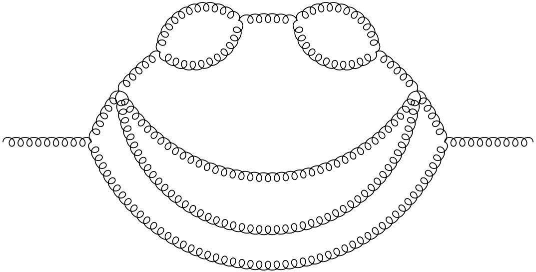
Anomalous dimensions at five loops

Andreas Maier

Thomas Luthe   Peter Marquard   York Schröder

15 December 2022



Motivation

- Anomalous dimensions are fundamental objects of quantum field theory.
- Good calculational testing ground
- Renormalisation constants needed for future calculations; gauge parameter cancellation provides important check
- Parameter evolution for precision phenomenology

Motivation

- Anomalous dimensions are fundamental objects of quantum field theory.
- Good calculational testing ground
- Renormalisation constants needed for future calculations; gauge parameter cancellation provides important check
- Parameter evolution for precision phenomenology

Example: $\Gamma(H \to b\bar{b}) \propto m_b(m_H)^2$

$$m_b(m_H) = m_b(m_b) \exp\left(\int_{\alpha_s(m_b)}^{\alpha_s(m_H)} \frac{d\alpha_s}{\alpha_s} \frac{\gamma_m}{\beta}\right)$$

5-loop running eliminates source of uncertainty

Theory framework

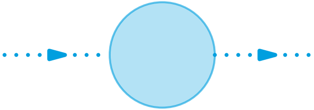
Yang-Mills theory coupled to fermions

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} + \sum_i \bar{\psi}_i \not{D}\psi_i - m\,\bar{\psi}_{N_\mathrm{f}}\psi_{N_\mathrm{f}}$$
$$+\,\partial^\mu \bar{c}^{\,a}\left(\partial_\mu c^a + g_s f^{abc} A^b_\mu c^c\right) - \frac{1}{2\xi_L}\left(\partial^\mu A_\mu\right)^2$$

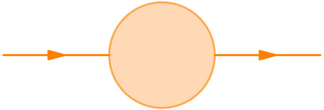
- N_f fermions (1 massive)
- General gauge group
- 3 + 2 independent renormalisation constants (anomalous dimensions):

$$\psi_0 = \sqrt{Z_2}\psi\,, \qquad A_0 = \sqrt{Z_3}A\,, \qquad c_0 = \sqrt{Z_3^c}c\,,$$
$$m_0 = Z_m m\,, \qquad g_{s,0} = \sqrt{Z_{\alpha_s}}\mu^\epsilon g_s\,.$$

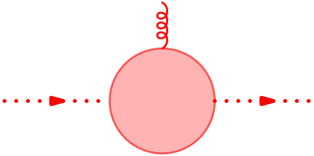
Green functions



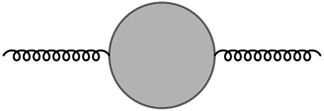
$\times\ (\sim 80\,000) \Rightarrow Z_3^c$



$\times\ (\sim 80\,000) \Rightarrow Z_2, Z_m$



$\times\ (\sim 1\,400\,000) \Rightarrow Z_1^{ccg} = \sqrt{Z_3} Z_3^c \sqrt{Z_{\alpha_s}}$



$\times\ (\sim 500\,000) \Rightarrow Z_3$

Infrared regularisation

- $\overline{\text{MS}}$ scheme: only interested in overall UV divergence
  $\rightarrow$ set (almost) all scales to 0
- Need to remove IR poles:
  - $R^\star$ operation $\Rightarrow$ 4-loop massless propagators
    [Chetyrkin, Tkachov, Smirnov 1982-85; Herzog, Ruijl 2017; Chetyrkin 2017]
  - Auxiliary mass $\Rightarrow$ 5-loop massive vacuum diagrams
    [Misiak, Münz 1994; Chetyrkin, Misiak, Münz 1997]

Infrared regularisation

- $\overline{\text{MS}}$ scheme: only interested in overall UV divergence
  $\rightarrow$ set (almost) all scales to 0
- Need to remove IR poles:
  - $R^\star$ operation $\Rightarrow$ 4-loop massless propagators

