Shakhov-like extension of the RTA in relativistic kinetic theory

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Zimányi School 2022

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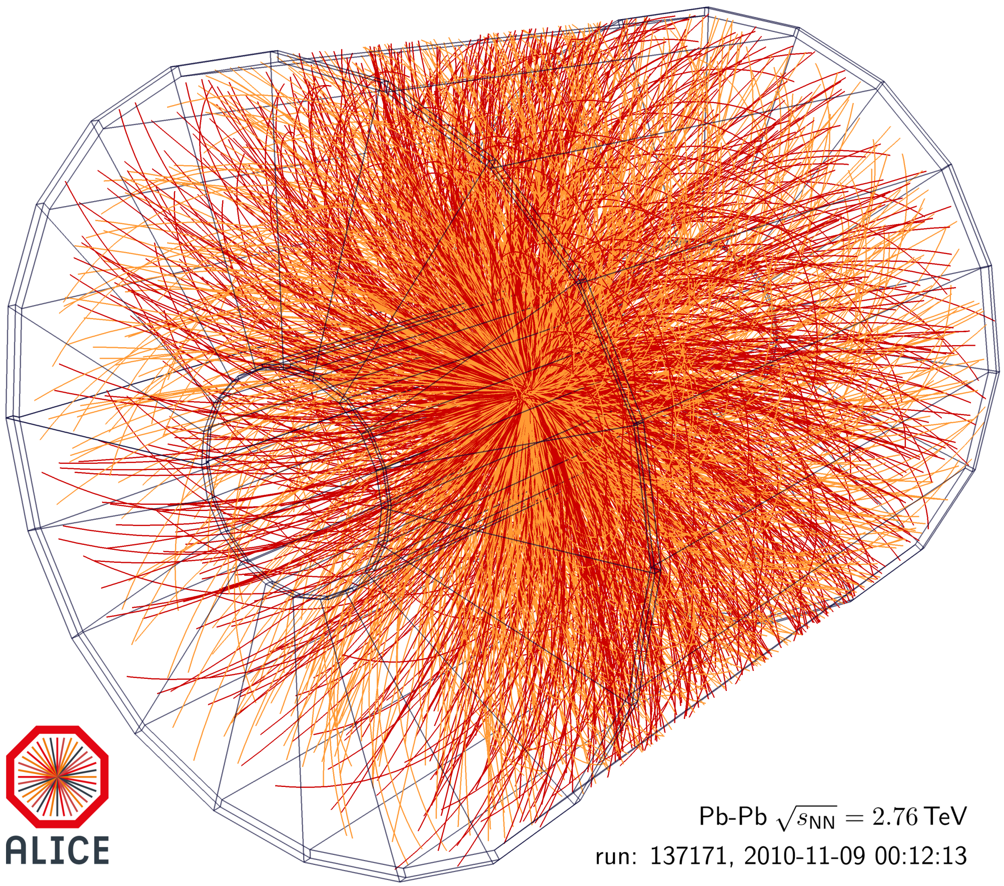
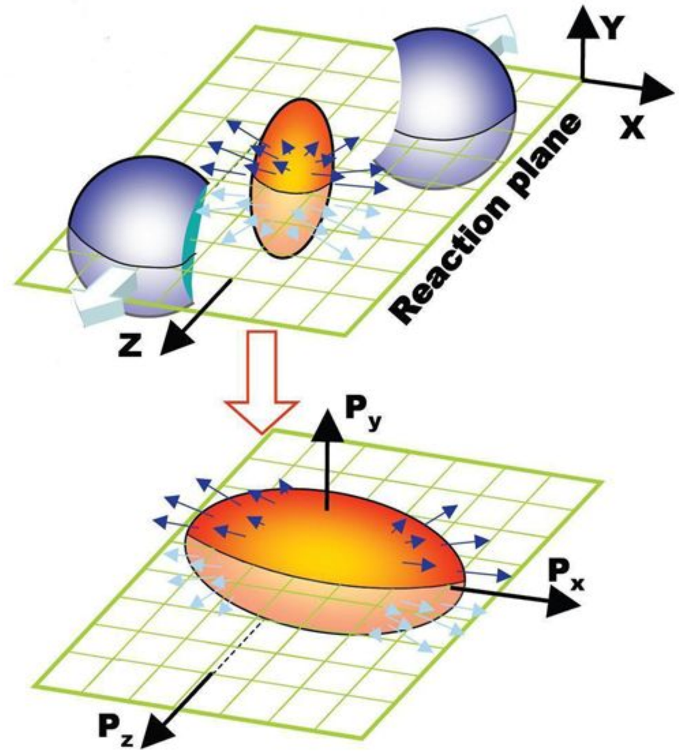
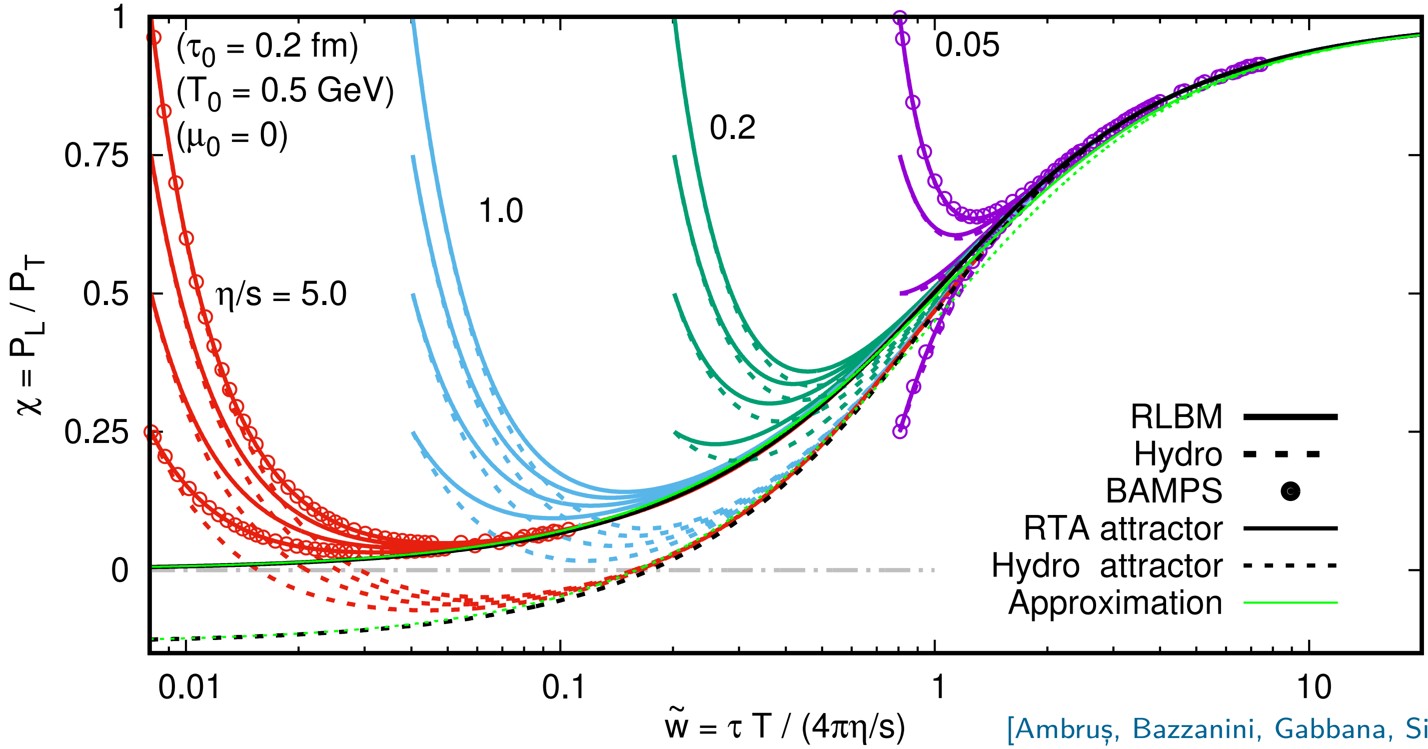




