

Phase transitions in relativistic meson systems

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Outline

- Introduction
- Selfinteracting real scalar field
- $\varphi^4 + \varphi^6$ model at $n_I = 0$
- Creation of the Bose-Einstein condensate: phase transitions of the 2nd and 1st order
- Thermodynamic mean-field model for a particle-antiparticle system
- Interacting gas of “pions” with conserved isospin: Selfconsistent solutions
- "Weak" attraction
- "Strong" attraction
- Concluding remarks

Motivation

Knowledge of the phase structure of meson systems in the regime of finite temperatures and isospin densities is crucial for understanding a wide range of phenomena, from nucleus-nucleus collisions to neutron stars, as well as cosmology. This field is an important part of hot and dense hadronic matter research. At the same time, the study of meson systems has its own specificity due to the possibility of the Bose-Einstein condensation of bosonic particles.

A sketch of the QCD phase diagram*

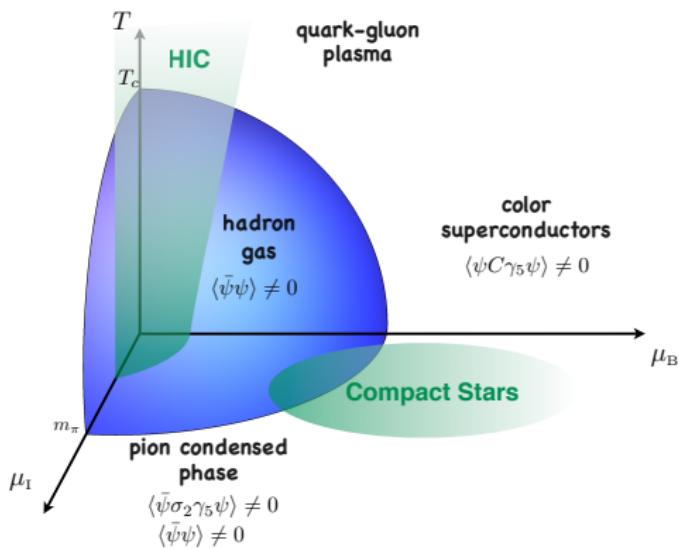


Figure: A sketch of the QCD phase diagram: a grand canonical description of the phases of hadronic matter as a function of the hadronic temperature, T , of the isospin chemical potential, μ_I , and of the baryonic chemical potentials, μ_B .

* M. Mannarelli, [arXiv:1908.02042 [hep-ph]]

Condensation of interacting scalar bosons

$$\mathcal{L}(x) = \frac{1}{2} \left[\partial_\mu \hat{\phi}(x) \partial^\mu \hat{\phi}(x) - m^2 \hat{\phi}^2(x) \right] + \mathcal{L}_{\text{int}}[\hat{\phi}^2(x)]$$

where $x = (t, \mathbf{r})$.

We adopt that:

$$\hat{\phi}(\mathbf{r}) = \phi_{\text{cond}} + \hat{\psi}(\mathbf{r}), \quad \text{where} \quad \langle \hat{\psi}(\mathbf{r}) \rangle = 0.$$

Here we use the famous Bogolyubov's decomposition of the field operator into two contributions [1]

$$\Psi(\mathbf{r}) = \frac{1}{\sqrt{V}} a_0 + \frac{1}{\sqrt{V}} \sum_{\mathbf{k} \neq 0} a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}/\hbar}.$$

Due to the argument that at $T \rightarrow 0$ in a nonperfect Bose-Einstein gas the number of particles on the ground state

$$N_0 = \langle a_0^+ a_0 \rangle \approx N,$$

one can treat a_0 and a_0^+ as classical values.

[1] N. Bogolubov, On the theory of superfluidity, Sov. J. Phys. **11**, 23 (1947).

M.M. Bogolyubov, *Lekcii z kvantovoi statystyky*, Kyiv, 1947 (Ukrainian).

N.N. Bogoliubov, *Lectures on Quantum Statistics*, Gordon and Breach, New York, 1967.

Condensation of interacting scalar bosons at finite temperatures

Heisenberg representation:

$$\hat{\phi}(x) = e^{iHt} \hat{\phi}(\mathbf{r}) e^{-iHt} = \phi_{\text{cond}} + \hat{\psi}(x) \quad \text{with} \quad \langle \hat{\psi}(x) \rangle = 0,$$

where

$$\left[\hat{\psi}(t, \mathbf{r}), \frac{\partial \hat{\psi}(t, \mathbf{r}')}{\partial t} \right] = \left[\hat{\phi}(t, \mathbf{r}), \frac{\partial \hat{\phi}(t, \mathbf{r}')}{\partial t} \right] = i\delta^3(\mathbf{r} - \mathbf{r}').$$

Hence, the quantum fluctuations of the field $\hat{\psi}(x)$ have the same commutation relation as the complete field $\hat{\phi}(x)$.