    [Chetyrkin, Tkachov, Smirnov 1982-85; Herzog, Ruijl 2017; Chetyrkin 2017]
  - Auxiliary mass $\Rightarrow$ 5-loop massive vacuum diagrams

    [Misiak, Münz 1994; Chetyrkin, Misiak, Münz 1997]

    + Straightforward
    – Genuine 5-loop calculation

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

$$\frac{1}{(k+q)^2} = \frac{1}{k^2 - M^2} - \frac{q^2 + 2kq + M^2}{(k^2 - M^2)(k+q)^2}$$

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

$$\frac{1}{(k+q)^2} = \frac{1}{k^2-M^2} - \frac{q^2+2kq+M^2}{(k^2-M^2)^2} + \frac{\left(q^2+2kq+M^2\right)^2}{(k^2-M^2)^2(k+q)^2}$$

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

$$\frac{1}{(k+q)^2} = \underbrace{\frac{1}{k^2-M^2} - \frac{q^2+2kq+M^2}{(k^2-M^2)^2}}_{\text{IR finite}} + \underbrace{\frac{\left(q^2+2kq+M^2\right)^2}{(k^2-M^2)^2(k+q)^2}}_{\text{less UV divergent}}$$

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

$$\frac{1}{(k+q)^2} = \underbrace{\frac{1}{k^2-M^2} - \frac{q^2+2kq+M^2}{(k^2-M^2)^2}}_{\text{IR finite}} + \underbrace{\frac{\left(q^2+2kq+M^2\right)^2}{(k^2-M^2)^2(k+q)^2}}_{\text{less UV divergent}}$$

- Iterate & drop UV finite terms

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

$$\frac{1}{(k+q)^2} = \underbrace{\frac{1}{k^2 - M^2} - \frac{q^2 + 2kq + M^2}{(k^2 - M^2)^2}}_{\text{IR finite}} + \underbrace{\frac{\left(q^2 + 2kq + M^2\right)^2}{(k^2 - M^2)^2 (k+q)^2}}_{\text{less UV divergent}}$$

- Iterate & drop UV finite terms
- Absorb M^2 in numerator into “gauge boson mass” counterterm $Z_{M^2} A^2$ cancels M^2 dependence in denominator

Infrared regularisation

Auxiliary mass

- Take integral with all subdivergences cancelled
- Apply exact decomposition (q external):

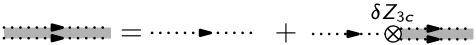
$$\frac{1}{(k+q)^2} = \underbrace{\frac{1}{k^2-M^2} - \frac{q^2+2kq+M^2}{(k^2-M^2)^2}}_{\text{IR finite}} + \underbrace{\frac{\left(q^2+2kq+M^2\right)^2}{(k^2-M^2)^2(k+q)^2}}_{\text{less UV divergent}}$$

- Iterate & drop UV finite terms
- Absorb M^2 in numerator into “gauge boson mass” counterterm $Z_{M^2}A^2$ cancels M^2 dependence in denominator
- Decomposition now corresponds to expansion around $q = 0$:

$$\frac{1}{(k+q)^2-M^2} = \frac{1}{k^2-M^2} - \frac{q^2+2kq}{(k^2-M^2)^2} + \dots$$

IR regularised propagators

$$\mathcal{D}^c(p) = \frac{i}{p^2} + \frac{i}{p^2} \cdot i\delta Z_{3c} p^2 \cdot \mathcal{D}^c(p)$$



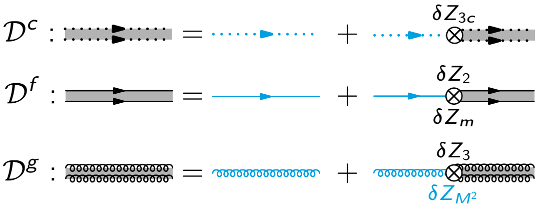
IR regularised propagators

$$\mathcal{D}^c(p) = \frac{i}{p^2 - M^2} + \frac{i}{p^2 - M^2} \cdot i\delta Z_{3c} p^2 \cdot \mathcal{D}^c(p)$$

$\mathcal{D}^c$:

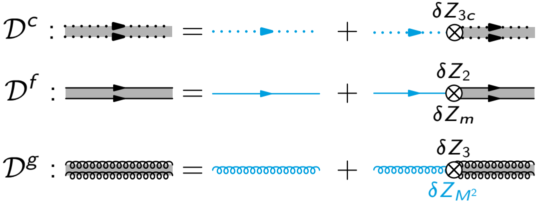
IR regularised propagators

$$\mathcal{D}^c(p) = \frac{i}{p^2 - M^2} + \frac{i}{p^2 - M^2} \cdot i\delta Z_{3c} p^2 \cdot \mathcal{D}^c(p)$$



IR regularised propagators

$$\mathcal{D}^c(p) = \frac{i}{p^2 - M^2} + \frac{i}{p^2 - M^2} \cdot i\delta Z_{3c} p^2 \cdot \mathcal{D}^c(p)$$



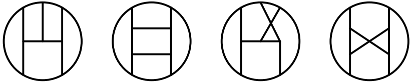
- Expand to linear order in $m \to 0$
- Expand around vanishing external momenta: massive vacuum diagrams

Calculational setup

- Generate diagrams (QGRAF [Nogueira 2006])
- Insert Feynman rules; calculate colour factors & traces, expand in q, m, ...
  (FORM [Vermaseren et al.], color [van Ritbergen, Schellekens, Vermaseren 1998])
- Reduce to master integrals via integration by parts [Chetyrkin, Tkachov 1981; Laporta 2000]
  (Crusher [Marquard, Seidel], Spades [Luthe])
  $\sim$ 35 000 000 scalar integrals $\to \sim$ 100 master integrals

$$M_i = \int dl_1 \cdots dl_5 \, \frac{1}{(k_1^2 - M^2)^{\alpha_1} \cdots (k_{15}^2 - M^2)^{\alpha_{15}}} \, , \qquad \alpha_i \in \{0, 1, 2, 3\}$$

At most 12 propagators:



Calculational setup

Master integrals

Calculation of master integrals $M_i = \int \frac{1}{D_1^{\alpha_1} \cdots D_{15}^{\alpha_{15}}}$

Integrals with ≤ 11 lines: optimised difference equations [Laporta 2000; Luthe, Schröder]

- Raise one denominator to symbolic power x:

$$M_i(x) = \int \frac{1}{D_1^{x} \cdots D_{15}^{\alpha_{15}}}$$

- Integration by parts $\Rightarrow$ Coupled first-order difference equations

$$M_i(x \pm 1) = \sum_j p_{ij}^{\pm}(d, x) M_j(x)$$

- Recurrence relations from factorial series ansatz

$$M_i(x) = \sum_{s=0}^{\infty} a_{i,s} \frac{\Gamma(x+1)}{\Gamma(x+s-d/2+1)}$$

- Evolve from boundary conditions at $x \to \infty \Rightarrow$ high-precision numerical solution

Calculational setup

Master integrals

Integrals with $N = 12$ lines: sector decomposition (FIESTA 4 [Smirnov 2016])

[Hepp 1966; Speer 1968; Binoth, Heinrich 2000; Bogner, Weinzierl 2007,. . .]

Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$

$$I(x) = \int \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}}$$

$$\propto \prod_{j=1}^{N} \left(\int_0^\infty dx_j \, x_j^{\alpha_j - 1} \right) \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{N_\alpha - 3d}}{\mathcal{F}^{N_\alpha - 5d/2}}$$

Calculational setup

Master integrals

Integrals with $N = 12$ lines: sector decomposition (FIESTA 4 [Smirnov 2016])

[Hepp 1966; Speer 1968; Binoth, Heinrich 2000; Bogner, Weinzierl 2007,...]

Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$

$$I(x) = \int \frac{1}{D_1^{\alpha_1} \cdots D_N^{\alpha_N}}$$

$$\propto \prod_{j=1}^{N} \left(\int_0^\infty dx_j \, x_j^{\alpha_j - 1} \right) \delta\left(1 - \sum_j x_j\right) \frac{\mathcal{U}^{N_\alpha - 3d}}{\mathcal{F}^{N_\alpha - 5d/2}}$$

2. Transform to sector integrals over hypercube
   - Split into *primary sectors* $s = 1, \ldots, N$ such that $x_s \geq x_j$
   - Rescale integration variables $x_j = t_j x_s$ with $t_j \in [0, 1]$

Calculational setup

Master integrals

Integrals with $N = 12$ lines: sector decomposition (FIESTA 4 [Smirnov 2016])

[Hepp 1966; Speer 1968; Binoth, Heinrich 2000; Bogner, Weinzierl 2007,...]

Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$
2. Transform to sector integrals over hypercube

$$I(x) \propto \sum_{s=1}^{N} \prod_{j=1}^{N-1} \left(\int_0^1 dt_j \, t_j^{\alpha_j - 1} \right) \frac{\mathcal{U}_s^{N_\alpha - 3d}}{\mathcal{F}_s^{N_\alpha - 5d/2}}$$

Calculational setup

Master integrals

Integrals with $N = 12$ lines: sector decomposition (FIESTA 4 [Smirnov 2016])

[Hepp 1966; Speer 1968; Binoth, Heinrich 2000; Bogner, Weinzierl 2007,...]

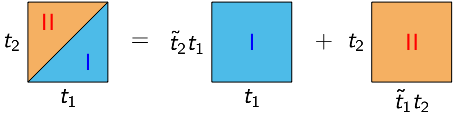
Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$
2. Transform to sector integrals over hypercube

$$I(x) \propto \sum_{s=1}^{N} \prod_{j=1}^{N-1} \left(\int_0^1 dt_j \, t_j^{\alpha_j - 1} \right) \frac{\mathcal{U}_s^{N_\alpha - 3d}}{\mathcal{F}_s^{N_\alpha - 5d/2}}$$

3. Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$

   Example: zero for $t_1 = t_2$



4. Expand around $d = 4$

Calculational setup

Master integrals

Integrals with $N = 12$ lines: sector decomposition (FIESTA 4 [Smirnov 2016])

[Hepp 1966; Speer 1968; Binoth, Heinrich 2000; Bogner, Weinzierl 2007,...]

Goal: numerical integrations over hypercube

1. Introduce Feynman parameters $\left(N_\alpha = \sum_j \alpha_j\right)$
2. Transform to sector integrals over hypercube

$$I(x) \propto \sum_{s=1}^{N} \prod_{j=1}^{N-1} \left(\int_0^1 dt_j \, t_j^{\alpha_j - 1} \right) \frac{\mathcal{U}_s^{N_\alpha - 3d}}{\mathcal{F}_s^{N_\alpha - 5d/2}}$$

3. Iteratively decompose until $\mathcal{F}_s, \mathcal{U}_s \neq 0$ for all $t_j = 0$
4. Expand around $d = 4$
5. Integrate $\Rightarrow$ low-precision numerical result

Calculational setup

Master integrals

From numerical to analytical result:

- 3 combinations of low-precision integral coefficients in renormalisation constants
  Fixed from
  1. $\frac{1}{\epsilon^2}$ poles of renormalisation constants analytically known from 4 loop
  2. Coefficient of spurious colour factor $d_{FAA}^{444} N_\mathrm{f}$ in Z_1^{ccg} vanishes
  3. Fix N_f coefficient in γ_m to SU(3) result
- Complete analytic results using PSLQ
  200 digits discovery, 250 digits confirmation

Results

Anomalous dimensions for general gauge group:

- γ_m, β at five loops

  [Luthe, Maier, Marquard, Schröder, 1612.05512 & 1709.07718]

- Gauge-dependent five-loop anomalous dimensions up to ξ^1 in expansion around Feynman gauge $\xi = 0$

  [Luthe, Maier, Marquard, Schröder 1701.07068 & 1709.07718]

- 4-loop anomalous dimensions in general covariant gauge

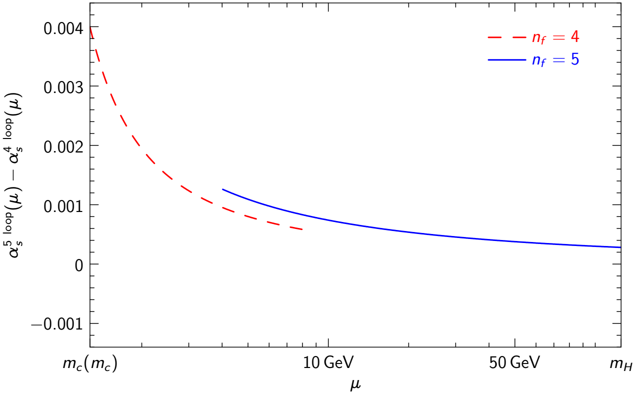
  [Luthe, Maier, Marquard, Schröder 1701.07068]