Hiroshi Masui (2008)

- Relativistic fluid dynamics is indispensible when studying the dynamics of the QGP fireball produced in HICs.
- Realistic models account for the QCD equation of state; realistic transport coefficients; chiral phase transition (hadronization).

Hydro vs Kinetic theory



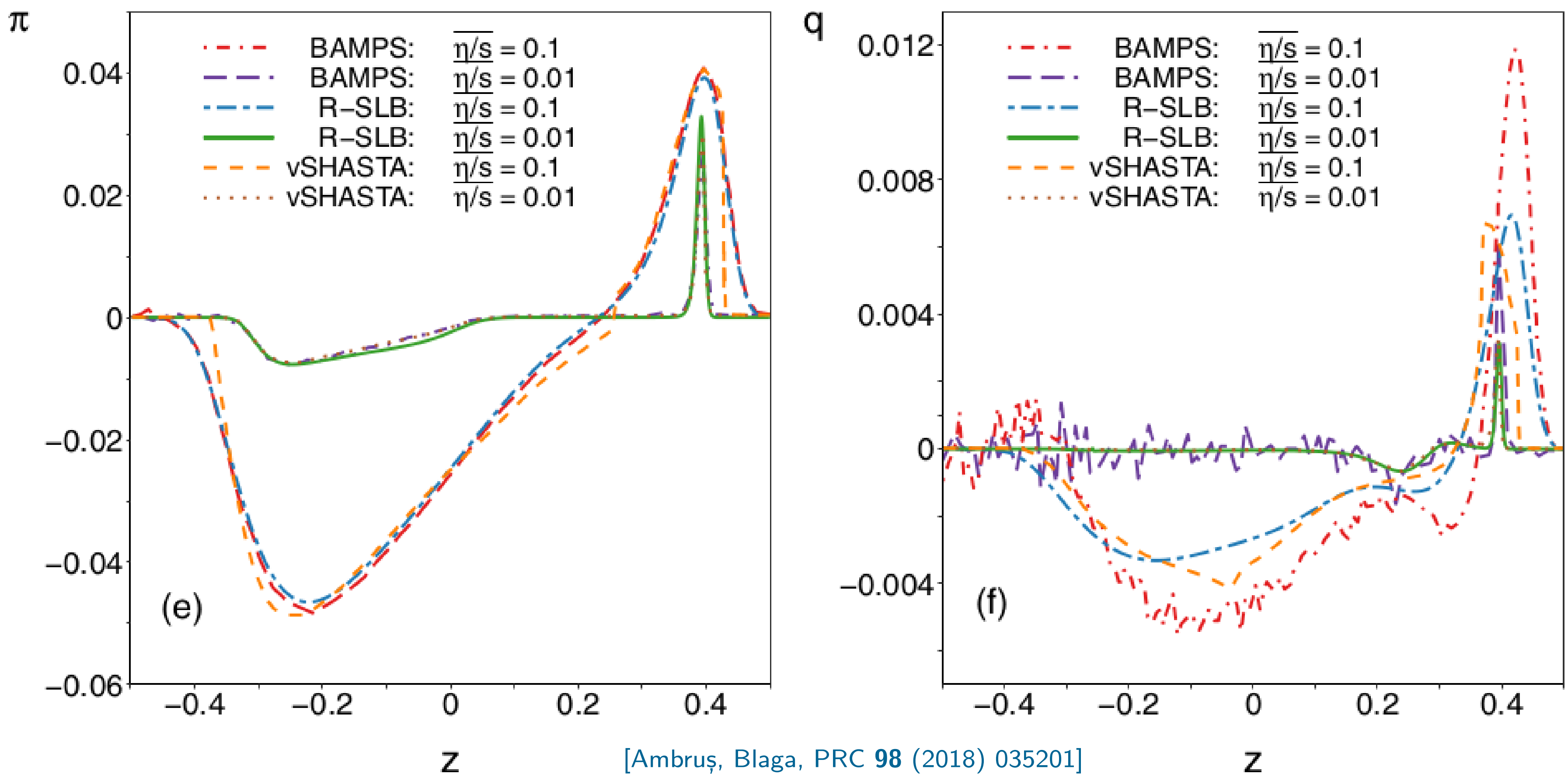
[Ambruș, Bazzanini, Gabbana, Simeoni, Succi, Nature Comput. Sci. **2** (2022) 641]

- Hydro takes the above into account, but it breaks down far from eq.
- Kinetic theory overcomes this limitation, but realistic simulations are expensive due to $C[f]$.

  AMPT: He, Edmonds, Lin, Liu, Molnar, Wang [PLB 753 (2016) 506]
  BAMPS: Greif, Greiner, Schenke, Schlichting, Xu [PRD 96 (2017) 091504]
- RTA: $C[f] \sim -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}) \Rightarrow 1-2$ o.m. faster than, e.g., BAMPS.

  VEA, Busuioc, Fotakis, Gallmeister, Greiner [PRD **104** (2021) 094022]
- τ_R fixes the IR limit of RTA by matching e.g. η to that of $C[f] \Rightarrow$ good agreement with BAMPS.

RTA vs BAMPS



[Ambruș, Blaga, PRC **98** (2018) 035201]

- τ_R governs all dissipative transport $\Rightarrow$ can fix only shear (η) or diffusion (κ), but not both.
- Fixing η via τ_R gives good agreement with BAMPS for $\pi^{\mu\nu}$ but q^μ is not captured correctly.
- **Aim of this work:** Extend RTA with extra parameters allowing multiple transport coefficients to be controlled independently.