Expansion over solutions of the Klein-Gordon equation:

$$\hat{\psi}(x) = \int_{|\mathbf{p}| \neq 0} \frac{d^3 p}{(2\pi)^3 2\omega_p} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^+ e^{ip \cdot x} \right) |_{p^0 = \omega_p}.$$

Hence,

$$[a_k, a_p^+] = (2\pi)^3 2\omega_p \delta^3(\mathbf{k} - \mathbf{p}), \quad [a_k, a_p] = 0.$$

Scalar density

$$\langle \hat{\phi}^2(x) \rangle = \langle \phi_{\text{cond}}^2 + 2\phi_{\text{cond}}\hat{\psi} + \hat{\psi}^2 \rangle = \phi_{\text{cond}}^2 + \langle \hat{\psi}^2 \rangle.$$

Condensation of interacting scalar bosons at finite temperatures [2]

Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2} \left[\partial_\mu \hat{\phi}(x) \partial^\mu \hat{\phi}(x) - m^2 \hat{\sigma}(x) \right] + \mathcal{L}_{\text{int}}[\hat{\sigma}(x)]$$

$$\hat{\sigma}(x) = \hat{\phi}^2(x)$$

Averaging in Grand Canonical Ensemble

$$\langle \hat{A} \rangle = \frac{1}{Z} \text{Tr} \left[e^{-\beta(\hat{H}-\mu\hat{N})} \hat{A} \right], \quad Z = \text{Tr} \left[e^{-\beta(\hat{H}-\mu\hat{N})} \right]$$

$$\sigma = \langle \hat{\sigma} \rangle, \quad \delta\hat{\sigma} = \hat{\sigma} - \sigma$$

The mean-field approximation (MFA)

$$\mathcal{L}_{\text{int}}(\hat{\sigma}) \simeq \mathcal{L}_{\text{int}}(\sigma) + \delta\hat{\sigma} \mathcal{L}'_{\text{int}}(\sigma) = \mathcal{L}_{\text{int}}(\sigma) + \hat{\sigma} \mathcal{L}'_{\text{int}}(\sigma) - \sigma \mathcal{L}'_{\text{int}}(\sigma)$$

The effective Lagrangian in the mean-field approximation

Effective Lagrangian density

$$\mathcal{L}(x) \simeq \frac{1}{2} \left[\partial_\mu \hat{\phi}(x) \partial^\mu \hat{\phi}(x) - M^2(\sigma) \hat{\phi}^2(x) \right] + P_{\text{ex}}(\sigma),$$

where

Definitions

$$P_{\text{ex}}(\sigma) \equiv \mathcal{L}_{\text{int}}(\sigma) - \sigma \frac{\partial \mathcal{L}_{\text{int}}(\sigma)}{\partial \sigma}, \quad U(\sigma) \equiv - \frac{\partial \mathcal{L}_{\text{int}}(\sigma)}{\partial \sigma},$$
$$\hat{M}^2(\sigma) = m^2 + 2 U(\sigma), \quad 2\hat{U}'(\sigma) = \Pi(\sigma).$$

Consistency relation

$$\sigma \frac{\partial U(\sigma)}{\partial \sigma} = \frac{\partial P_{\text{ex}}(\sigma)}{\partial \sigma}$$

Hamiltonian density in the mean-field approximation

$$\hat{\pi}(x) = \partial_t \hat{\phi}(x), \quad [\hat{\phi}(t, \mathbf{r}), \hat{\pi}(t, \mathbf{r}')] = i\delta^3(\mathbf{r} - \mathbf{r}')$$

Effective Hamiltonian density $\hat{\mathcal{H}} = \hat{\pi} \partial_t \hat{\phi} - \mathcal{L}$

$$\hat{\mathcal{H}} \simeq \frac{1}{2} \left[\hat{\pi}^2(x) + \nabla \hat{\phi}(x) \cdot \nabla \hat{\phi}(x) + M^2(\sigma) \hat{\phi}^2(x) \right] - P_{\text{ex}}(\sigma),$$

Using solutions of the Klein-Gordon equation

$$\partial^\mu \partial_\mu \hat{\phi} + M^2(\sigma) \hat{\phi} = 0$$

one can represent the scalar field $\hat{\phi}(x)$ as

$$\hat{\phi}(x) = g \int \frac{d^3 k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} [a_{\mathbf{k}} e^{-ik \cdot x} + a_{\mathbf{k}}^+ e^{ik \cdot x}]$$

where $k^0 = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M^2(\sigma)}$.

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^+] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \quad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = [a_{\mathbf{k}}^+, a_{\mathbf{k}'}^+] = 0$$

Hamiltonian in the mean-field approximation

The Hamiltonian operator in the MFA

$$\hat{H} = \int d^3x \hat{\mathcal{H}} = V \left[g \int \frac{d^3k}{(2\pi)^3} \omega_{\mathbf{k}} a_{\mathbf{k}}^+ a_{\mathbf{k}} - P_{\text{ex}}(\sigma) \right]$$

