Perturbative approaches in relativistic kinetic theory and the emergence of first-order hydrodynamics

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Introduction

- Fluid dynamics: effective theory for collective behavior in systems out of equilibrium in the regime where there is a wide separation of scales $L_{\text{micro}} \ll L_{\text{macro}}$

(Credit: Jorge Porto
https://www.waves.com.br/expedicao/arraigal-do-cabo-de-gala/)
Hydrodynamic variables and EoMs

- Basic hydrodynamics EoMs: local conservation of net charge, energy and momentum
  \[ \partial_\mu N^\mu = 0 \quad \partial_\mu T^{\mu\nu} = 0 \]

- Simplest model: ideal fluid (all cells in local equilibrium)
  \[ N^\mu_E = n_0 u^\mu , \]
  \[ T^{\mu\nu}_E = \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu} \]

- Dissipative fluids: Fictitious local equilibrium state
  \[ N^\mu = N^\mu_E + \tilde{N}^\mu = (n_0 + \delta n) u^\mu + \nu^\mu \]
  \[ T^{\mu\nu} = T^{\mu\nu}_E + \tilde{T}^{\mu\nu} = (\varepsilon_0 + \delta \varepsilon) u^\mu u^\nu - (P_0 + \Pi) \Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} \]
Matching conditions

- The separation is not unique – \((\alpha, \beta, u^\mu)\) are not uniquely defined out of equilibrium. 

\[
\begin{align*}
\frac{\mu}{T} & = \frac{1}{T} \\
\end{align*}
\]

- Prescriptions are used to make the definitions – the matching conditions. They usually restrict the conserved tensors.

- The most used in HIC is the Landau one: “the comoving observer should see no heat flux”

Define a fictitious equilibrium state so that we can define \(T\) and \(\mu\) with equilibrium EoS.

\[
\begin{align*}
\delta n &= \delta \varepsilon = 0 \\
\varepsilon &= \varepsilon_0(\mu, T) \\
n &= n_0(\mu, T)
\end{align*}
\]

\[h^\mu = 0\]

Landau and Lifshitz: Fluid Mechanics - Volume 6 (Course of Theoretical Physics), 1987
Matching conditions

- The separation is not unique – $\left( \alpha, \beta, u^\mu \right)$ are not uniquely defined out of equilibrium

\[
\begin{align*}
\mu & \quad \frac{1}{T} \\
\frac{1}{T} & \quad \frac{1}{T}
\end{align*}
\]

- Prescriptions are used to make the definitions – the matching conditions. They usually restrict the conserved tensors

- The astrophysics community uses mostly Eckart matching: “$u^\mu$ is the velocity of [one of the] matter currents”

Define a fictitious equilibrium state so that we can define $T$ and $\mu$ as before

\[
\begin{align*}
\delta n &= \delta \varepsilon = 0 \\
\varepsilon &= \varepsilon_0(\mu, T) \\
n &= n_0(\mu, T)
\end{align*}
\]

\[\nu^\mu = 0\]
Hydrodynamic variables and EoMs

- \( \partial_{\mu} N^\mu = 0 \quad \partial_{\mu} T^{\mu\nu} = 0 \) 5 Eqs for 14 variables (Landau)

- Constitutive relations/further dynamical equations must be derived
  - Navier-Stokes (Landau):
    - \( \Pi = -\zeta \theta \quad \theta \equiv \partial_{\mu} u^\mu \)
    - \( \nu^\mu = \kappa \nabla^\mu \alpha \)
    - \( \nabla^\mu = \Delta^{\mu\nu} \partial_\nu \)
    - \( \pi^{\mu\nu} \equiv 2\eta \sigma^{\mu\nu} \)
    - \( \sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} \partial_\alpha u_\beta \)

  Only space-like derivatives!