Agreement with other approaches

[Baikov, Chetyrkin, Kühn; Herzog, Ruijl, Ueda, Vermaseren, Vogt; Chetyrkin, Falcioni, Herzog, Vermaseren]

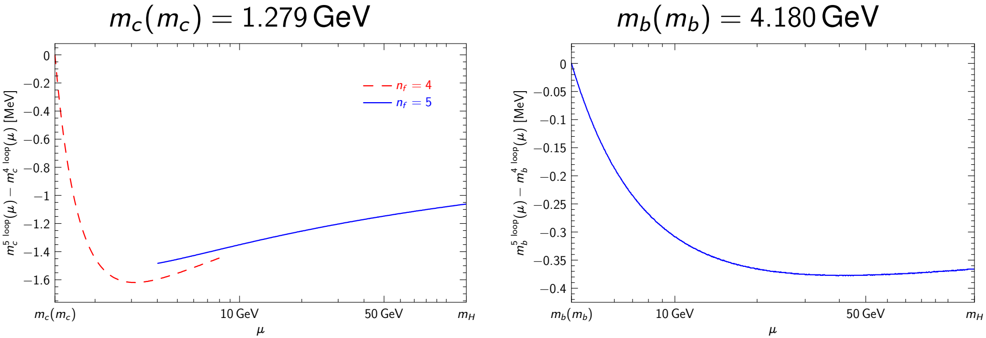
Strong coupling evolution

Difference between four- and five-loop evolution from $\alpha_s^{(n_f=3)}(m_\tau) = 0.328$:



Heavy quark mass evolution

Difference between four- and five-loop evolution from



- Uncertainty at $\mu = m_H$ under control:
  - $\Delta m_c(m_H) = 1.1\,\text{MeV}$
  - $\Delta m_b(m_H) = 0.4\,\text{MeV}$

Conclusions

- Computing anomalous dimensions at five loops is feasible[1]
- Well-established method
  e.g. Abelian Higgs model at four loops [Ihrig, Zerf, Marquard, Herbut, Scherer 2019]
- Evolution of strong coupling and quark masses is well under control

[1] Barring problems with γ_5

Backup

Conventions

Compact notation for colour factors:

$$a = \frac{C_A\, g_s^2}{16\pi^2}\,, \qquad n_f = \frac{N_\mathrm{f}\, T_\mathrm{F}}{C_\mathrm{A}}\,, \quad c_f = \frac{C_\mathrm{F}}{C_\mathrm{A}}\,,$$

$$d_1 = \frac{[\mathrm{sTr}(T^a T^b T^c T^d)]^2}{D_\mathrm{A} T_\mathrm{F}^2 C_\mathrm{A}^2}\,,$$

$$d_2 = \frac{\mathrm{sTr}(T^a T^b T^c T^d)\ \mathrm{sTr}(F^a F^b F^c F^d)}{D_\mathrm{A} T_\mathrm{F} C_\mathrm{A}^3}\,,$$

$$d_3 = \frac{[\mathrm{sTr}(F^a F^b F^c F^d)]^2}{D_\mathrm{A} C_\mathrm{A}^4}$$

β function

$$\beta = -a\left(b_0 + b_1\, a + b_2\, a^2 + b_3\, a^3 + b_4\, a^4 + \ldots\right),$$

$$3^5\, b_4 = b_{44}\, n_f^4 + b_{43}\, n_f^3 + b_{42}\, n_f^2 + b_{41}\, n_f + b_{40}$$

$$\begin{aligned}
b_{44} &= -8(107 + 144\zeta_3)c_f + 4(229 - 480\zeta_3)\,, \\
b_{43} &= -6(4961 - 11424\zeta_3 + 4752\zeta_4){c_f}^2 - 48(46 + 1065\zeta_3 - 378\zeta_4)c_f \\
&\quad + 1728(55 - 123\zeta_3 + 36\zeta_4 + 60\zeta_5)d_1 \\
&\quad - 3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)\,, \\
b_{42} &= -54(2509 + 3216\zeta_3 - 6960\zeta_5){c_f}^3 \\
&\quad + 9(94749/2 - 28628\zeta_3 + 10296\zeta_4 - 39600\zeta_5){c_f}^2 \\
&\quad + 25920(13 + 16\zeta_3 - 40\zeta_5)c_f\, d_1 \\
&\quad + 3(5701/2 + 79356\zeta_3 - 25488\zeta_4 + 43200\zeta_5)c_f \\
&\quad - 864(115 - 1255\zeta_3 + 234\zeta_4 + 40\zeta_5)d_2 \\
&\quad - 432(1347 - 2521\zeta_3 + 396\zeta_4 - 140\zeta_5)d_1 \\
&\quad + 843067/2 + 166014\zeta_3 - 8424\zeta_4 - 178200\zeta_5\,,
\end{aligned}$$