In the MFA the equilibrium momentum distribution coincides with that of an ideal gas of bosons with the effective mass $M(\sigma)$

$$n_{\mathbf{k}}(\sigma) \equiv \langle a_{\mathbf{k}}^+ a_{\mathbf{k}} \rangle = (e^{\beta \omega_{\mathbf{k}}} - 1)^{-1}, \quad \beta = 1/T, \quad k_B = 1, \quad \mu = 0,$$

$$\omega_{\mathbf{k}} = \sqrt{M^2(\sigma) + \mathbf{k}^2} \quad \text{with} \quad M^2(\sigma) = m^2 + 2U(\sigma)$$

The scalar density $\sigma = \langle \hat{\phi}^2 \rangle$ is obtained by direct calculation

$$\sigma = g \int \frac{d^3k}{(2\pi)^3} \frac{n_{\mathbf{k}}(\sigma)}{\omega_{\mathbf{k}}}, \quad \sigma = \sigma_{\text{cond}} + g \int \frac{d^3k}{(2\pi)^3} \frac{n_{\mathbf{k}}(\sigma)}{\omega_{\mathbf{k}}} \Big|_{M^2(\sigma)=0}$$
$$\langle \hat{\phi}^2 \rangle = \phi_{\text{cond}}^2 + \langle \hat{\psi}^2 \rangle$$

Pressure, energy density, entropy density

Pressure $P = T \ln Z/V$

$$P = P_{\text{kin}}(M, T) + P_{\text{ex}}(\sigma), \quad P_{\text{kin}}(M, T) = \frac{g}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{\omega_{\mathbf{k}}} n_{\mathbf{k}}(\sigma)$$

Energy density, entropy density $s = (\varepsilon + P)/T$

$$\varepsilon = \varepsilon_{\text{cond}} + g \int \frac{d^3 k}{(2\pi)^3} \omega_{\mathbf{k}} n_{\mathbf{k}}(\sigma) - P_{\text{ex}}(\sigma)$$

$$s = s_{\text{cond}} + \frac{g}{T} \int \frac{d^3 k}{(2\pi)^3} \left(\omega_{\mathbf{k}} + \frac{k^2}{3\omega_{\mathbf{k}}} \right) n_{\mathbf{k}}(\sigma)$$

Bosonic system with $\varphi^4 + \varphi^6$ interaction

Mean field and excess pressure

$$\begin{aligned}\mathcal{L}_{\text{int}}(\hat{\phi}^2(x)) &= \frac{a}{4} \hat{\phi}^4(x) - \frac{b}{6} \hat{\phi}^6(x), \\ \sigma &= \langle \hat{\phi}^2(x) \rangle,\end{aligned}$$

$$M^2(\sigma) = m^2 + 2U(\sigma) = m^2 - a\sigma + b\sigma^2, \quad P_{\text{ex}}(\sigma) = -\frac{a}{4}\sigma^2 + \frac{b}{3}\sigma^3$$

$$n_k(\sigma) = \left[e^{\sqrt{k^2+M^2(\sigma)}/T} - 1 \right]^{-1}, \quad M^2(\sigma) = m^2 - a\sigma + b\sigma^2 \geq 0$$

The onset of the scalar condensate: $M^2(\sigma) = m^2 - a\sigma + b\sigma^2 = 0$.To parameterize the attraction coefficient a , we introduce dimensionless κ :

$$\kappa = a/(2m\sqrt{b}) \quad \rightarrow \quad a = \kappa a_c, \quad a_c = 2m\sqrt{b}$$

Solution of equation $M^2(\sigma) = m^2 - a\sigma + b\sigma^2 = 0$

$$\sigma_{1,2} = \frac{m}{\sqrt{b}} \left(\kappa \mp \sqrt{\kappa^2 - 1} \right)$$

Thermodynamic mean-field model [3]: $F(T, n) = F_0 + F_{\text{int}}$

$$\varphi^4 + \varphi^6 \rightarrow n^2 + n^3$$

$$p(T, \mu) = \frac{g}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{\mathbf{k}^2}{\sqrt{m^2 + \mathbf{k}^2}} f(E_k(n), \mu) + P_{\text{ex}}(n),$$

$$f(E, \mu) = \left\{ \exp \left[\frac{E - \mu}{T} \right] - 1 \right\}^{-1}, \quad E_k(n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

► $n = g \int \frac{d^3 k}{(2\pi)^3} f(E_k(n)) , \quad n = n_{\text{cond}} + g \int \frac{d^3 k}{(2\pi)^3} f(E_{\text{kin}}) , \quad \mu = 0$

$$E_{\text{kin}} = \sqrt{m^2 + \mathbf{k}^2} - m, \quad n \frac{\partial U(n)}{\partial n} = \frac{\partial P_{\text{ex}}(n)}{\partial n}$$

Parametrization of the interaction [4]

$$P_{\text{ex}}(n) = -\frac{1}{2} A n^2 + \frac{2}{3} B n^3 \quad \rightarrow \quad U(n) = -A n + B n^2$$

[3] D. Anchishkin, V. Vovchenko, J. Phys. G **42**, 105102 (2015).

[4] D. Anchishkin, I. Mishustin, and H. Stoecker, J. Phys. G **46**, 035002 (2019).