  - Linearly acausal and unstable EoMs
  - Possible solutions: IS-like theory; **BDNK theory**

Pichon, Ann. de l'I.H.P. Phys. théor. 2, 21 (1965)
Hiscock, Lindblom PRD 31, 725 (1985)

BDN - PRD, 98(10):104064, (2018); PRD 100(10):104020, (2019); PRX 12 2 021044 (2022)
Idea: Modified constitutive relations now with *time-like* derivatives \[ D = u \cdot \partial \]

General matching:

\[
\Pi = \zeta^{(\alpha)} D\alpha - \zeta^{(\beta)} \frac{D\beta}{\beta} - \zeta^{(\theta)} \theta,
\]

\[
\delta n = \xi^{(\alpha)} D\alpha - \xi^{(\beta)} \frac{D\beta}{\beta} - \xi^{(\theta)} \theta,
\]

\[
\delta \varepsilon = \chi^{(\alpha)} D\alpha - \chi^{(\beta)} \frac{D\beta}{\beta} - \chi^{(\theta)} \theta,
\]

\[
\nu^\mu = \kappa^{(\alpha)} \nabla^\mu \alpha - \kappa^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + Du^\mu \right),
\]

\[
h^\mu = \lambda^{(\alpha)} \nabla^\mu \alpha - \lambda^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + Du^\mu \right),
\]

\[
\pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \quad \nabla^\mu = \Delta^{\mu\nu} \partial_\nu; \quad \sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} \partial_\alpha u_\beta
\]

- EoMs are causal, hyperbolic and have non-negative entropy production even w/ GR. [BDN PRX 12 2 021044]

- Non-linear causality requires \[ \delta \varepsilon \neq 0 \quad h^\mu \neq 0 \]

Unusual definitions of the equilibrium state!
How do the coeffs. depend on \( \mu, T \)?
Microscopic derivation of BDNK from the Boltzmann eqn.

Chain (Navier-Stokes): compatibility conditions imply

\[ f_p = \sum_{i=0}^{\infty} \epsilon^i f_p^{(i)}, \quad Df_p = \sum_{i=0}^{\infty} \epsilon^i (Df_p)^{(i)} \]

"Inhomogeneity must be orthogonal to zero-mode space"

At O(1) \[ \int dP p^\nu \left( p^\mu \partial_\mu f_0 p = f_0 p \hat{L}_p \right) \Rightarrow \text{Euler eqns} \]

To derive BDNK, we propose a novel perturbation method

\[ \epsilon \int dP \mathcal{P}^{(\ell)}(\beta E_p) p_{\{\mu_1 \cdots \mu_\ell\}} p^\mu \partial_\mu f_p = \int dP \mathcal{P}^{(\ell)}(\beta E_p) p_{\{\mu_1 \cdots \mu_\ell\}} C[f_p], \]

Basis, not necessarily orthogonal, conserved quantities explicitly excluded from perturbation procedure, when present.
Microscopic derivation of BDNK from the Boltzmann eqn.

- Using \( P^{(\ell)}_n (\beta E_p) = (\beta E_p)^n, \ n = 0, 1, \cdots \)

\[
\begin{align*}
\int dP p^{\mu} \partial_{\mu} f_p &= 0, \\
\int dP E_p p^{\mu} \partial_{\mu} f_p &= 0, \\
\int dP p_{(\mu)} p^{\alpha} \partial_{\alpha} f_p &= 0,
\end{align*}
\]

\[
\varepsilon \int dP (\beta E_p)^n p^{\mu} \partial_{\mu} f_p = \int dP (\beta E_p)^n C[f_p], \quad n = 2, 3, 4, \cdots,
\]

\[
\varepsilon \int dP (\beta E_p)^n p_{(\mu)} p^{\alpha} \partial_{\alpha} f_p = \int dP (\beta E_p)^n p_{(\mu)} C[f_p], \quad n = 1, 2, 3, \cdots,
\]

\[
\varepsilon \int dP (\beta E_p)^n p_{(\mu_1} \cdots p_{\mu_\ell)} p^{\mu} \partial_{\mu} f_p = \int dP (\beta E_p)^n p_{(\mu_1} \cdots p_{\mu_\ell)} C[f_p],
\]

\( n = 0, 1, 2, \cdots, \) for \( \ell \geq 2, \)

