Propagation properties of spin degrees of freedom within the framework of relativistic hydrodynamics with spin

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Abstract

The propagation properties of spin degrees of freedom are analyzed in the framework of relativistic hydrodynamics with spin based on the de Groot–van Leeuwen–van Weert definitions of the energy-momentum and spin tensors. We derive the analytical expression for the spin wave velocity for arbitrary statistics and show that it goes to half the speed of light in the ultra-relativistic limit. We find that only the transverse degrees of freedom propagate, analogously to electromagnetic waves [1].

Canonical currents

Starting from the Dirac Lagrangian,

$$\mathcal{L}_D(x) = \frac{i\hbar c}{2}\bar{\psi}(x)\gamma^\mu \overleftrightarrow{\partial}_\mu \psi(x) - mc^2\bar{\psi}(x)\psi(x),$$

the canonical energy-momentum and spin tensors can be obtained as:

$$T^{\mu\nu}_{\rm can} = \frac{i\hbar c}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}^\nu\psi,$$
$$S^{\lambda,\mu\nu}_{\rm can} = -\frac{\hbar c}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}\gamma_\alpha\gamma_5\psi,$$

while the total angular momentum tensor reads

$$J^{\lambda,\mu\nu}_{\rm can} = x^\mu T^{\lambda\nu}_{\rm can} - x^\nu T^{\lambda\mu}_{\rm can} + S^{\lambda,\mu\nu}_{\rm can}.$$

Since the total angular momentum is conserved, we have

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can} = T^{[\nu\mu]}_{\rm can}.$$

Pseudogauge transformations

One can define a new pair of tensors $T^{\mu\nu}$ and $S^{\lambda,\mu\nu}$ connected to the canonical ones through the so-called pseudo-gauge transformations [2]

$$T^{\mu\nu} = T^{\mu\nu}_{\rm can} + \frac{1}{2}\partial_\lambda(\Phi^{\lambda,\mu\nu} + \Phi^{\nu,\mu\lambda} + \Phi^{\mu,\nu\lambda})$$
$$S^{\lambda,\mu\nu} = S^{\lambda,\mu\nu}_{\rm can} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho}$$

where the superpotentials $\Phi^{\lambda,\mu\nu}$ and $Z^{\mu\nu,\lambda\rho}$ satisfy $\Phi^{\lambda,\mu\nu} = -\Phi^{\lambda,\nu\mu}$ and $Z^{\mu\nu,\lambda\rho} = -Z^{\nu\mu,\lambda\rho} = -Z^{\mu\nu,\rho\lambda}$. The newly defined tensors preserve the total energy, linear momentum, and angular momentum obtained by integrating over a closed or infinite hypersurface:

$$P^\nu = \int_\Sigma d\Sigma_\mu T^{\mu\nu}, \qquad J^{\lambda\nu} = \int_\Sigma d\Sigma_\mu J^{\mu,\lambda\nu}.$$

The conservation laws are unchanged: $\partial_\mu T^{\mu\nu} = 0$, $\partial_\lambda J^{\lambda,\mu\nu} = 0$. In de Groot-van Leeuwen-van Weert (GLW) pseudogauge, $\Phi^{\lambda,\mu\nu} = \frac{i\hbar^2}{4m}\bar{\psi}(\sigma^{\lambda\mu}\overleftrightarrow{\partial}^\nu - \sigma^{\lambda\nu}\overleftrightarrow{\partial}^\mu)\psi$, $Z^{\mu\nu\lambda\rho} = 0$, which gives:

$$T^{\mu\nu}_{\rm GLW} = -\frac{\hbar^2}{4m}\bar{\psi}\overleftrightarrow{\partial}^\mu\overleftrightarrow{\partial}^\nu\psi,$$

$$S^{\lambda,\mu\nu}_{\rm GLW} = \frac{i\hbar^2}{4m}\left(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}^\lambda\psi - \partial_\rho\epsilon^{\mu\nu\lambda\rho}\bar{\psi}\gamma^5\psi\right).$$

Relativistic hydrodynamics with spin

Since GLW form of energy-momentum tensor is symmetric: $T^{\nu\mu}_{\rm GLW}(x) = T^{\mu\nu}_{\rm GLW}(x)$, conservation of total angular momentum $\partial_\mu J^{\mu,\alpha\beta}(x) = \partial_\mu(x^\alpha T^{\mu\beta}(x) - x^\beta T^{\mu\alpha}(x)) + \partial_\mu S^{\mu,\alpha\beta}(x) = 0$ implies spin conservation $\partial_\mu S^{\mu,\alpha\beta}_{\rm GLW}(x) = 0$. Using classical treatment of spin [3], one can obtain well-known perfect-fluid form of energy-momentum tensor whereas the spin tensor (for the general statistic) reads as