Selfconsistent solutions

$$m = m_\pi = 140 \text{ MeV}, g = 3, \mu = 0$$

$$\sigma_{\lim} = \sigma_{\text{th}}(M=0, T) = \frac{g}{12} T^2, \quad n_{\lim} = g \int \frac{d^3 k}{(2\pi)^3} \left\{ \exp \left[\frac{\sqrt{m^2 + k^2} - m}{T} \right] - 1 \right\}^{-1}$$

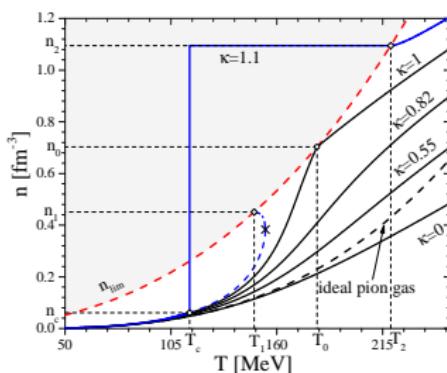
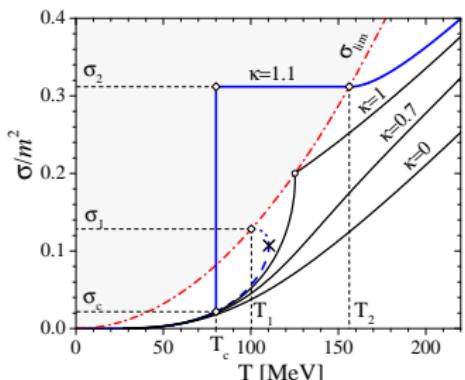


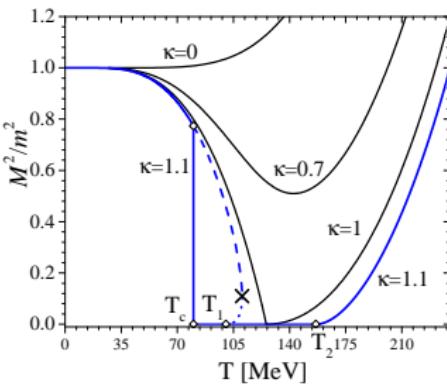
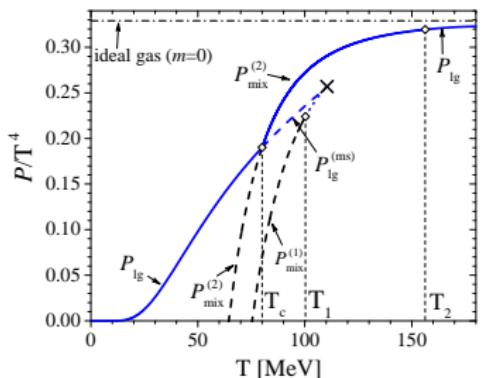
Figure: Left panel: Scalar density vs temperature, $b = 25 m_\pi^{-2}$, $a = \kappa 2m\sqrt{b}$.

Right panel: Particle-number density vs temperature, $B = 10m_\pi b^2$ with $b = \frac{16\pi}{3} r_0^3$, $r_0 \approx 0.3 \text{ fm}$.

- $\kappa < 1$ is the "weak" attraction,
- $\kappa > 1$ is the "strong" attraction.

Scalar condensate: The pressure and effective mass at $\mu = 0$

$$p = \frac{g\pi^2}{90} T^4 + P_{\text{ex}}(\sigma_2), \quad M^2(\sigma) = m^2 + 2 U(\sigma)$$



- *Left panel:* Pressure vs temperature for the supercritical case $\kappa = 1.1$.
- *Right panel:* Effective mass vs temperature.

The mean-field model for the system of particles and antiparticles with $n_l \neq 0$

- It is assumed that in general the free-energy density $\Phi = F/V$ of the two-component system looks like:

$$\Phi(T, n_1, n_2) = \Phi_1^{(0)}(T, n_1) + \Phi_2^{(0)}(T, n_2) + \Phi_{\text{int}}(n),$$

$$n = n_1 + n_2, \quad \mu_j^{(0)} = \frac{\partial \Phi_j^{(0)}(T, n_j)}{\partial n_j}, \quad j = 1, 2$$

We introduce:

$$\blacktriangleright \quad U(n) \equiv \frac{\partial \Phi_{\text{int}}(n)}{\partial n},$$

$$\blacktriangleright \quad P_{\text{ex}}(n) \equiv n \left[\frac{\partial \Phi_{\text{int}}(n)}{\partial n} \right]_T - \Phi_{\text{int}}$$

$$\Rightarrow p(T, n_1, n_2) = p_1^{(0)} + p_2^{(0)} + P_{\text{ex}}(n), \quad \text{where} \quad p_j^{(0)} = \mu_j^{(0)} n_j - \Phi_j^{(0)}$$

$$n \frac{\partial U(n)}{\partial n} = \frac{\partial P_{\text{ex}}(n)}{\partial n}$$

The mean-field model for the system of particles and antiparticles with $n_l \neq 0$

