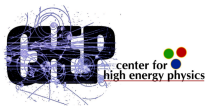


A NEW WAY OF RESUMMING QCD AT A FINITE CHEMICAL POTENTIAL

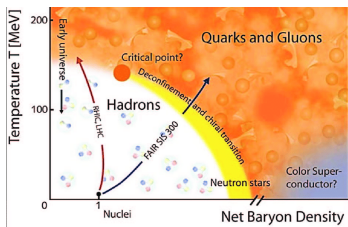
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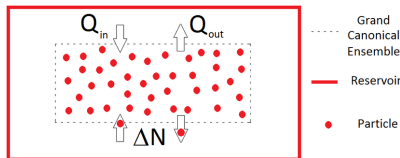


Motivation and Introduction



- The QCD phase diagram still remains to be **conclusively conjectured**
- Still in quest of conclusive evidences like a possible phase transition, **QCD critical point**, phases like **color superconductivity**, quarkyonic phases, extreme QCD (**neutron stars**)
- To answer, QCD **Equation of state** (EoS) is **crucial** to know
- Adopt a **thermodynamic** approach
- Observe the behaviour of thermodynamic observables with changing μ by remaining in a **non-perturbative** regime

Partition function and sign problem



- The ensemble : a **grand canonical ensemble** of quarks (u, d, s) in **thermal equilibrium** with a **reservoir** at temperature T
- Partition function $\mathcal{Z}(\mu, V, T) = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}U \exp(iS_{QCD}[\bar{\Psi}, \Psi, U, \mu, V, T]) \rightarrow \mathcal{Z}(\mu, T) \sim \int \mathcal{D}U e^{-S_g[U, T]} [\det \mathcal{M}(\mu, T, U)]$, with $\mathcal{M} \rightarrow$ fermion matrix
- Complex** $\det \mathcal{M}(\mu)$ **inhibits** Monte-Carlo importance sampling
- With **reweighting** measure at $\mu = 0$, we get **real** measure but observable becomes **complex** and **sign problem** comes due to phaseangle $\theta(\mu)$:

$$\frac{\det \mathcal{M}(\mu)}{\det \mathcal{M}(0)} = \left| \frac{\det \mathcal{M}(\mu)}{\det \mathcal{M}(0)} \right| e^{i\theta(\mu)}$$
- Decreasing** $\langle \cos \theta \rangle$ with **increasing** μ , breakdown near **$\langle \cos \theta \rangle \approx 0$**
- One way-around is **Taylor expansion** around $\mu = 0$

Taylor Expansion: Use of Random volume sources

- With $\hat{\mu} \equiv \mu/T$, the Taylor expansion of **excess pressure** $\Delta P = P(\mu) - P(0)$ and **number density** \mathcal{N} to $\mathcal{O}(\mu^N)$ is given as

$$\frac{\Delta P}{T^4} = \sum_{n=1}^{N/2} C_{2n} \hat{\mu}^{2n}, \quad C_{2n} = \frac{1}{(2n)!} \frac{\partial^{2n}}{\partial \hat{\mu}^{2n}} \left[\frac{\Delta P}{T^4} \right] \bigg|_{\mu=0} \quad (1)$$

$$\frac{\mathcal{N}}{T^3} = \frac{\partial}{\partial \hat{\mu}} \left[\frac{\Delta P}{T^4} \right] = \sum_{n=1}^{N/2} 2n C_{2n} \hat{\mu}^{2n-1} \quad (2)$$

- CP symmetry** of QCD \rightarrow eqn.(1) **even** and eqn.(2) **odd** in μ
- To calculate N th order Taylor coefficient, we need to evaluate terms such as $\langle D_1^{P_1} D_2^{P_2} \cdots D_N^{P_N} \rangle$, where \mathcal{M} is fermion matrix and

$$D_n = \frac{D_n}{n!} = \frac{1}{n!} \frac{\partial^n}{\partial \hat{\mu}^n} \ln \det \mathcal{M}(T, \hat{\mu}) \bigg|_{\mu=0} \quad \text{where} \quad \sum_{k=1}^N \mathbf{k} \cdot \mathbf{P}_k = N \quad (3)$$

- Slow** convergence and **non-monotonic** behaviour of Taylor series for different N and $T \rightarrow$ **need** to do tedious calculations of higher order $C_n \rightarrow$ **motivates** exponential resummation of **lower** order Taylor series