BGK model

- In non-relativistic kinetic theory, the RTA was proposed by Bhatnagar, Gross and Krook (BGK): [Bhatnagar, Gross, Krook, Phys. Rev. **94** (1954) 511]

$$C_{\rm BGK}[f] = -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \qquad f_{0\mathbf{k}} = \frac{n e^{-\boldsymbol{\xi}^2/2mk_BT}}{(2\pi m k_B T)^{3/2}}, \tag{1}$$

where $\boldsymbol{\xi} = \boldsymbol{p} - m\boldsymbol{u}$ is the peculiar momentum.
- Applying the Chapman-Enskog expansion gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} = -\tau_R \left(\frac{\partial}{\partial t} + \frac{\boldsymbol{k}}{m}\cdot\nabla\right) f_{0\mathbf{k}}. \tag{2}$$

- At first order, $\pi_{ij} = T_{ij} - P\delta_{ij}$ and $\boldsymbol{q}$ are

$$\pi_{ij} = \int d^3k \frac{\xi_i \xi_j}{m} \delta f_{\mathbf{k}} \simeq -2\eta\sigma_{ij}, \tag{3a}$$

$$\boldsymbol{q} = \int d^3k \frac{\boldsymbol{\xi}^2}{2m}\frac{\boldsymbol{\xi}}{m}\delta f_{\mathbf{k}} \simeq -\lambda\nabla T, \tag{3b}$$

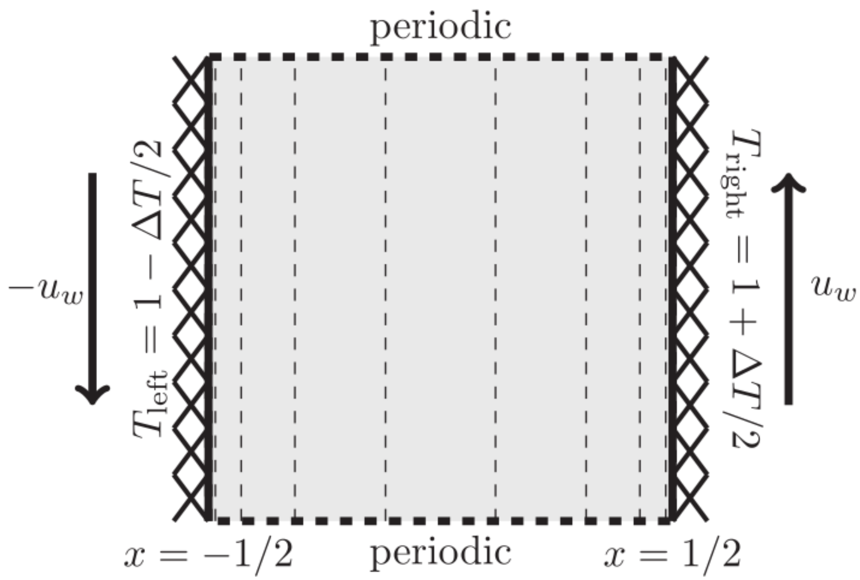
where $\sigma_{ij} = \partial_{(i}u_{j)} - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij}$ is the shear tensor, while

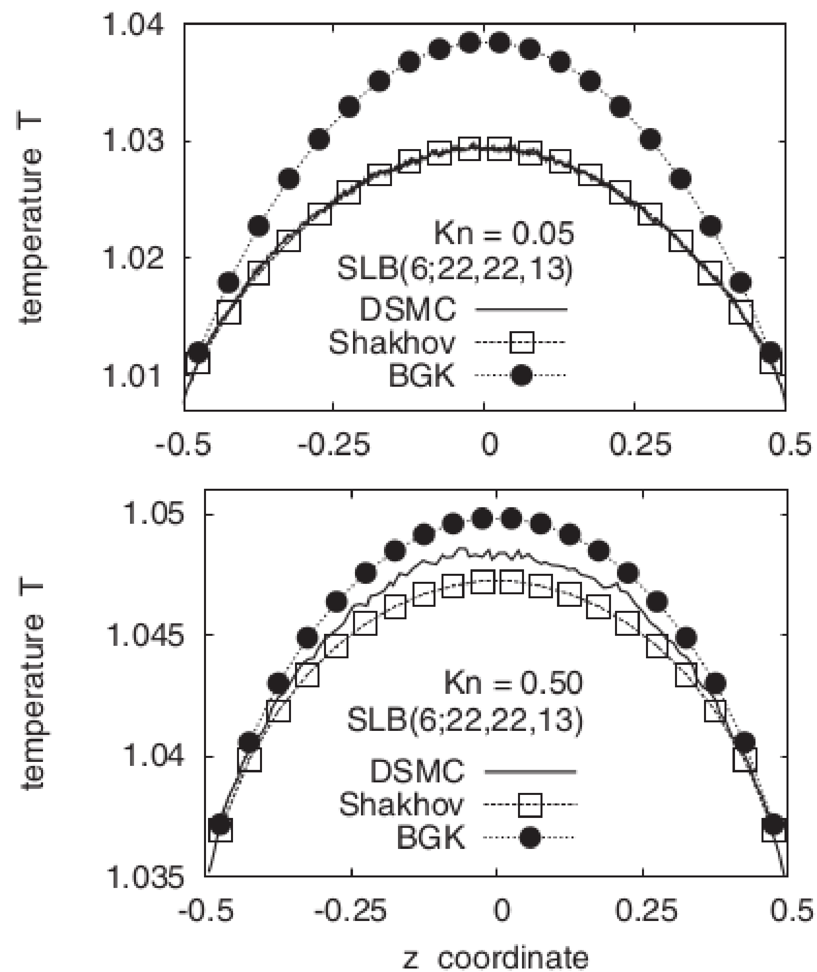
$$\eta = \tau_R P, \qquad \lambda = c_p \tau_R P, \tag{4}$$

where $c_p = 5k_B/2m \equiv$ specific heat at constant P of the monatomic ideal gas.

Shakhov model

[Shakhov, Fluid Dyn. **3** (1968) 112]





[Ambruș, Sofonea, PRE **86** (2012) 016708]

- In BGK, $\mathrm{Pr} = \dfrac{c_p \eta}{\lambda} = 1$.
- Hard-sphere ideal gas: $\mathrm{Pr} = \frac{2}{3}$.
- The Shakhov model employs $C_{\mathrm{S}}[f] = -\dfrac{1}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}})$ with

$$f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}}(1 + \mathbb{S}_{\mathbf{k}}), \quad \mathbb{S}_{\mathbf{k}} = \frac{1 - \mathrm{Pr}}{P k_B T}\left(\frac{\boldsymbol{\xi}^2}{5 m k_B T} - 1\right) \boldsymbol{q} \cdot \boldsymbol{\xi}. \qquad (5)$$

- Comparison with DSMC validates Shakhov for small Kn.