Due to the opposite sign of the charge (for details, see [3]):

$$\mu_1 = -\mu_2 \equiv \mu_l$$

Therefore, the Euler relation includes only the isospin number density,
 $n_l = n^{(-)} - n^{(+)}$:

$$\varepsilon + p = T s + \mu_l n_l$$

$$\begin{aligned} p &= -T \int \frac{d^3 k}{(2\pi)^3} \ln \left[1 - \exp \left(-\frac{\omega_k + U(n) - \mu_l}{T} \right) \right] - \\ &\quad - T \int \frac{d^3 k}{(2\pi)^3} \ln \left[1 - \exp \left(-\frac{\omega_k + U(n) + \mu_l}{T} \right) \right] + P_{\text{ex}}(n), \end{aligned}$$

$$\omega_k = \sqrt{m^2 + \mathbf{k}^2}$$

Particle-antiparticle system with conservation of isospin (charge) [5]

- Canonical Ensemble: the canonical variables (T, n_I)

$$\begin{aligned}\blacktriangleright \quad n &= \int \frac{d^3 k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) + f_{\text{BE}}(E(k, n), -\mu_I)], \\ \blacktriangleright \quad n_I &= \int \frac{d^3 k}{(2\pi)^3} [f_{\text{BE}}(E(k, n), \mu_I) - f_{\text{BE}}(E(k, n), -\mu_I)],\end{aligned}$$

where

$$f_{\text{BE}}(E, \mu) = \left[\exp \left(\frac{E - \mu}{T} \right) - 1 \right]^{-1}, \quad E(k, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n)$$

Parametrization of the interaction $\varphi^4 + \varphi^6 \rightarrow n^2 + n^3$

$$P_{\text{ex}}(n) = -\frac{1}{2} A n^2 + \frac{2}{3} B n^3 \quad \rightarrow \quad U(n) = -A n + B n^2$$

[5] D. Anchishkin, V. Gnatovskyy, D. Zhuravel, and V. Karpenko, Selfinteracting Particle-Antiparticle System of Bosons, Phys. Rev. C **105**, 045205 (2022); arXiv: 2102.02529 [nucl-th].

Particle-antiparticle system with conservation of isospin (charge) [5]

Parameterization of the attraction coefficient:

$$U(n) + m = 0$$

$$n_1 = \sqrt{\frac{m}{B}} \left(\kappa - \sqrt{\kappa^2 - 1} \right), \quad n_2 = \sqrt{\frac{m}{B}} \left(\kappa + \sqrt{\kappa^2 - 1} \right),$$

$$\kappa \equiv \frac{A}{2\sqrt{mB}}, \quad A_c = 2\sqrt{mB}, \quad A = \kappa A_c$$

The condensed state can be developed just under the necessary condition:

► $m + U(n) - \mu_I = 0$

If one of the components of a two-component system is in the condensate phase:

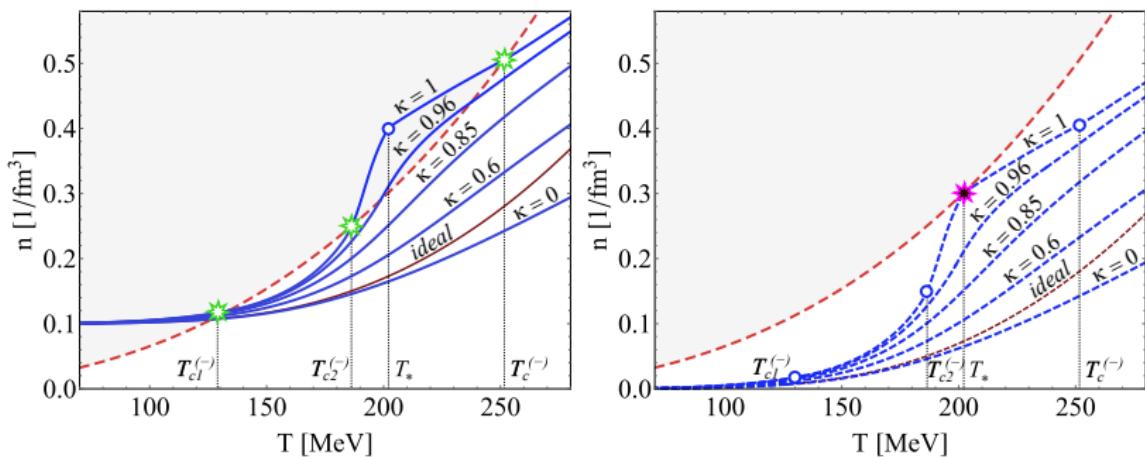
► $n = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) + \int \frac{d^3 k}{(2\pi)^3} f_{\text{BE}}(E(k, n), -\mu_I) \Big|_{\mu_I=m+U(n)},$

► $n_I = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) - \int \frac{d^3 k}{(2\pi)^3} f_{\text{BE}}(E(k, n), -\mu_I) \Big|_{\mu_I=m+U(n)},$

where

$$n_{\text{lim}}(T) = \int \frac{d^3 k}{(2\pi)^3} f_{\text{BE}}(\omega_k, \mu_I) \Big|_{\mu_I=m}$$