Exponential Resummation

- The exponential resummed estimate of dimensionless excess pressure $\Delta P_N^R(T, \mu)/T^4$ to all orders in μ for $D_n (1 \leq n \leq N)$ is given by [S. Mondal, S. Mukherjee, P. Hegde, Phys. Rev. Lett **128**, 022001 (2022)]

$$\frac{\Delta P_N^R(T, \mu)}{T^4} = \frac{1}{VT^3} \ln \left\langle \text{Re} \left[\exp \left(\sum_{n=1}^N D_n \hat{\mu}^n \right) \right] \right\rangle \quad (4)$$

$$\frac{\mathcal{N}_N^R(T, \mu)}{T^3} = \frac{\partial}{\partial(\mu/T)} \left[\frac{\Delta P_N^R}{T^4} \right] \quad (5)$$

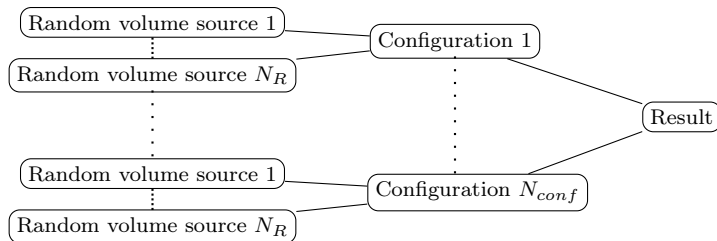
- CP symmetry** : \mathcal{Z} must be **real**, implying that **every** configuration estimate of \mathcal{Z} must be real and so, extract the **real** part of the exponential.
- D_1 and D_2 can be expressed as follows:

$$D_1 = \text{tr} \left[\mathcal{M}^{-1} \frac{\partial \mathcal{M}}{\partial \mu} \right], \quad D_2 = \text{tr} \left[\mathcal{M}^{-1} \frac{\partial^2 \mathcal{M}}{\partial \mu^2} \right] - \text{tr} \left[\mathcal{M}^{-1} \frac{\partial \mathcal{M}}{\partial \mu} \mathcal{M}^{-1} \frac{\partial \mathcal{M}}{\partial \mu} \right]$$

- \mathcal{M}^{-1} **cannot be evaluated exactly**, for which we need to therefore **estimate** D_n using N_R random volume sources for every configuration



Scheme of the structure



Stochastic Estimate of Trace

- The D_n in eqn.(4) are replaced with \bar{D}_n (estimates of D_n) as follows

$$\bar{D}_n = \frac{1}{N_R} \sum_{r=1}^{N_R} D_n^{(r)}$$

- These \bar{D}_n lead to **biased estimates** (series expansion of eqn.(4))

$$(\bar{D}_n)^m = \left[\frac{1}{N_R} \sum_{r=1}^{N_R} D_n^{(r)} \right]^m = \left[\left(\frac{1}{N_R} \right)^m \sum_{r_1=1}^{N_R} \dots \sum_{r_m=1}^{N_R} D_n^{(r_1)} \dots D_n^{(r_m)} \right] \quad (6)$$

$$\approx \text{Biased estimate} + \sum_{\substack{r_1 \neq \dots \neq r_m = 1}}^{N_R} \dots \sum_{r_m=1}^{N_R} D_n^{(r_1)} \dots D_n^{(r_m)}$$

- These biased estimates are **replaced** with **unbiased** estimates order-by-order through **cumulant expansion** of exponential resummed series [S. Mitra, P. Hegde, C. Schmidt, Phys Rev D.106.034504, arXiv:2205.08517]
- But, we **lose** partition function due to truncation of the series
- This motivates us towards **unbiased exponential resummation** to achieve unbiased thermodynamics



μ basis

In μ basis, we have

$$P_{ub}^{\mu,N} = \frac{1}{VT^3} \ln \mathcal{Z}_{ub}^{\mu,N}, \quad \mathcal{Z}_{ub}^{\mu,N} = \langle e^{A_N(\mu)} \rangle, \quad A_N(\mu) = \sum_{n=1}^N \mu^n \frac{C_n}{n!} \quad (7)$$