Anderson-Witting model

- The Anderson & Witting RTA reads [Anderson, Witting, Physica **74** (1974) 466]

$$k^\mu \partial_\mu f_{\mathbf{k}} = C_{\rm AW}[f], \quad C_{\rm AW}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \tag{6}$$

  where $E_{\mathbf{k}} = k^\mu u_\mu$ and $f_{0\mathbf{k}}$ is the equilibrium distribution.
- The macroscopic quantities N^μ and $T^{\mu\nu}$ are obtained from $f_{\mathbf{k}}$ via

$$N^\mu = \int dK\, k^\mu\, f_{\mathbf{k}}, \quad T^{\mu\nu} = \int dK\, k^\mu k^\nu f_{\mathbf{k}}, \tag{7}$$

  with $dK = g\, d^3k/[k_0(2\pi)^3]$ being the Lorentz-covariant integration measure and g is the degeneracy factor.
- Imposing $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ requires Landau matching:

$$n = n_0, \quad e = e_0, \quad T^{\mu\nu} u_\nu = e u^\mu, \tag{8}$$

- In equilibrium, we have

$$N_0^\mu = n u^\mu, \quad T_0^{\mu\nu} = e u^\mu u^\nu - P \Delta^{\mu\nu}, \tag{9}$$

  where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.

Chapman-Enskog expansion

- We are now interested to obtain constitutive relations for the non-equilibrium quantities

$$N^\mu - N_0^\mu = V^\mu, \quad T^{\mu\nu} - T_0^{\mu\nu} = -\Pi\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{10}$$

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}}, \tag{11}$$

such that

$$\Pi = -\zeta_R \theta, \quad V^\mu = \kappa_R \nabla^\mu \alpha, \quad \pi^{\mu\nu} = 2\eta_R \sigma^{\mu\nu}. \tag{12}$$

- ζ_R, κ_R and η_R are given by

$$\zeta_R = \frac{m^2}{3}\tau_R \alpha_0^{(0)}, \quad \kappa_R = \tau_R \alpha_0^{(1)}, \quad \eta_R = \tau_R \alpha_0^{(2)}. \tag{13}$$

where $\alpha_0^{(\ell)}$ are τ_R-independent thermodynamic functions.

Shakhov-like extension

[Ambruș, Molnár, under review]

- ζ, η and κ are governed by the same parameter, τ_R.
- We consider a Shakhov-like extension:

$$C_{\rm S}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{{\rm S}\mathbf{k}}), \tag{14}$$

where $f_{{\rm S}\mathbf{k}} \to f_{0\mathbf{k}}$ as $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}} \to 0$.

- The cons. eqs. $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ imply:

$$u_\mu N^\mu = u_\mu N^\mu_{\rm S}, \quad u_\nu T^{\mu\nu} = u_\nu T^{\mu\nu}_{\rm S}, \tag{15}$$

which allows for plenty of degrees of freedom (δn, δe, W^μ, etc).

- For simplicity, we stick to the Landau matching conditions:
$\delta n = \delta e = 0$, $T^{\mu\nu} u_\nu = e u^\mu$.

Shakohv-like extension

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu f_{0\mathbf{k}}, \tag{16}$$

  leading to

$$\Pi - \Pi_{\mathrm{S}} = -\zeta_R \theta, \quad V^\mu - V^\mu_{\mathrm{S}} = \kappa_R \nabla^\mu \alpha, \quad \pi^{\mu\nu} - \pi^{\mu\nu}_{\mathrm{S}} = 2\eta_R \sigma^{\mu\nu}. \tag{17}$$

- We seek to replace ζ_{R} etc by independent transport coefficients:

$$\begin{aligned} \Pi &\simeq -\zeta_{\mathrm{S}}\theta, & V^\mu &\simeq \kappa_{\mathrm{S}} \nabla^\mu \alpha, & \pi^{\mu\nu} &\simeq 2\eta_{\mathrm{S}} \sigma^{\mu\nu}, \\ \zeta_{\mathrm{S}} &= \frac{\tau_\Pi}{\tau_R} \zeta_R, & \kappa_{\mathrm{S}} &= \frac{\tau_V}{\tau_R} \kappa_R, & \zeta_{\mathrm{S}} &= \frac{\tau_\pi}{\tau_R} \eta_R. \end{aligned} \tag{18}$$

- Eq. (18) can be obtained from Eq. (17) when

$$\Pi_{\mathrm{S}} = \Pi \left(1 - \frac{\tau_\Pi}{\tau_R}\right), \quad V^\mu_{\mathrm{S}} = V^\mu \left(1 - \frac{\tau_V}{\tau_R}\right), \\ \pi^{\mu\nu}_{\mathrm{S}} = \pi^{\mu\nu} \left(1 - \frac{\tau_\pi}{\tau_R}\right). \tag{19}$$

Minimal $\delta f_{\mathrm{S}\mathbf{k}}$

- Writing $f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathrm{S}\mathbf{k}}$, we require:

$$\begin{pmatrix}\rho_{\mathrm{S},0}\\ \rho_{\mathrm{S},1}\\ \rho_{\mathrm{S},2}\end{pmatrix} = \int dK \begin{pmatrix}1\\ E_{\mathbf{k}}\\ E_{\mathbf{k}}^2\end{pmatrix} \delta f_{\mathrm{S}\mathbf{k}} = \begin{pmatrix}-3\Pi_{\mathrm{S}}/m^2\\ 0\\ 0\end{pmatrix},$$
$$\begin{pmatrix}\rho^{\mu}_{\mathrm{S},0}\\ \rho^{\mu}_{\mathrm{S},1}\end{pmatrix} = \int dK \begin{pmatrix}1\\ E_{\mathbf{k}}\end{pmatrix} k^{\langle\mu\rangle} \delta f_{\mathrm{S}\mathbf{k}} = \begin{pmatrix}V^{\mu}_{\mathrm{S}}\\ 0\end{pmatrix},$$
$$\rho^{\mu\nu}_{\mathrm{S},0} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta f_{\mathbf{k}} = \pi^{\mu\nu}_{\mathrm{S}}. \qquad (20)$$

- Thus, $\delta f_{\mathbf{k}} = f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathbb{S}_{\mathbf{k}}$ with can be written as[1]

$$\mathbb{S}_{\mathbf{k}} = -\frac{3\Pi}{m^2}\left(1 - \frac{\tau_R}{\tau_\Pi}\right)\mathcal{H}^{(0)}_{\mathbf{k}0} + k_\mu V^\mu \left(1 - \frac{\tau_R}{\tau_V}\right)\mathcal{H}^{(1)}_{\mathbf{k}0} + k_\mu k_\nu \pi^{\mu\nu}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\mathcal{H}^{(2)}_{\mathbf{k}0}, \qquad (21)$$

where the functions $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ are identical to those used in constructing $\delta f_{\mathbf{k}}$ in the 14-moment approximation.

[DNMR, PRD **85** (2012) 114047]

[1] $\tilde{f} = 1 - af$ and $a = 0$, 1 and -1 for classical, F-D and B-E statistics, respectively.