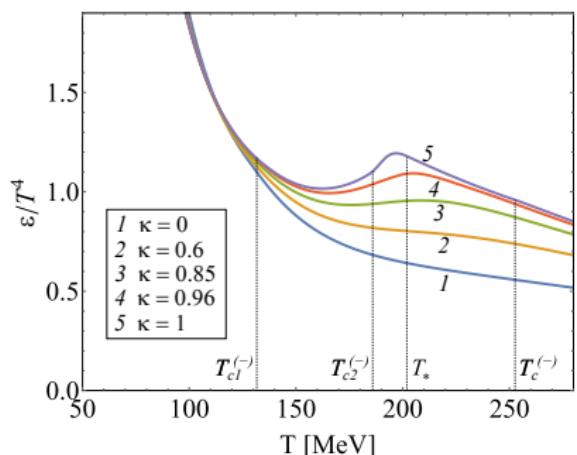
"Weak" attraction: 2nd order phase transitions generated by the π^- -meson subsystem



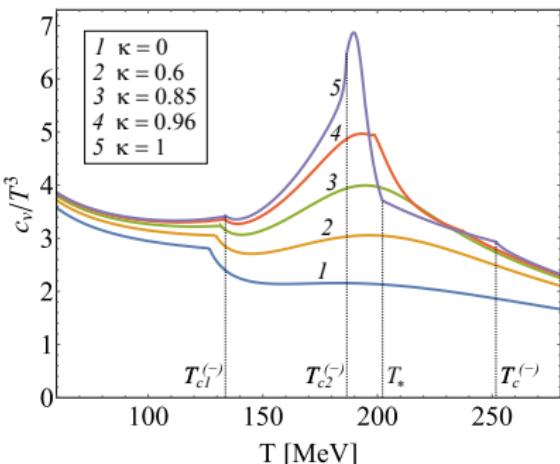
Left panel: The particle-number densities $n^{(-)}$ of π^- mesons vs temperature at $n_l = 0.1 \text{ fm}^{-3}$.

Right panel: The particle-number densities $n^{(+)}$ of π^+ mesons vs temperature. The "dark" star indicates a point-like phase transition of the 2nd order generated by the π^+ -meson subsystem.

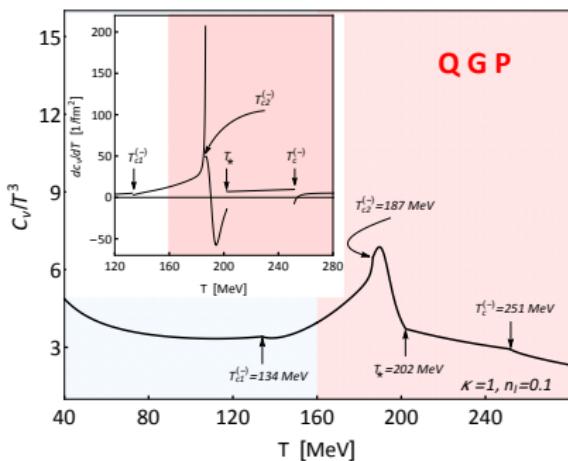
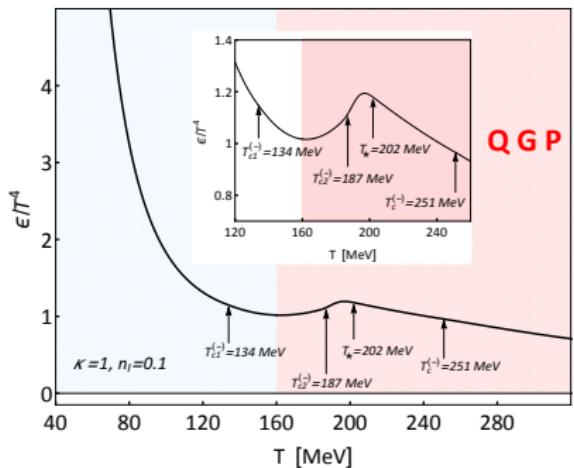
"Weak" attraction: Energy density and Heat capacity



Left panel: Energy density vs temperature at $n_l = 0.1 \text{ fm}^{-3}$.



Right panel: Heat capacity vs temperature.

Energy density and Heat capacity at $\kappa = 1$ 

Left panel: Energy density vs temperature at $n_l = 0.1 \text{ fm}^{-3}$.

Right panel: Heat capacity vs temperature.

