Equation of state at not-too-large $\hat{\mu}_B$ from lattice QCD

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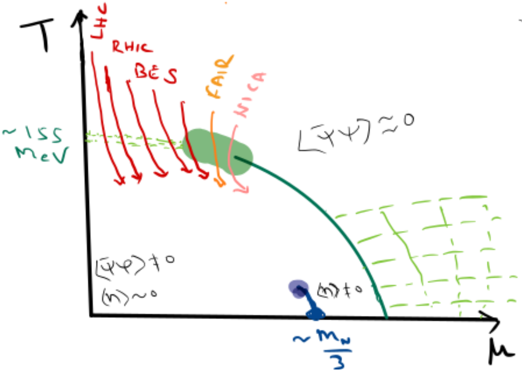
March 31, 2023

Introduction



While LHC explores QGP at $\hat{\mu}_B \sim 0$, experiments like BES at RHIC, CBM at FAIR, NICA at JINR explore strongly interacting matter at finite $\hat{\mu}_B$.

For analyzing them, one needs EoS at finite $(T, \hat{\mu}_B)$.

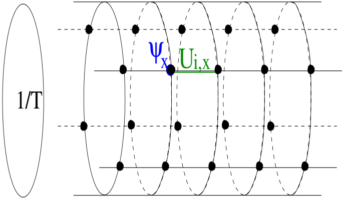
Critical point in $(T, \hat{\mu}_B)$ plane?

Neutron star explores physics at large $\hat{\mu}_B$, small T.

QCD at finite μ, T: Lattice

$$\mathcal{Z} = \int \mathcal{D}\mathcal{U}\,\mathcal{D}(\bar\Psi,\Psi)\; e^{-S_G(U) + \int \bar\Psi\,(\not{D}+m)\,\Psi + \mu\int \bar\Psi\,\gamma_0\,\gamma_5\,\Psi}$$
$$= \int \mathcal{D}\mathcal{U}\; e^{-S_G(U)}\;\det\left(\not{D} + m + \mu\gamma_0\right)$$

For Monte Carlo simulations, we would like to use the integrand as a probability function: must be non-negative.



$$\gamma_5\,(\not{D} + m)\,\gamma_5 = (\not{D} + m)^\dagger$$
$$\Rightarrow \det(\not{D} + m) = \det(\not{D} + m)^\dagger$$

So for two degenerate flavors, we get

$$\text{Determinant} = |\det(\not{D} + m)|^2$$

Sign problem

- Does not work in presence of μ:

$$\gamma_5 \ (\not{D} + m + \mu\gamma_0) \ \gamma_5 = (\not{D} + m - \mu\gamma_0)^\dagger$$

Sign problem!

Not possible to do numerical Monte Carlo simulations in the presence of a baryon number chemical potential.

- Note that for imaginary $\mu = i\mu_i$, no sign problem.
- For a chiral density chemical potential $\mu_5\gamma_0\gamma_5$, no sign problem.

  Though the axial current is not conserved.
- For an isospin chemical potential $\hat{\mu}_I \gamma_0 \frac{\tau_3}{2}$,

$$\text{Determinant} = \det\left(\not{D} + m + \frac{\hat{\mu}_I}{2}\gamma_0\right)\left(\not{D} + m - \frac{\hat{\mu}_I}{2}\gamma_0\right)$$

$$\gamma_5 \left(\not{D} + m + \frac{\hat{\mu}_I}{2}\gamma_0\right) \gamma_5 = \left(\not{D} + m - \frac{\hat{\mu}_I}{2}\gamma_0\right)^\dagger$$

$\Rightarrow$ no sign problem.

Methods at finite $\hat{\mu}_B$

- Calculate observables at small $\hat{\mu}_B$ by doing a Taylor expansion in $\hat{\mu}_B$, whose coefficients can be evaluated at $\hat{\mu}_B$=0.
  Gavai & Gupta (2003), Allton et al. (2003)
- Multiparameter reweighting:

$$Z(\beta,\mu) \;=\; \int \mathcal{D}\mathcal{U}\; e^{-S_G(\beta')}\;\det[\beta',\hat{\mu}_B=0]\times\left\{e^{-\left(S_G(\beta)-S_G(\beta')\right)}\;\frac{\det[\beta,\hat{\mu}_B]}{\det[\beta',\hat{\mu}_B=0]}\right\}$$

  Fodor & Katz (2002)
- Use simulations at imaginary $\hat{\mu}_B = i\mu_i$:
  - Analytical continuation to $(\hat{\mu}_B)_r$.
    de Forcrand & Philipsen (2002)
  - Find the Taylor expansion coefficients.
    D'Elia, Gagliardi, Sanfilippo (2017)
  - Taylor expansion around $i\mu_i$.
    Borsanyi, et al., PRL 126 (2021) 232001.
    Dimopoulos, et al., PRD 105 (2022) 034513.
- Other methods, like complex Langevin, Lefschetz thimbles, have given interesting results on simpler systems.