First-order model

- Specifically, $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ must satisfy:

$$\int dK\, f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_{\mathbf{k}} \\ E_{\mathbf{k}}^2 \end{pmatrix} \mathcal{H}^{(0)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$\frac{1}{3}\int dK\, f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \begin{pmatrix} 1 \\ E_{\mathbf{k}} \end{pmatrix} (\Delta^{\alpha\beta}k_\alpha k_\beta)\mathcal{H}^{(1)}_{\mathbf{k}0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\frac{2}{15}\int dK\, f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}(\Delta^{\alpha\beta}k_\alpha k_\beta)^2\mathcal{H}^{(2)}_{\mathbf{k}0} = 1. \tag{22}$$

- The lowest-order polynomials satisfying these relations are

$$\mathcal{H}^{(0)}_{\mathbf{k}0} = \frac{G_{33} - G_{23}E_{\mathbf{k}} + G_{22}E_{\mathbf{k}}^2}{J_{00}G_{33} - J_{10}G_{23} + J_{20}G_{22}},$$
$$\mathcal{H}^{(1)}_{\mathbf{k}0} = \frac{J_{31}E_{\mathbf{k}} - J_{41}}{J_{21}J_{41} - J_{31}^2}, \quad \mathcal{H}^{(2)}_{\mathbf{k}0} = \frac{1}{2J_{42}}, \tag{23}$$

where $G_{nm} = J_{n0}J_{m0} - J_{n-1,0}J_{m+1,0}$, while

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!}\int dK\, E_{\mathbf{k}}^{n-2q}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}. \tag{24}$$

Entropy production

- In kinetic theory, the entropy current is given by

$$S^{\mu} = -\int dK\, k^{\mu} \left(f_{\mathbf{k}} \ln f_{\mathbf{k}} + \frac{1}{a} \tilde{f}_{\mathbf{k}} \ln \tilde{f}_{\mathbf{k}} \right) . \tag{25}$$

- In the Shakhov model, $k^{\mu} \partial_{\mu} f = C_{\mathrm{S}}[f]$ and

$$\begin{aligned} \partial_{\mu} S^{\mu} &= -\int dK\, C_{\mathrm{S}}[f] \ln \frac{f_{\mathbf{k}}}{\tilde{f}_{\mathbf{k}}} \\ &= \frac{1}{\tau_R} \int dK\, E_{\mathbf{k}} (\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}}) \ln \frac{f_{\mathbf{k}}}{\tilde{f}_{\mathbf{k}}} . \end{aligned} \tag{26}$$

- Analyzing the entropy production $\partial_{\mu} S^{\mu}$ for generic $f_{\mathbf{k}}$ is difficult, but one can estimate it close to equilibrium.

Entropy production

- When $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}$ and $\mathbb{S}_{\mathbf{k}} = \delta f_{\mathrm{S}\mathbf{k}} / f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}$ are small,

$$\partial_\mu S^\mu \simeq \int dK \, \frac{E_{\mathbf{k}}}{\tau_R} \left[(\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}}) \ln \frac{f_{0\mathbf{k}}}{\tilde{f}_{0\mathbf{k}}} + \delta f_{\mathbf{k}} (\phi_{\mathbf{k}} - \mathbb{S}_{\mathbf{k}}) \right] . \qquad (27)$$

- Using $\ln(f_{0\mathbf{k}} / \tilde{f}_{0\mathbf{k}}) = \alpha - \beta E_{\mathbf{k}}$, the first term vanishes,

$$\int dK \, (f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}})(\alpha E_{\mathbf{k}} - \beta E_{\mathbf{k}}^2) = 0, \qquad (28)$$

...by virtue of $u_\mu (N^\mu - N^\mu_{\mathrm{S}}) = u_\mu u_\nu (T^{\mu\nu} - T^{\mu\nu}_{\mathrm{S}}) = 0$.

- Approximating $\phi_{\mathbf{k}} - \mathbb{S}_{\mathbf{k}} \simeq -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu (\alpha - \beta E_{\mathbf{k}})$ leads to

$$\partial_\mu S^\mu \simeq \frac{\beta}{\zeta_{\mathrm{S}}} \Pi^2 - \frac{1}{\kappa_{\mathrm{S}}} V_\mu V^\mu + \frac{\beta}{2\eta_{\mathrm{S}}} \pi_{\mu\nu} \pi^{\mu\nu} \geq 0. \qquad (29)$$

- Thus, for small deviations from equilibrium, the Shakhov model satisfies the second law of thermodynamics.

Example 1: Bjorken flow

- We first consider the $0+1$-dimensional boost-invariant Bjorken expansion of a classical ideal gas of massive particles ($a=0$) with particle non-conservation ($\alpha=0$).
- In this case, $T^{\mu\nu}=\mathrm{diag}(e,P_T,P_T,\tau^{-2}P_L)$ with

$$P_T = P + \Pi - \frac{\pi_d}{2}, \quad P_L = P + \Pi + \pi_d. \tag{30}$$

- In second-order fluid dynamics, we have:

[Denicol, Florkowski, Ryblewski, Strickland, PRC **90** (2014) 044905]

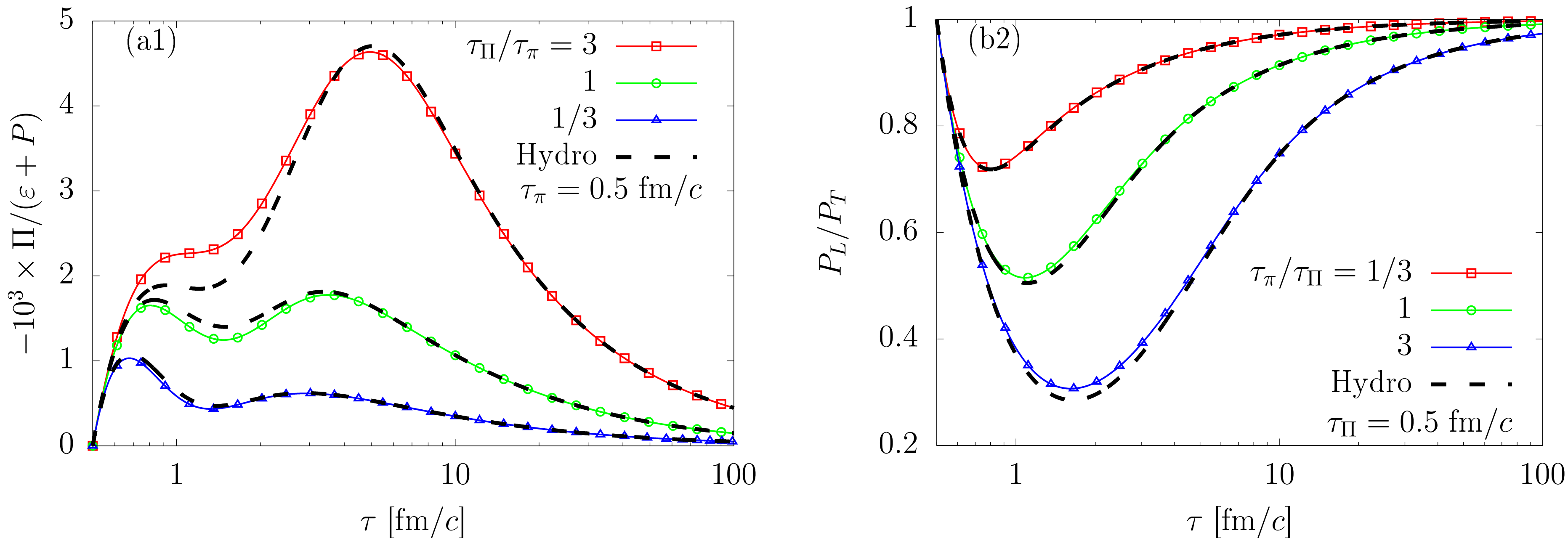
$$\tau\dot{e} + e + P_L = 0, \tag{31a}$$