A particle-antiparticle boson system under "strong" attraction, $\kappa > 1$

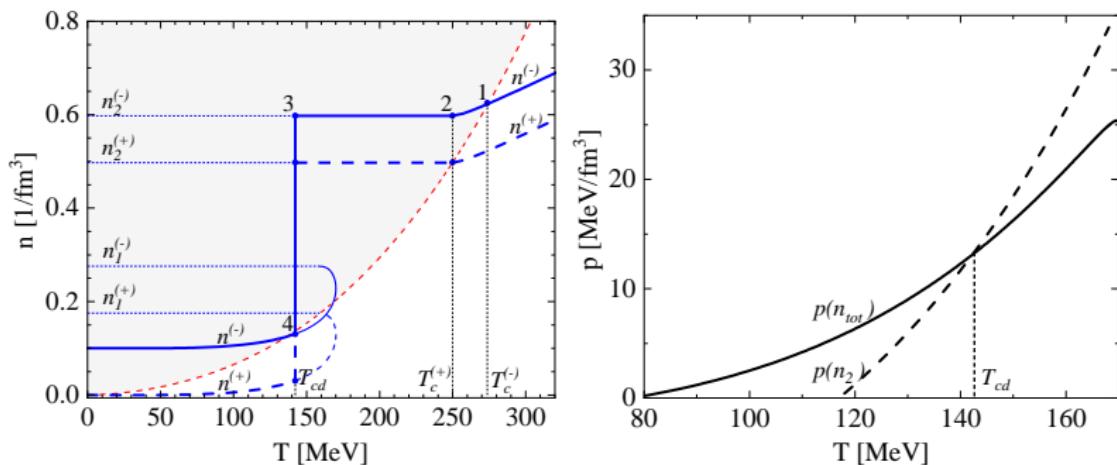
$$n = n_{\text{cond}}^{(-)}(T) + \int \frac{d^3 k}{(2\pi)^3} \left[\exp \left(\frac{\omega_k + U(n) - \mu_I}{T} \right) - 1 \right]^{-1}$$
$$+ n_{\text{cond}}^{(+)}(T) + \int \frac{d^3 k}{(2\pi)^3} \left[\exp \left(\frac{\omega_k + U(n) + \mu_I}{T} \right) - 1 \right]^{-1}$$

□ Necessary conditions for condensation of both components at the same time:

$$m + U(n) - \mu_I = 0,$$
$$m + U(n) + \mu_I = 0$$

Which leads to:

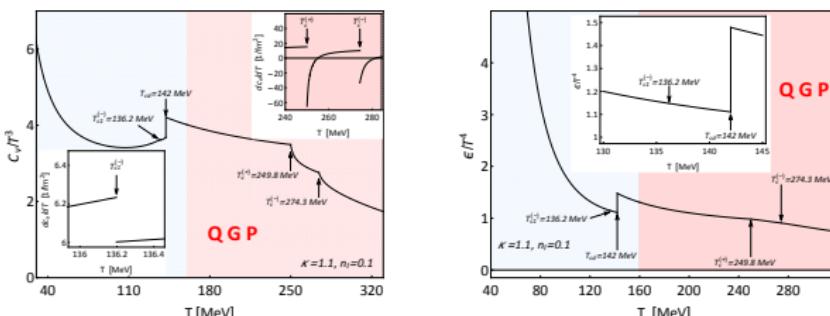
► $\mu_I = 0, \quad U(n) + m = 0$

A particle-antiparticle boson system under "strong" attraction, $\kappa > 1$ 

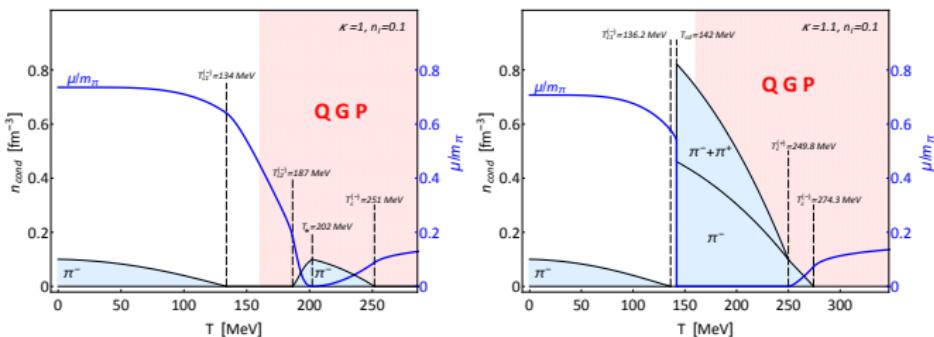
Left panel: Densities $n^{(-)}$ and $n^{(+)}$ vs temperature at $n_I = 0.1 \text{ fm}^{-3}$ and a fixed attraction parameter $\kappa = 1.1$. The quantities $n_1^{(\pm)}$, $n_2^{(\pm)}$ correspond to the roots n_1, n_2 . *Right panel:* The total pressure versus temperature.

$$U(n) + m = 0 \quad \rightarrow \quad n_2 = \sqrt{\frac{m}{B}} \left(\kappa + \sqrt{\kappa^2 - 1} \right), \quad n_1 = \sqrt{\frac{m}{B}} \left(\kappa - \sqrt{\kappa^2 - 1} \right),$$

$$\text{where} \quad n_2^{(-)} + n_2^{(+)} = n_2, \quad \kappa \equiv \frac{A}{2\sqrt{mB}}, \quad A = \kappa A_c, \quad A_c = 2\sqrt{mB}$$

Heat capacity and Energy density vs temperature, a "strong" attraction, $\kappa = 1.1$ 

Density of the condensate and the chemical potential at $n_l = 0.1 \text{ fm}^{-3}$.