β function

$$
\begin{aligned}
b_{41} = & -81(4157/2 + 384\zeta_3)c_f{}^4 + 81(11151 + 5696\zeta_3 - 7480\zeta_5)c_f{}^3 \\
& - 3(548732 + 151743\zeta_3 + 13068\zeta_4 - 346140\zeta_5)c_f{}^2 \\
& - 25920(3 - 4\zeta_3 - 20\zeta_5)c_f d_2 \\
& + (8141995/8 + 35478\zeta_3 + 73062\zeta_4 - 706320\zeta_5)c_f \\
& + 216(113 - 2594\zeta_3 + 396\zeta_4 + 500\zeta_5)d_3 \\
& + 216(1414 - 15967\zeta_3 + 2574\zeta_4 + 8440\zeta_5)d_2 \\
& - 5048959/4 + 31515\zeta_3 - 47223\zeta_4 + 298890\zeta_5\,, \\
b_{40} = & -162(257 - 9358\zeta_3 + 1452\zeta_4 + 7700\zeta_5)d_3 \\
& + 8296235/16 - 4890\zeta_3 + 9801\zeta_4/2 - 28215\zeta_5
\end{aligned}
$$

Fermion mass anomalous dimension

$$\gamma_m(a) = -\, c_f\, a \left(3 + \gamma_{m1}\, a + \gamma_{m2}\, a^2 + \gamma_{m3}\, a^3 + \gamma_{m4}\, a^4 + \ldots\right),$$

$$6^5\, \gamma_{m4} = \gamma_{m44}\, (4n_f)^4 + \gamma_{m43}\, (4n_f)^3 + \gamma_{m42}\, (4n_f)^2 + \gamma_{m41}\, (4n_f) + \gamma_{m40}$$

$$\begin{aligned}
\gamma_{m44} &= -\, 6(65 + 80\zeta_3 - 144\zeta_4)\,, \\
\gamma_{m43} &= 3(4483 + 4752\zeta_3 - 12960\zeta_4 + 6912\zeta_5)c_f \\
&\quad + (18667/2 + 32208\zeta_3 + 29376\zeta_4 - 55296\zeta_5)\,, \\
\gamma_{m42} &= 9(45253 - 230496\zeta_3 + 48384\zeta_3^2 + 70416\zeta_4 \\
&\qquad + 144000\zeta_5 - 86400\zeta_6)c_f{}^2 \\
&\quad + (375373 + 323784\zeta_3 - 1130112\zeta_3^2 + 905904\zeta_4 \\
&\qquad - 672192\zeta_5 + 129600\zeta_6)c_f \\
&\quad - 864(431 - 1371\zeta_3 + 432\zeta_4 + 420\zeta_5)d_1 \\
&\quad + 4(13709 + 394749\zeta_3 + 173664\zeta_3^2 - 379242\zeta_4 \\
&\qquad - 119232\zeta_5 + 162000\zeta_6)\,,
\end{aligned}$$