Taylor Expansion in $\hat{\mu}_B$

Calculate observables at small $\hat{\mu}_B$ by expanding in $\hat{\mu}_B$:

$$P(\hat{\mu}_B, T) = \sum_n \frac{\hat{\mu}_B^n}{n!} \chi_n^B$$

where $P(\hat{\mu}_B = 0, T) = \chi_0^B$ and the generalized baryon number susceptibilities $\chi^B_{n=2,4,6,...}$ are obtained from derivatives of P.

Gavai & Gupta 2003; Allton et al. 2003

One can write the series as an expansion for baryon susceptibility

$$\chi_B \equiv \chi_{2B} = \frac{\partial^2 P}{\partial \mu_B^2}$$

The convergence of the above series has been suggested as a way to look for the QCD critical point in $(\hat{\mu}_B, T)$ plane.

Gavai & Gupta 2005

Caveat: Only 3-4 terms in series, higher coefficients noisier.

Quark number susceptibilities

One can introduce a chemical potential for each flavor of quark; e.g., for two flavors,

$$\Delta P(\hat{\mu}_B, T) \;=\; P(\hat{\mu}_B, T) \;-\; P(0, T) \;=\; \sum_{n_u n_d} \chi_{n_u n_d} \, \frac{\mu_u^{n_u}}{n_u!} \, \frac{\mu_d^{n_d}}{n_d!}$$

The generalized Quark number susceptibilities (QNS)

$$\chi_{n_u\, n_d} = \frac{\partial^{n_u + n_d} P}{\partial \mu_u^{n_u} \, \partial \mu_d^{n_d}}$$

can be easily connected to susceptibilities of baryon number and charge quantum number.
For three flavors, one has also μ_s.
Free theory:

$$P(T, \{\mu_f\};\, m_f = 0) \;=\; 2(N_c^2 - 1)\, \tfrac{\pi^2}{90}\, T^4 \;+\; 4\, N_c\, N_f \left[\tfrac{7}{8}\, \tfrac{\pi^2}{90}\, T^4 \;+\; \tfrac{\mu_f^2 T^2}{24} \;+\; \tfrac{\mu_f^4}{48\pi^2} \right]$$

Quark number susceptibilities

What do susceptibilities tell us?

- $\chi_{n_u\,n_d}$ have been widely studied in the lattice for understanding properties of QGP, long predating the finite μ_q usage.
  Gottlieb et al, PRL 59 ('87) 2247.
  Gavai & Gupta, PRD 68 ('03) 034506; Allton et al., PRD 68 ('03) 014507.
- χ_{20} is connected to fluctuation of baryon number, and shows a rise at T_c.
- Correlation observables like χ_{11} can be used to explore the nature of the QGP phase.
- Similarly, higher order susceptibilities show very characteristics properties at T_c, associated with the deconfining or chiral symmetry restoring nature of the transition.
  Ejiri, Karsch, Redlich (2006)
- Alternately, one can use the chemical potentials for baryon number, electric charge and strangeness number.

Quark number susceptibilities

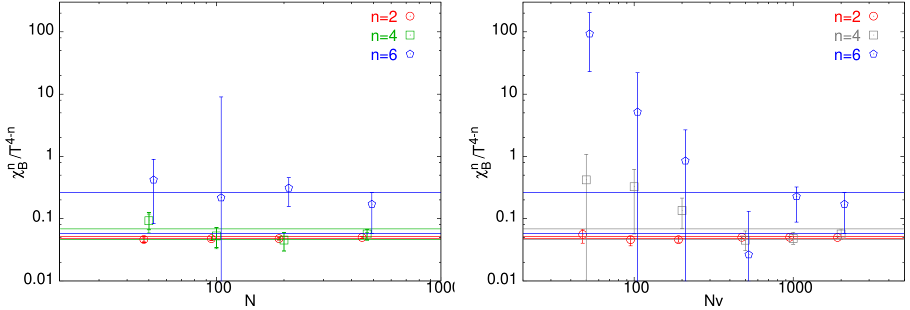
- χ_n^B and their ratios can be connected to event-by-event fluctuations of baryon numbers: have been used for phenomenology of heavy ion collisions, in particular, to study freezeout.

  Gavai & Gupta, PLB (2011). Gupta et al., Science (2012). Bazavov et al., PRL (2012); Borsanyi et al. (2013)

- Calculation of higher order terms complicated.
- Derivatives introduce terms like $\mathrm{Tr}(M^{-1}M)^n$, where M is the quark matrix.
- Higher terms: many inversions. Accuracy difficult to maintain.
- The problem becomes more severe as one goes to lighter quarks.

Statistics, statistics...

One needs to accurately calculate susceptibilities of various order. Requires high statistics, and large number of vectors to invert.

