha} + aM_{-\alpha}U_{1-\alpha} + 2a\alpha M_{-\alpha}U_{1-\alpha} \pm \big[4a^2 M_{1-\alpha}U_{-\alpha}^2 + M_{-\alpha}M_{1-\alpha}U_{-\alpha}U_{1-\alpha}\{4a^2 + 8a^2\alpha\} + M_{-\alpha}U_{1-\alpha}\{a^2 + 4a^2\alpha\} + 4a^2\alpha^2 M_{-\alpha}U_{1-\alpha} + 8c_2 b\alpha M_{-\alpha}U_{1-\alpha}^2 - 8c_1 b\alpha M_{-\alpha}M_{1-\alpha}U_{1-\alpha} + 8c_2 bM_{1-\alpha}U_{-\alpha}U_{1-\alpha} - 8c_1 bM_{1-\alpha}U_{-\alpha} + 4c_2 bM_{-\alpha}U_{1-\alpha} - 4bc_1 M_{-\alpha}M_{1-\alpha}U_{1-\alpha}\big]^{\frac{1}{2}}\Big) / (b[2\alpha M_{-\alpha}U_{1-\alpha} + 2M_{1-\alpha}U_{-\alpha} + M_{-\alpha}U_{1-\alpha}]).$$

with the notation of:

$$M_{1-\alpha} := M\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right), \quad U_{1-\alpha} := U\left(1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right),$$

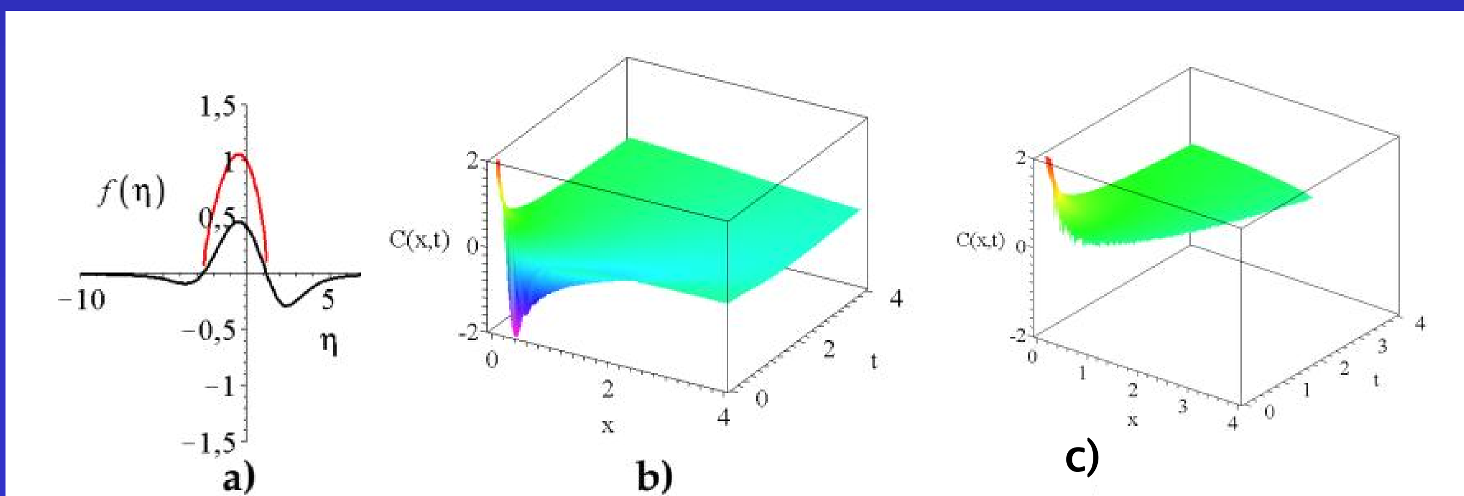
$$M_{-\alpha} := M\left(-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right), \quad U_{-\alpha} := U\left(-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right)$$

The obtained solutions

For α = ½ it looks much simpler

$$f = -\frac{a e^{\frac{\eta^2}{4D}}\sqrt{-\frac{1}{D}} \pm \sqrt{-e^{\frac{\eta^2}{4D}}\left[a^2 e^{\frac{\eta^2}{4D}} - 4bc_1 D\sqrt{-\frac{\pi}{D}} + 4Dbc_2\sqrt{-\frac{\pi}{D}}\,erf\left\{\frac{1}{2}\sqrt{\frac{-1}{D}}\,\eta\right\}\right]/D}}{2be^{\frac{\eta^2}{4D}}\sqrt{-\frac{1}{D}}},$$

Has a very delicate property:



Considering just the second order term a= 0, we have a solution with compact support [red line in a) and c)]

The obtained solutions

These are published already:

L. Mátyás and I.F. Barna
"General Self-Similar Solutions of Diffusion Equation and Related Constructions"
Romanian Journal of Physics 67, (2022) 101
https://arxiv.org/abs/2104.09128

I.F. Barna and L. Mátyás
"Advanced analytic self-similar solutions of regular and irregular diffusion equations"
MDPI Mathematics 10, (2022) 3281
http://arxiv.org/abs/2204.04895

Connection with many numerical method, e.g.

Á. Nagy, I. Omle, H. Kareem, E. Kovács, I.F. Barna and G. Bognár
"Stable, explicit, leapfrog-hopscotch algorithms for the diffusion equation"
Computation 9, (2021) 92

Summary & Outlook

Defined the self-similar Ansatz and solved the regular diffusion equation, found new class of solutions

Tried other Ansätze, most of them do not give new results, but be can even get a solution with a compact support

Work is in progress to derive solutions for diffusion equations where the diffusion coefficient D depends on time D(t), space D(x) or even on concentration D(C) (in many cases the solutions are Kummer's of Whitakker functions)
It we add some source terms we have the reaction-diffusion equation which is also interesting

Thank you for

your attention!

Questions, Remarks, Comments?...