$$\tau\dot{\Pi} + \left(\frac{\delta_{\Pi\Pi}}{\tau_\Pi} + \frac{\tau}{\tau_\Pi}\right)\Pi + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi_d = -\frac{\zeta}{\tau_\Pi},$$

$$\tau\dot{\pi}_d + \left(\frac{\delta_{\pi\pi}}{\tau_\pi} + \frac{\tau_{\pi\pi}}{3\tau_\pi} + \frac{\tau}{\tau_\pi}\right)\pi_d + \frac{2\lambda_{\pi\Pi}}{3\tau_\pi}\Pi = -\frac{4\eta}{3\tau_\pi}. \tag{31b}$$

- We employ the Shakhov model to control ζ independently from η.

Shakhov model: ζ vs. η



- Setting $\tau_R = \tau_\Pi$ for definiteness, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}} \left[1 + \frac{\beta^2 k_\mu k_\nu \pi^{\mu\nu}}{2(e+P)} \left(1 - \frac{\tau_\Pi}{\tau_\pi} \right) \right]. \quad (32)$$

- Left panel: τ_π is fixed and τ_Π is varied using the Shakhov model.
- Right panel: τ_Π is fixed and τ_π is varied using the Shakhov model.
- $m = 1$ GeV; $\tau_0 = 0.5$ fm; $\beta_0^{-1} = 0.6$ GeV; For $\tau_\pi = 0.5$ fm, $4\pi\eta/s \simeq 3.3$ at $\tau = \tau_0$.

Example 2: Sound waves damping

- We now consider an infinitesimal perturbation propagating in an ultrarelativistic fluid at rest.
- Writing $u^\mu \simeq (1, 0, 0, \delta v)$, $e = e_0 + \delta e$ and $n = n_0 + \delta n$, we have

$$\begin{aligned} \partial_t \delta n + n_0 \partial_z \delta v + \partial_z \delta V &= 0, \\ \partial_t \delta e + (e_0 + P_0) \partial_z \delta v &= 0, \\ (e_0 + P_0) \partial_t \delta v + \partial_z \delta P + \partial_z \delta \pi &= 0, \\ \tau_V \partial_t \delta V + \delta V + \kappa \partial_z \delta \alpha - \ell_{V\pi} \partial_z \delta \pi &= 0, \\ \tau_\pi \partial_t \delta \pi + \delta \pi + \frac{4\eta}{3} \partial_z \delta v + \ell_{\pi V} \partial_z \delta V &= 0, \end{aligned} \tag{33}$$

where $\delta V = V^z$ and $\delta \pi = \pi^{zz}/\gamma^2$.

- In RTA, $\ell_{V\pi} = \ell_{\pi V} = 0$. [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]
- We track the time evolution of the amplitudes

$$\widetilde{\delta V} = \frac{2}{L} \int_0^L dz \, \delta V \, \cos(kz), \quad \widetilde{\delta \pi} = \frac{2}{L} \int_0^L dz \, \delta \pi \, \sin(kz). \tag{34}$$

- We employ the Shakhov model to control κ independently from η.

Sound waves: linear modes

- Inserting $A(t,x) = A_0 + \int_{-\infty}^{\infty} dk \sum_\omega e^{-i(\omega t - kz)} \delta A_\omega(k)$ gives

$$\begin{pmatrix} -3\frac{\omega}{k} & 4P_0 & 0 & 0 & 0 \\ 1 & -\frac{4\omega}{k}P_0 & 1 & 0 & 0 \\ 0 & \frac{4\eta}{3} & -\frac{i}{k} - \frac{\omega}{k}\tau_\pi & 0 & 0 \\ 0 & n_0 & 0 & -\frac{\omega}{k} & 1 \\ -\frac{3\kappa}{P_0} & 0 & -\ell_{V\pi} & \frac{4\kappa}{n_0} & -\frac{i}{k} - \frac{\omega}{k}\tau_V \end{pmatrix} \begin{pmatrix} \delta P_\omega(k) \\ \delta v_\omega(k) \\ \delta \pi_\omega(k) \\ \delta n_\omega(k) \\ \delta V_\omega(k) \end{pmatrix} = 0.$$

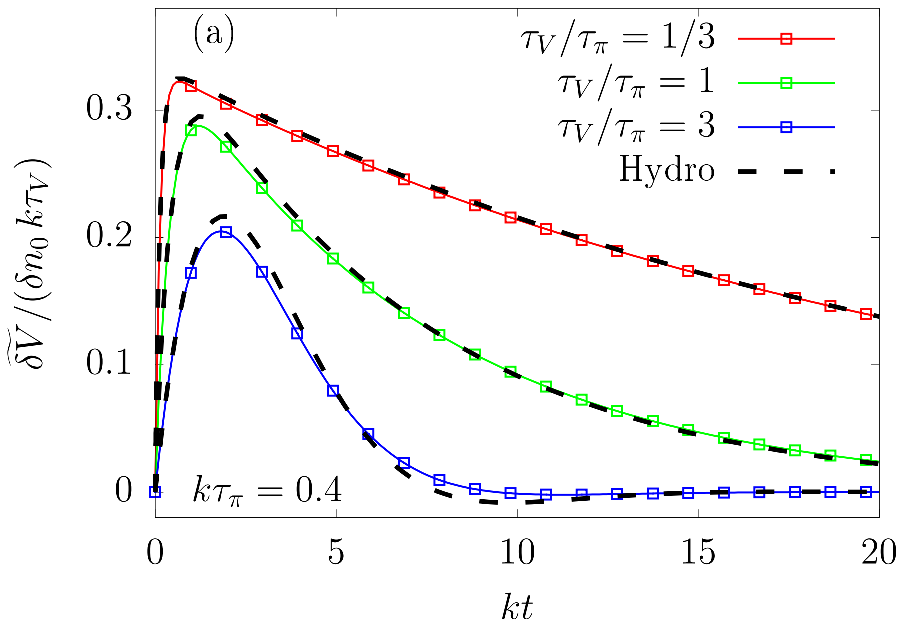
- Thanks to $\ell_{V\pi} = \ell_{\pi V} = 0$, the shear and diffusion sectors decouple:

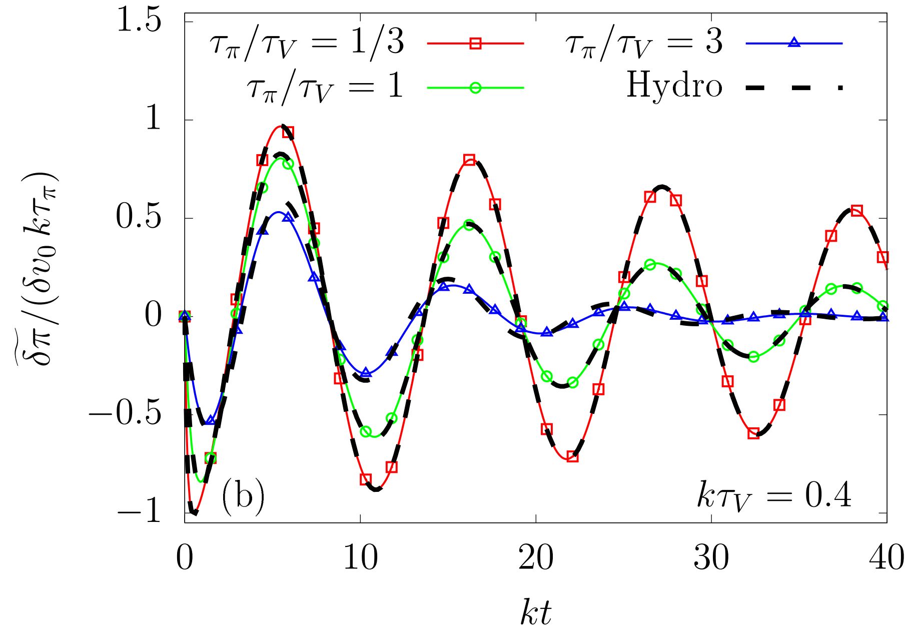
$$(k^2 - 3\omega^2)(1 - i\omega\tau_\pi) - \frac{ik^2\omega}{P_0}\eta = 0, \quad \omega(1 - i\omega\tau_V) + \frac{4ik^2}{n_0}\kappa = 0.$$

- The shear and diffusion modes are:

$$\begin{aligned} &\omega_a^\pm = \pm|k|c_{s;a} - i\xi_a, \qquad \omega_\eta = -i\xi_\eta; \qquad \omega_\kappa^\pm = -i\xi_\kappa^\pm, \\ &c_{s;a} \simeq \frac{1}{\sqrt{3}}, \quad \xi_a \simeq \frac{k^2\eta}{6P_0}, \quad \xi_\eta \simeq \frac{1}{\tau_\pi} - \frac{k^2\eta}{3P_0}, \\ &\xi_\kappa^- \simeq \frac{4k^2\kappa}{n_0}, \qquad \xi_\kappa^+ \simeq \frac{1}{\tau_V} - \frac{4k^2\kappa}{n_0}. \end{aligned} \tag{35}$$

Shakhov model: κ vs. η





- Setting $\tau_R = \tau_\pi$ for definiteness, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}}\left[1 + \frac{k_\mu V^\mu}{P}(\beta E_{\mathbf{k}} - 5)\left(1 - \frac{\tau_\pi}{\tau_V}\right)\right]. \qquad (36)$$

- At initial time, $n(0,z) = n_0 + \delta n_0 \cos(kz)$ and $v(0,z) = \delta v_0 \sin(kz)$.
- The approximate solution is

[Ambruș, PRC **97** (2018) 024914.]

$$\widetilde{\delta V} \simeq \frac{4k\kappa\delta n_0}{\tau_V n_0}\frac{e^{-\xi_\kappa^+ t} - e^{-\xi_\kappa^- t}}{\xi_\kappa^+ - \xi_\kappa^-},$$

$$\widetilde{\delta\pi} \simeq -\frac{4\eta}{3}\delta v_0\left\{e^{-\xi_a t}\left[\cos(kc_s t) - \frac{\xi_a}{kc_s}\sin(kc_s t)\right] - e^{-t/\tau_\pi}\right\}. \qquad (37)$$

Conclusions

- The Shakhov model was generalized for the relativistic Anderson-Witting RTA, allowing ζ, κ and η to be controlled independently.
- Numerical simulations of the Bjorken flow and of sound waves damping confirmed that the model is robust.
- The Shakhov model can be straightforwardly extended to higher orders, allowing also the second-order transport coefficients to be controlled.
- This work was supported through a grant of the Ministry of Research, Innovation and Digitization, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2021-1707, within PNCDI III.

Appendix

Arbitrary Shakhov matrix

- The model can be extended to control 2^{nd}-order transport coeffs..
- Systematic extensions can be obtained by writing in general

$$\mathbb{S}_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \sum_{n=-s_\ell}^{N_\ell} \rho_{\mathrm{S};n}^{\mu_1\cdots\mu_\ell} E_{\mathbf{k}}^{-s_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \widetilde{\mathcal{H}}_{\mathbf{k},n+s_\ell}^{(\ell)}, \tag{38}$$

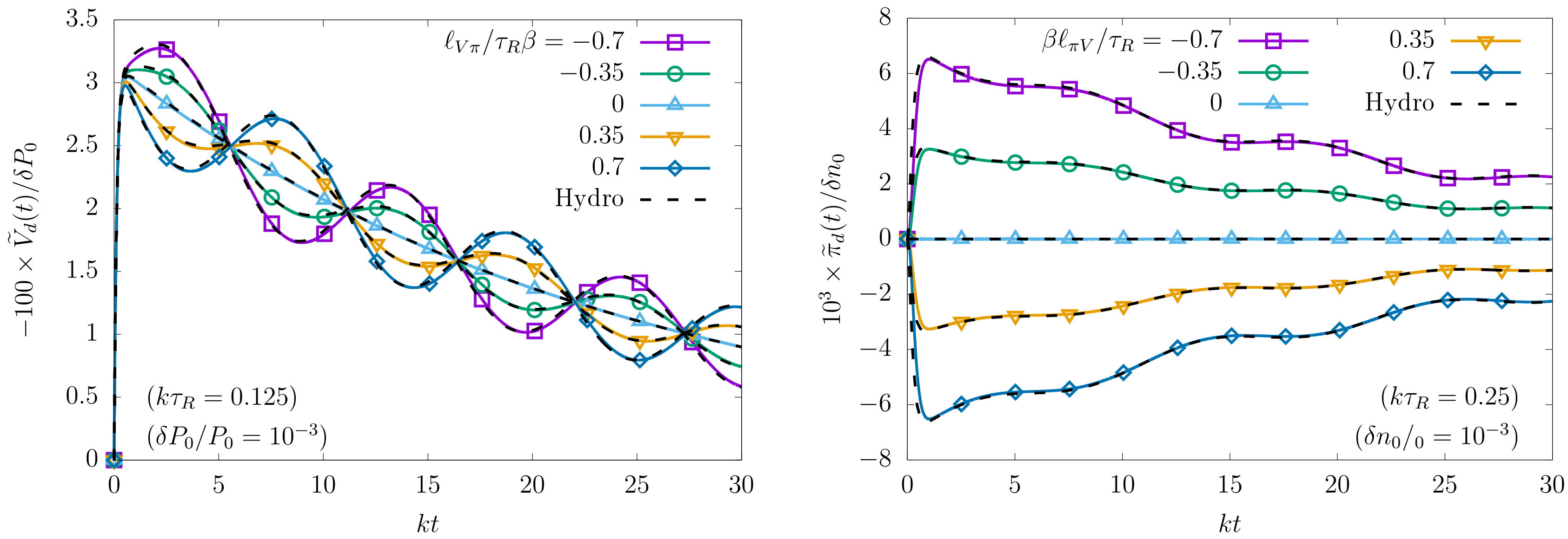
  where $N_\ell \equiv$ expansion order and $s_\ell \equiv$ basis-shift allowing to access negative-order moments.
- The Shakhov irreducible moments are taken as