For $N = 4$, we have (unbiased contributions upto $\mathcal{O}(\mu_B^4)$ and $\mathcal{O}(\mu_I^8)$)

$$C_1 = \overline{D_1},$$

$$C_2 = \overline{D_2} + \left(\overline{D_1^2} - (\overline{D_1})^2 \right)$$

$$C_3 = \overline{D_3} + 3 \left(\overline{D_2 D_1} - (\overline{D_2}) (\overline{D_1}) \right) + \left(\overline{D_1^3} - 3 (\overline{D_1^2}) (\overline{D_1}) + 2 (\overline{D_1})^3 \right)$$

$$\begin{aligned} C_4 = & \overline{D_4} + 3 \left(\overline{D_2^2} - (\overline{D_2})^2 \right) + 4 \left(\overline{D_3 D_1} - (\overline{D_3}) (\overline{D_1}) \right) + \\ & 6 \left(\overline{D_2 D_1^2} - (\overline{D_2}) (\overline{D_1^2}) \right) - 12 \left((\overline{D_2 D_1}) (\overline{D_1}) - (\overline{D_2}) (\overline{D_1})^2 \right) + \\ & \left(\overline{D_1^4} - 4 (\overline{D_1^3}) (\overline{D_1}) + 12 (\overline{D_1^2}) (\overline{D_1})^2 - 6 (\overline{D_1})^4 - 3 (\overline{D_1^2})^2 \right) \end{aligned}$$

$\overline{D_m^p D_n^q} \rightarrow \text{unbiased } (p+q)^{th} \text{ power of } D_m \text{ and } D_n, \text{ for integers } m, n, p, q \geq 0$

Cumulant or X basis

In **cumulant** or \mathbf{X} basis, we define a new variable $X_N = \sum_{n=1}^N \frac{\mu^n}{n!} D_n$

$$P_{ub,M}^{X,N} = \frac{1}{VT^3} \ln \mathcal{Z}_{ub,M}^{X,N}, \quad \mathcal{Z}_{ub,M}^{X,N} = \left\langle e^{Y_M^N(X)} \right\rangle, \quad Y_M^N(X) = \sum_{n=1}^M \frac{\mathcal{L}_n(X_N)}{n!} \quad (8)$$

\mathbf{M} cumulants, highest order derivative $\rightarrow \mathbf{N}$. For $M = 4$, we have

$$\mathcal{L}_1(X_N) = (\overline{X_N})$$

$$\mathcal{L}_2(X_N) = \left[(\overline{X_N^2}) - (\overline{X_N})^2 \right]$$

$$\mathcal{L}_3(X_N) = \left[(\overline{X_N^3}) - 3 (\overline{X_N^2}) (\overline{X_N}) + 2 (\overline{X_N})^3 \right]$$

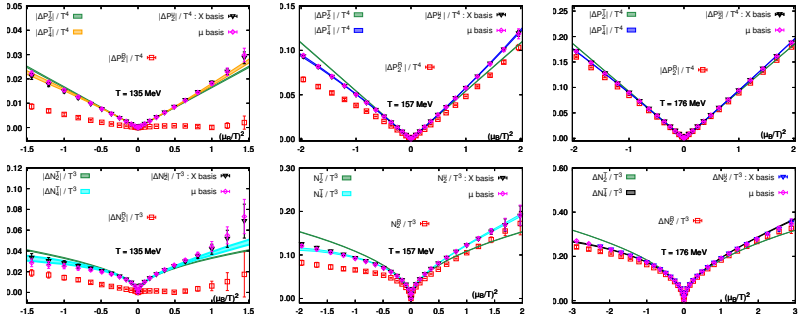
$$\begin{aligned} \mathcal{L}_4(X_N) = & \left[(\overline{X_N^4}) - 4 (\overline{X_N^3}) (\overline{X_N}) + 12 (\overline{X_N^2}) (\overline{X_N})^2 \right. \\ & \left. - 6 (\overline{X_N})^4 - 3 (\overline{X_N^2})^2 \right] \end{aligned}$$