The sail-like shaded area indicates the condensate states created by π^- -mesons and by π^+ -mesons at the same time. The gap of the chemical potential at $T = T_{cd}$ reflects phase transition of the first order which create the condensate of π^- and π^+ mesons.

Discussion. Grand Canonical Ensemble for the condensate phase - I ("weak" attr-n)

$$\blacktriangleright n = n_{\text{cond}}^{(-)}(T) + \left[\int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k(n), \mu_I) + \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k(n), -\mu_I) \right]_{m+U(n)-\mu_I=0},$$

$$\blacktriangleright n_I = n_{\text{cond}}^{(-)}(T) + \left[\int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k(n), \mu_I) - \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(E_k(n), -\mu_I) \right]_{m+U(n)-\mu_I=0},$$

where $E_k(n) = \omega_k + U(n)$, $U(n) = -m + \mu_I \rightarrow$

$$\blacktriangleright \langle n \rangle = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) + \int \frac{d^3k}{(2\pi)^3} \left[\exp \left(\frac{\omega_k - m + 2\mu_I}{T} \right) - 1 \right]^{-1},$$

$$\blacktriangleright \langle n_I \rangle = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T) - \int \frac{d^3k}{(2\pi)^3} \left[\exp \left(\frac{\omega_k - m + 2\mu_I}{T} \right) - 1 \right]^{-1}$$

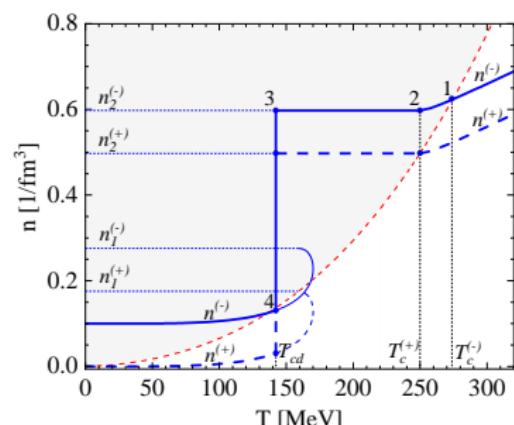
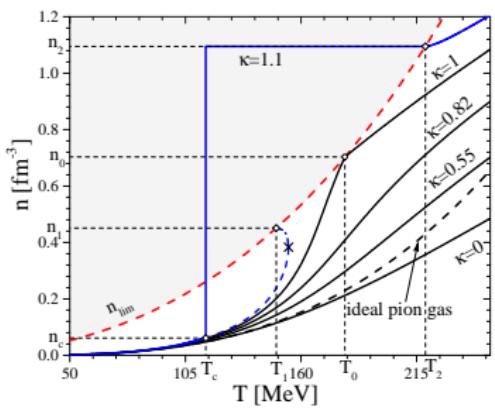
$$\blacktriangleright \langle n^{(-)} \rangle = n_{\text{cond}}^{(-)}(T) + n_{\text{lim}}(T), \quad \text{where } n_{\text{lim}}(T) = \int \frac{d^3k}{(2\pi)^3} f_{\text{BE}}(\omega_k, \mu_I) \Big|_{\mu_I=m},$$

$$\blacktriangleright \langle n^{(+)} \rangle = \int \frac{d^3k}{(2\pi)^3} \left[\exp \left(\frac{\omega_k - m + 2\mu_I}{T} \right) - 1 \right]^{-1}$$

The variables (T, μ_I) get control under $\langle n^{(+)} \rangle$.

To control the value of $\langle n^{(-)} \rangle$ is additionally required value of n_I .

Discussion. Grand Canonical Ensemble for the condensate phase - II ("strong" attr-n)

Canonical variables in the Canonical Ensemble: (T, n_I) .Canonical variables in the Grand Canonical Ensemble: (T, μ_I) .**Left panel:** Particle-number density vs temperature at $n_I = 0$,

Condensate phase : $\mu_I = 0, \rightarrow n^{(-)} = n^{(+)}$

Right panel: Particle-number density vs temperature at $n_I = 0.1 \text{ fm}^{-3}$,

Condensate phase : $\mu_I = 0, \rightarrow n^{(-)} - n^{(+)} = n_I$

Discussion and Conclusions

- The intersections of the particle density curves with the critical curve are related to 2nd order phase transitions in the system.
- At the point where a curve of particle density touches a critical curve, there exists a point-like or virtual phase transition of the 2nd order, i.e., a phase transition without setting the order parameter.
- The meson system develops a 1st order phase transition for sufficiently strong attractive interactions via forming a Bose condensate and releasing the latent heat. Prediction of the model is that a constant total density of particles characterizes the condensed phase.
- The Grand Canonical Ensemble is not suitable for describing a multi-component system in the condensate phase, even if only one of the components is in the condensate state.
GCE is unable to describe the condensate state because the chemical potential μ_i is involved in the condensate-formation conditions, so it cannot be a free variable when the system is in the condensate phase.

Selfinteracting scalar field
○○○○○○○○○○○○○○

"Weak" attraction
○○○

"Strong" attraction
○○○

Concluding remarks
○○○●○

Thank you for attention !

Function