$$\rho_{\mathrm{S};r}^{\mu_1\cdots\mu_\ell} = \sum_{n=-s_\ell}^{N_\ell} \left(\delta_{rn} - \tau_R \mathcal{A}_{\mathrm{S};rn}^{(\ell)}\right) \rho_n^{\mu_1\cdots\mu_\ell}. \tag{39}$$

  with arbitrary entries $\mathcal{A}_{\mathrm{S};rn}^{(\ell)}$ defined for $-s_\ell \le r, n \le N_\ell$.
- The irreducible moments $C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell}$ of the collision term can be written as

$$C_{\mathrm{S};r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell}, \quad \mathcal{A}_{rn}^{(\ell)} = \begin{pmatrix} \frac{1}{\tau_R}\delta_{rn} & \mathcal{A}_{<;rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{\mathrm{S};rn}^{(\ell)} & 0 \\ 0 & \mathcal{A}_{>;rn}^{(\ell)} & \frac{1}{\tau_R}\delta_{rn} \end{pmatrix}. \tag{40}$$

$(N_1, N_2, s_1, s_2) = (1, 0, 0, 1)$ model



- We consider a simple extension of the tenso matrix to cover the $r = -1$ row.
- Setting $\mathcal{A}_{\rm S}^{(1)} = 1/\tau_V$ and

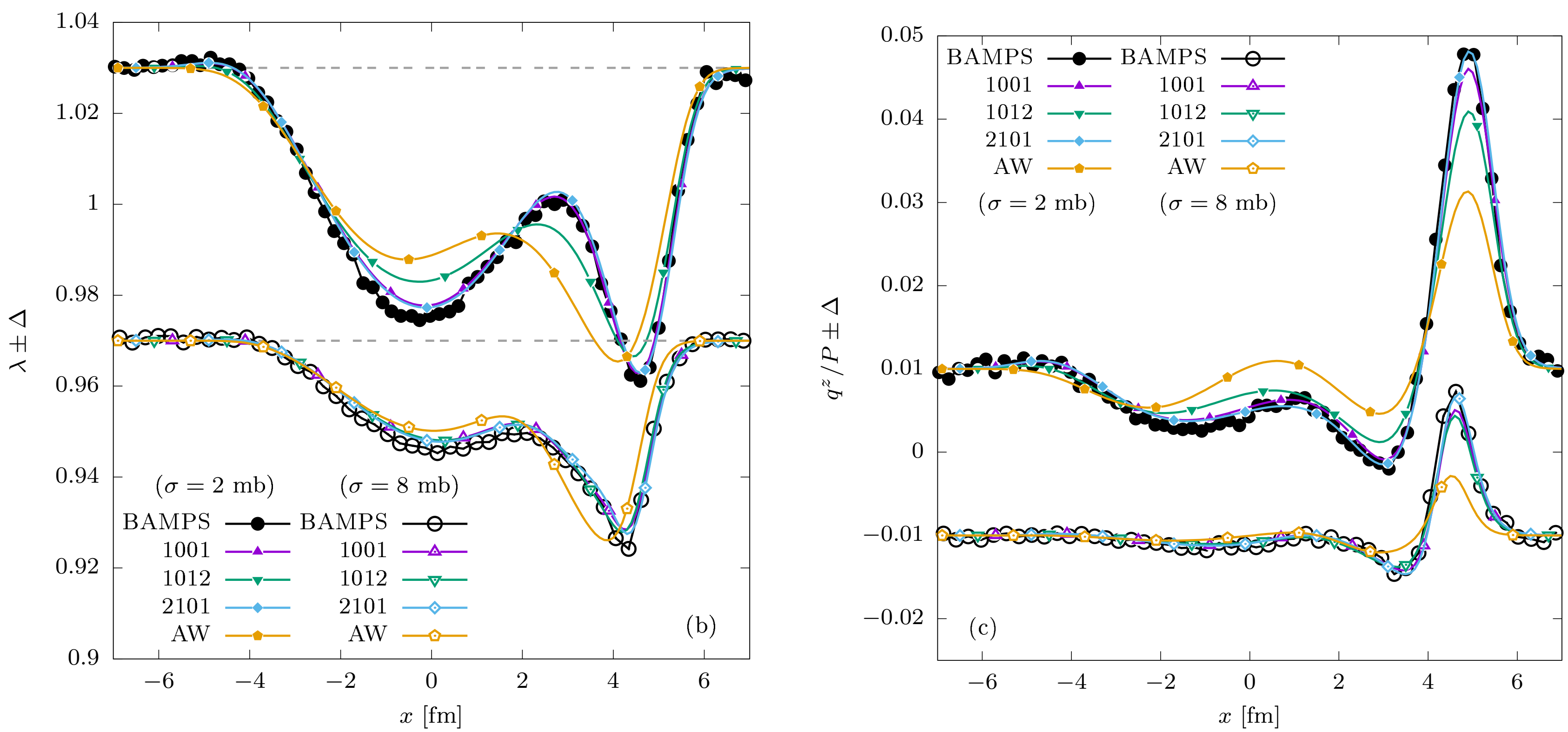
$$\mathcal{A}_{\rm S}^{(2)} = \frac{1}{\tau_\pi H(H + L_{V\pi}L_{\pi V})}\begin{pmatrix} H - L_{\pi V} & \frac{\beta}{4}(HL_{V\pi} + L_{\pi V}) \\ -\frac{4}{\beta}L_{\pi V} & H + L_{\pi V} \end{pmatrix}, \tag{41}$$

allows $\ell_{V\pi}$ and $\ell_{\pi V}$ to be controlled independently via

$$L_{V\pi} = \frac{4}{\beta\tau_V}\ell_{V\pi}, \qquad L_{\pi V} = \frac{5\beta}{8\tau_\pi}\ell_{\pi V}, \qquad H = \frac{5\eta}{4\tau_\pi P}, \tag{42}$$

Comparison to BAMPS

[DNBMXRG, PRD **89** (2014) 074005]



- We can go to higher orders, giving us sufficient free parameters to tune all second-order transport coefficients.
- Setting them to match those for hard spheres gives good agreement to BAMPS.