- Another choice \( P^{(\ell)}_m (x) = \frac{x^{m-m_\ell}}{(1 + x)^{N-n_\ell}}, \ m = 1, \cdots N \)

Inspired in
Arnold, Moore, and Yaffe, JHEP 11, 001 (2000);
Arnold, Moore, and Yaffe, JHEP 01, 030 (2003)
Transport coefficients

- First order: two classes of matching conditions

\[
\begin{align*}
(i) & \quad \delta n \equiv 0 \quad \rho_s = \int dPE_p^s \delta f_p \equiv 0 \quad \nu^\mu \equiv 0 \\
\delta \varepsilon &= \chi^{(\alpha)} D \alpha - \chi^{(\beta)} \left( \frac{D \beta}{\beta} - \frac{1}{3} \theta \right), \\
\delta n &= \xi^{(\alpha)} D \alpha - \xi^{(\beta)} \left( \frac{D \beta}{\beta} - \frac{1}{3} \theta \right), \\
\end{align*}
\]

\[
h^{\mu} = \chi^{(\alpha)} \nabla^\mu \alpha - \chi^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + Du^\mu \right), \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},
\]

Massless limit, relaxation time approximation, constant relaxation time

GSR, Denicol, Noronha PRL 127, 042301 (2021)

(i) 
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<thead>
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<tr>
<td>(\chi^{(\alpha)}/(P_0 \tau_R) ) (s = 3)</td>
<td>1.50</td>
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<tr>
<td>(\chi^{(\beta)}/(P_0 \tau_R) ) (s = 3)</td>
<td>7.50</td>
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<td>(\chi^{(\alpha)}/(P_0 \tau_R) ) (s = 4)</td>
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<td>(\chi^{(\beta)}/(P_0 \tau_R) ) (s = 4)</td>
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<td>(\xi^{(\alpha)}/(P_0 \tau_R) ) (s = 4)</td>
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<tr>
<td>(\xi^{(\beta)}/(P_0 \tau_R) ) (s = 4)</td>
<td>-3.00</td>
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For both classes

No obvious physical interpretation for both “Exotic Eckart”
Attractor structure of BDNK theory in Bjorken flow

- $\delta n \equiv 0$, $\delta \varepsilon \neq 0$ – both hydro and early time attractors

(i) Dynamics independent of ‘s’

\[ \rho_s = \left\langle E^s_p \phi_p \right\rangle_0 \equiv 0 \]

Inspired in BDN PRD 100(10):104020, (2019)

Late time attractor

Pullback attractor

Attractor structure of BDNK theory in Bjorken flow

- $\delta n \neq 0$, $\delta \epsilon \equiv 0$ No pullback attractor

(ii) $\rho_s = \left\langle E^s_p \phi_p \right\rangle_0 \equiv 0$

's'-dependent dynamics
Comparison of attractor structures in Bjorken flow

- We compare the evolution under the Boltzmann moment equation, IS, and BDNK in Bjorken flow for the alternative matching conditions

\[ \delta n \equiv 0, \delta \varepsilon \neq 0 \quad \rho_s = \left\langle E_p^s \phi_p \right\rangle_0 \equiv 0 \]

\[ s=3 \]

\[ s=4 \]
Comparison of attractor structures in Bjorken flow

- We compare the evolution under the Boltzmann moment equation, IS, and BDNK in Bjorken flow for the alternative matching conditions

\[ \delta n \neq 0, \delta \varepsilon \equiv 0 \quad \rho_s = \langle E_p^s \phi_p \rangle_0 \equiv 0 \]
Conclusions

- We proposed a novel perturbative procedure to derive BDNK hydrodynamics from Kinetic Theory;
- We analytically compute the attractors of BDNK theory and compared it with Boltzmann moments and IS EoMs in Bjorken flow;

- We intend to generalize to other backgrounds and for momentum-dependent relaxation times;
- Improve hydro IS-like truncation;
- Application in thermal mass models;

THANK YOU FOR THE ATTENTION!