Fermion mass anomalous dimension

$$
\begin{aligned}
\gamma_{m41} = & -54(48797 - 247968\zeta_3 + 24192\zeta_4 + 444000\zeta_5 - 241920\zeta_7){c_f}^3 \\
& - 18(406861 + 216156\zeta_3 - 190080\zeta_3^2 + 254880\zeta_4 - 606960\zeta_5 \\
& \quad - 475200\zeta_6 + 362880\zeta_7){c_f}^2 \\
& - 62208(11 + 154\zeta_3 - 370\zeta_5) c_f d_1 \\
& + (753557 + 15593904\zeta_3 - 3535488\zeta_3^2 - 6271344\zeta_4 \\
& \quad - 17596224\zeta_5 + 1425600\zeta_6 + 1088640\zeta_7) c_f \\
& + 1728(3173 - 6270\zeta_3 + 1584\zeta_3^2 + 2970\zeta_4 - 13380\zeta_5) d_1 \\
& + 1728(380 - 5595\zeta_3 - 1584\zeta_3^2 - 162\zeta_4 + 1320\zeta_5) d_2 \\
& - 2(4994047 + 11517108\zeta_3 - 57024\zeta_3^2 - 5931900\zeta_4 \\
& \quad - 15037272\zeta_5 + 4989600\zeta_6 + 3810240\zeta_7) \,,
\end{aligned}
$$

Fermion mass anomalous dimension

$$
\begin{aligned}
\gamma_{m40} = {} & 972(50995 + 6784\zeta_3 + 16640\zeta_5){c_f}^4 \\
& - 54(2565029 + 1880640\zeta_3 - 266112\zeta_4 - 1420800\zeta_5){c_f}^3 \\
& + 108(2625197 + 1740528\zeta_3 - 125136\zeta_4 - 2379360\zeta_5 \\
& \qquad - 665280\zeta_7){c_f}^2 \\
& + 373248(141 + 80\zeta_3 - 530\zeta_5)c_f d_2 \\
& - 8(25256617 + 16408008\zeta_3 + 627264\zeta_3^2 - 812592\zeta_4 \\
& \qquad - 40411440\zeta_5 + 3920400\zeta_6 - 5987520\zeta_7)c_f \\
& - 6912(9598 + 453\zeta_3 + 4356\zeta_3^2 + 1485\zeta_4 - 26100\zeta_5 \\
& \qquad - 1386\zeta_7)d_2 \\
& + 5184(537 + 2494\zeta_3 + 5808\zeta_3^2 + 396\zeta_4 - 7820\zeta_5 - 1848\zeta_7)d_3 \\
& + 4(22663417 + 10054464\zeta_3 + 1254528\zeta_3^2 - 1695276\zeta_4 \\
& \qquad - 41734440\zeta_5 + 7840800\zeta_6 + 5987520\zeta_7)
\end{aligned}
$$

Gauge-dependent anomalous dimensions

Ghost-gluon vertex

$$
\begin{aligned}
\gamma_1^{ccg} &= -a(1-\xi)\left[\tfrac{1}{2} + \tfrac{6-\xi}{8}\, a + \gamma_{12}^{ccg} a^2 + \gamma_{13}^{ccg} a^3 + \gamma_{14}^{ccg} a^4 + \ldots\right], \\
2^{14}\, 3^5\, \gamma_{14}^{ccg} &= \gamma_{143}^{ccg}\,(16n_f)^3 + \gamma_{142}^{ccg}\,(16n_f)^2 + \gamma_{141}^{ccg}\,(16n_f) + \gamma_{140}^{ccg}\,, \\
\gamma_{14\{0,1,2\}}^{ccg} &= \gamma_{14\{0,1,2\}0}^{ccg} + \xi\, \gamma_{14\{0,1,2\}1}^{ccg} + \mathcal{O}(\xi^2)\,, \\
\gamma_{1421}^{ccg} &= 2(7855 - 22464\zeta_3 + 3240\zeta_4)\,, \\
\gamma_{1411}^{ccg} &= 93312(56 - 31\zeta_3 - 14\zeta_4)c_f \\
&\quad + 5184(820\zeta_3 - 78\zeta_3^2 - 171\zeta_4 - 660\zeta_5 + 225\zeta_6)d_3 \\
&\quad + 216(1247753/108 + 24604\zeta_3 - 66\zeta_3^2 + 1491\zeta_4 \\
&\qquad - 8760\zeta_5 + 1575\zeta_6)\,, \\
\gamma_{1401}^{ccg} &= 5184(1986 - 74900\zeta_3 + 4992\zeta_3^2 + 10044\zeta_4 + 52440\zeta_5 - 24000\zeta_6 + 25137\zeta_7)d_3 \\
&\quad - 216(39394519/54 + 616864\zeta_3 + 11472\zeta_3^2 - 36984\zeta_4 \\
&\qquad - 718836\zeta_5 + 81300\zeta_6 + 29925\zeta_7)
\end{aligned}
$$