$$S^{\lambda,\mu\nu} = \int {\rm dP}\ {\rm dS}\ p^\lambda\, s^{\mu\nu}\left[f^+_{\rm eq} + f^-_{\rm eq}\right] \simeq -\frac{2\mathfrak{s}^2}{3m^2}\sum_{\sigma=\pm}\int {\rm dP}\, p^\lambda f'^\sigma_{\rm eq}\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}{}_\alpha\right) = S^{\lambda,\mu\nu}_{\rm ph} + S^{\lambda,\mu\nu}_\Delta,$$

where $S^{\lambda,\mu\nu}_{\rm ph} = (\mathcal{A}_1+\mathcal{A}_3)U^\lambda\omega^{\mu\nu}$, $S^{\lambda,\mu\nu}_\Delta = (2\mathcal{A}_1-\mathcal{A}_3)U^\lambda U^\alpha U^{[\mu}\omega^{\nu]}{}_\alpha + \mathcal{A}_3(\Delta^{\lambda\alpha}U^{[\mu}\omega^{\nu]}{}_\alpha + U^\lambda\Delta^{\alpha[\mu}\omega^{\nu]}{}_\alpha + U^\alpha\Delta^{\lambda[\mu}\omega^{\nu]}{}_\alpha)$.

The thermodynamic functions $\mathcal{A}_1$ and $\mathcal{A}_3$ can be obtained as

$$\mathcal{A}_1 = \frac{\mathfrak{s}^2}{9}\left[\left(\frac{\partial\mathcal{N}}{\partial\xi}\right)_\beta - \frac{2}{m^2}\left(\frac{\partial\mathcal{E}}{\partial\beta}\right)_\xi\right],\ \mathcal{A}_3 = \frac{2\mathfrak{s}^2}{9}\left[\left(\frac{\partial\mathcal{N}}{\partial\xi}\right)_\beta + \frac{1}{m^2}\left(\frac{\partial\mathcal{E}}{\partial\beta}\right)_\xi\right].$$

Here, $\omega_{\mu\nu}$ is an anti-symmetric tensor of rank 2, having 6 degrees of freedom. With respect to U^μ, $\omega_{\mu\nu}$ can be decomposed as

$$\omega_{\mu\nu} = \kappa_\mu U_\nu - \kappa_\nu U_\mu + \epsilon_{\mu\nu\alpha\beta}U^\alpha\omega^\beta.$$

$\kappa^\mu = \omega^{\mu\nu}U_\nu$ is the electric field component. $\omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}U_\nu\omega_{\alpha\beta}$ is the magnetic field component. Since $\kappa^\mu U_\mu = \omega^\mu U_\mu = 0$, in the LRF they take the form

$$\kappa^\mu_{\rm LRF} = (0, C_{\kappa X}, C_{\kappa Y}, C_{\kappa Z}), \quad \omega^\mu_{\rm LRF} = (0, C_{\omega X}, C_{\omega Y}, C_{\omega Z}).$$

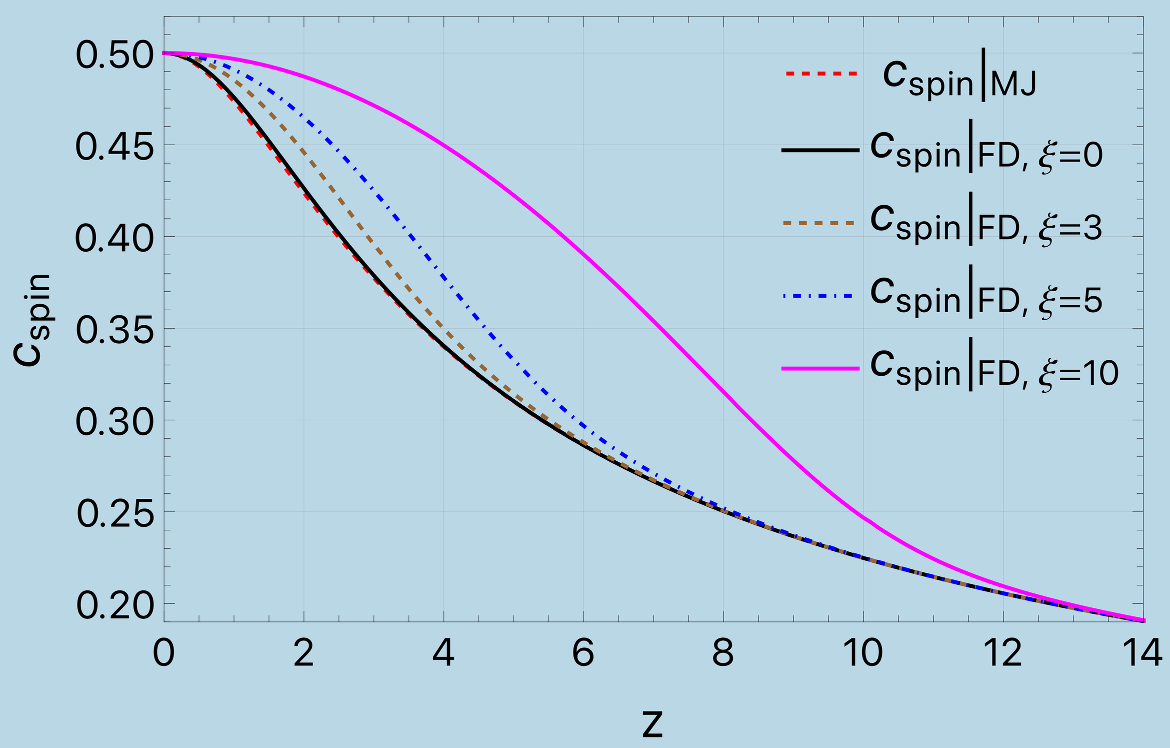
Propagation properties of spin degrees of freedom