Reproduces **exactly** first M cumulants of unbiased cumulant expansion

Captures **more higher-order contributions** over μ basis



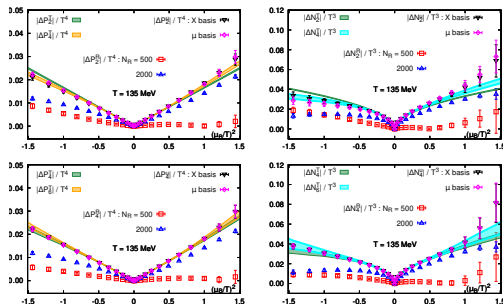
Results



$\Delta P_2/T^4$ (top row) and \mathcal{N}_2/T^3 (bottom row) plots in $(\mu_B/T)^2$ for 135 (left column), 157 (middle column) and 176 (right column) MeV

- We have used $\mathcal{O}(500)$ volume sources per configuration for **both** biased and unbiased exponential resummed results
- Captures **higher-order** Taylor series even for lowest $T = 135$ MeV,
- Also, **rapid** convergence from lowest $N = 2$. All these hold true for **all** three temperatures in three **distinct phases** of phase diagram

2000 random vectors for D_1

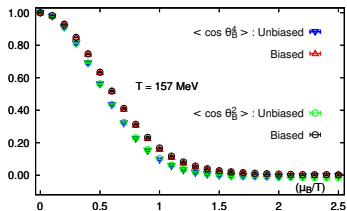
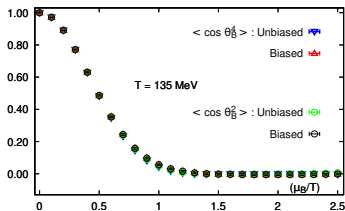


$\Delta P_{2,4}/T^4$ (left) and $\mathcal{N}_{2,4}/T^3$ (right) plots in both bases for $T = 135$ MeV

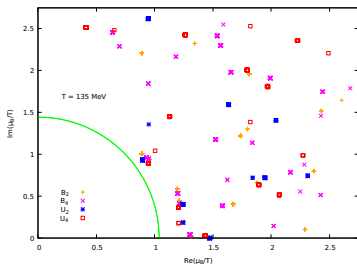
[S. Mitra, P. Hegde, arXiv:2209.11937]

- For **Biased** resummed results, we have used **2000** random volume sources for **D_1** and **500** random volume sources for **other D_n** . For **unbiased** results, we have used **500** random volume sources for **all D_n**
- Higher order** Taylor series captured with **just $\mathcal{O}(500)$** volume sources, **saving** appreciable **computational time** and **data storage space**

Phasefactor



Phasefactor plots for $T = 135 \text{ MeV}$ (left) and 157 MeV (right)



Roots of Z_2 and Z_4 in complex μ_B plane at 135 MeV

Conclusions

- **Essence** of cumulant expansion analysing biased and unbiased estimates
- **Replacing** biased with unbiased powers cause truncated ER **approach** QNS
- The very idea of an **unbiased exponential resummation**
- Not entire, but **partially** unbiased results upto $\mathcal{O}(\mu^N)$ working in two bases
- Works **well** for problematic, yet important **low T** regime
- **Saves** computational time and data storage space (**convenience**)
- A **newly defined** reweighting factor and \mathcal{Z} altogether
- **Retrieve back** phasefactor and roots of \mathcal{Z} in the **complex μ** plane
- **Get back** thermodynamics **preserving** partial unbiasedness
- Inductively, we approach **Towards** the much-coveted **unbiased exponential resummation to all orders** in the limit of an **infinite cumulant series** which is identical to an **all-ordered Taylor series**



Computational Setup

For all the above calculations, we have used the following data:

- **Physical** quark mass values for u, d, s quarks, where $m_u = m_d = m_s/27$. This sets the crossover $T \approx 157$ MeV at $\mu_B = 0$
- Chosen $T = 135, 157, 176$ MeV, characterizing **hadronic**, **crossover**, **quark gluon plasma** phases respectively.
- Temperatures are chosen so that $(T_{\text{hadron}} + T_{\text{plasma}})/2 \approx T_{\text{crossover}}$
- Considered lattice with $N_\sigma = 32$ spatial sites and $N_\tau = 8$ temporal sites
- Fermion action used is a **2 + 1** flavored **HISQ (Heavily Improved Staggered Quarks)** action
- For μ_I , we have used gauge ensemble having **20K** configurations and **100K** configurations for μ_B
- Because, there **is** a sign problem for μ_B . **No** sign problem for μ_I



THANK YOU SO MUCH
FOR YOUR PATIENCE AND ATTENTION !!!

Köszönöm