In an unpolarized fluid at rest, $U^\mu = (1,0,0,0)$ and $\omega^{\mu\nu} = 0$. Considering small perturbations along z, we look for oscillations in $\omega^{\mu\nu}$. Since there is no linear coupling between N^μ, $T^{\mu\nu}$ and $\omega^{\mu\nu}$, the fluctuations in the spin sector do not influence those in the fluid sector. In this approximation, we have $S^{\lambda,\mu\nu}_{\rm ph} = (\mathcal{A}_1+\mathcal{A}_3)g^{t\lambda}\omega^{\mu\nu}$, $S^{\lambda,\mu\nu}_\Delta = 2(\mathcal{A}_1-2\mathcal{A}_3)g^{t\lambda}g^{t[\mu}\omega^{\nu]t} + \mathcal{A}_3(g^{t[\mu}\omega^{\nu]\lambda} + g^{\lambda[\mu}\omega^{\nu]t} - g^{t\lambda}\omega^{\mu\nu})$. Taking the divergence of the above equations leads to: $\partial_t C_{\kappa i} - \frac{1}{2}\epsilon^{tijz}\partial_z C_{\omega j} = 0$, $\partial_t C_{\omega i} - \frac{\mathcal{A}_3}{2\mathcal{A}_1}\epsilon^{tijz}\partial_z C_{\kappa j} = 0$. The longitudinal components do not propagate, since $\partial_t C_{\kappa Z} = \partial_t C_{\omega Z} = 0$. The transverse dofs $\mathcal{C} \in \{C_{\kappa X}, C_{\kappa Y}, C_{\omega X}, C_{\omega Y}\}$ obey:

$$\left(\frac{\partial^2}{\partial t^2} - c^2_{\rm spin}\frac{\partial^2}{\partial z^2}\right)\mathcal{C} = 0,$$

where the speed of the spin wave satisfies:

$$c^2_{\rm spin} = -\frac{1}{4}\frac{\mathcal{A}_3}{\mathcal{A}_1} = \frac{1}{4}\frac{(\partial\mathcal{E}/\partial T)_\xi - z^2(\partial\mathcal{N}/\partial\xi)_T}{(\partial\mathcal{E}/\partial T)_\xi + \frac{z^2}{2}(\partial\mathcal{N}/\partial\xi)_T}.$$



Ideal gas limit: $c^2_{\rm spin} = \frac{1}{4}\frac{K_3(z)}{K_3(z)+\frac{z}{2}K_2(z)}$, (depends only on $z = m/T$).

Fermi-Dirac gas limit: $c^2_{\rm spin} = \frac{\frac{1}{4}\sum_{\ell=1}^\infty\frac{(-1)^{\ell+1}}{\ell}\cosh(\ell\xi)K_3(\ell z)}{\sum_{\ell=1}^\infty\frac{(-1)^{\ell+1}}{\ell}\cosh(\ell\xi)[K_3(\ell z)+\frac{\ell z}{2}K_2(\ell z)]}$, (depends on z and ξ).

For small $z = m/T \ll 1$:

$$\text{MJ:}\quad c_{\rm spin} = \frac{1}{2}\left[1 - \frac{z^2}{16} + O(z^4)\right], \qquad \text{FD:}\quad c_{\rm spin} = \frac{1}{2}\left[1 - \frac{15z^2}{4\pi^2}\frac{1+\frac{3\xi^2}{\pi^2}}{7+30\frac{\xi^2}{\pi^2}+15\frac{\xi^4}{\pi^4}} + O(z^4)\right].$$

For large $z = m/T \gg 1$: $c_{\rm spin} \simeq \frac{1}{\sqrt{2z}}$.

Noting that $C_{\kappa,Z}$ and $C_{\omega,Z}$ do not propagate. The linearly polarized solutions can be written as

$$C_{\boldsymbol{\kappa}} = C_0{\rm Re}[e^{-ik(c_{\rm spin}t-z)}](\boldsymbol{e}_1\cos\theta + \boldsymbol{e}_2\sin\theta), \quad C_{\boldsymbol{\omega}} = 2c_{\rm spin}C_0{\rm Re}[e^{-ik(c_{\rm spin}t-z)}](\boldsymbol{e}_1\sin\theta - \boldsymbol{e}_2\cos\theta),$$

where the analogy to the EM waves is evident since $C_{\boldsymbol{\omega}} = 2c_{\rm spin}\hat{\boldsymbol{n}}\times C_{\boldsymbol{\kappa}}$.

References

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