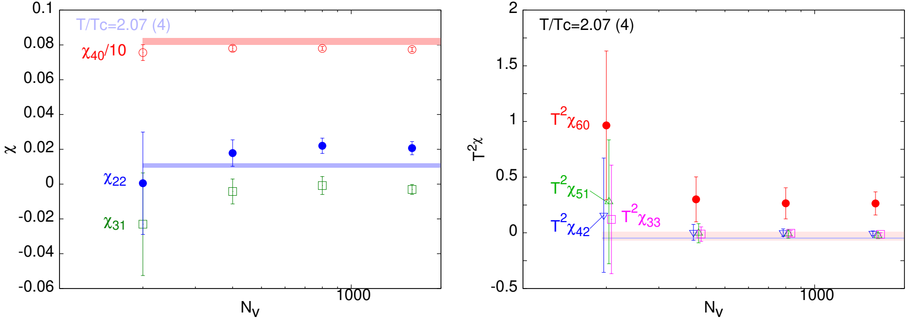


Datta, Gavai & Gupta, PRD 95 (2017) 054512

The problem becomes more severe as one goes to lighter quarks.

Statistics, statistics...

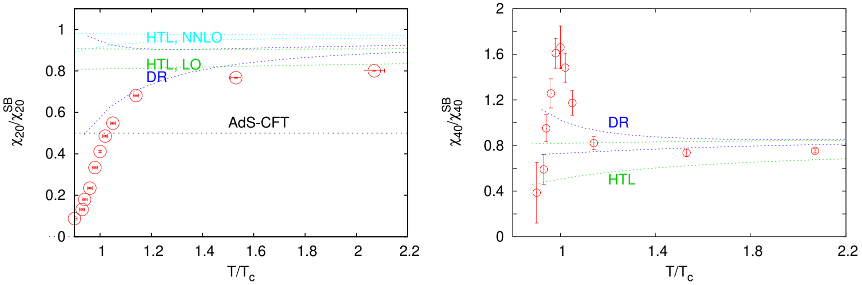
One needs to accurately calculate higher order susceptibilities.
Higher order susceptibilities are more difficult.



Datta, Gavai & Gupta, PRD 95 (2017) 054512

The problem becomes more severe as one goes to lighter quarks.

$\chi_{20,40}$

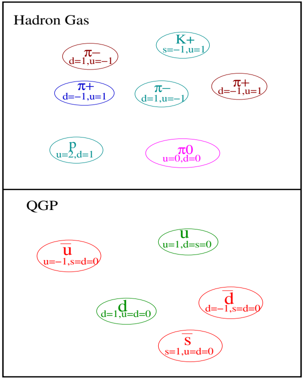


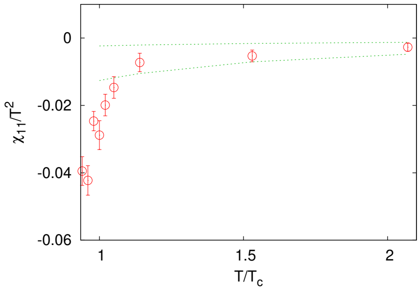
Datta, Gavai, Gupta, 2017, ICNFP2018 (MDPI Proc 13 (2019) 5).
LO HTL, DR: Andersen,Mogliacci,Strickland,Su,Vuorinen, JHEP 12 (2013) 055.
NLO HTL: Haque, Mustafa, Strickland, JHEP 07 (2013) 184.

- Comes to $\sim$ 20% of free theory by 2 T_c. χ_{40} in good agreement with perturbation theory, χ_{20} a little below.
- Calculations with standard staggered at $N_t = 8$ lattices: cutoff effect.
- χ_{40} peak at $T \sim T_c$.

Quantities like χ_{ud} probe correlations between quantum numbers: very different between bound state phase and free quarks.

Koch, Majumdar & Randrup, PRL 95(05) 182301





Datta, Gavai, Gupta (2017)
DR: Andersen, et al., JHEP (2013)

Leading contribution at $g^6 \log g$

In heavy ion collision experiments, we have three chemical potentials to take care of, corresponding to the baryon number, charge, and strangeness number.
We can write a similar Taylor series in the corresponding chemical potentials:

$$\frac{P}{T^4} = \sum_{lmn} \frac{\chi^{BQS}_{lmn}}{l!\, m!\, n!}\, \hat{\mu}_B^l\, \hat{\mu}_Q^m\, \hat{\mu}_S^n$$

Alternately, one could have used chemical potentials for u, d, s quark number.

$$\begin{aligned} \mu_u &= \frac{1}{3}\hat{\mu}_B + \frac{2}{3}\hat{\mu}_Q \\ \mu_d &= \frac{1}{3}\hat{\mu}_B - \frac{1}{3}\hat{\mu}_Q \\ \mu_s &= \frac{1}{3}\hat{\mu}_B - \frac{1}{3}\hat{\mu}_Q - \hat{\mu}_S \end{aligned}$$

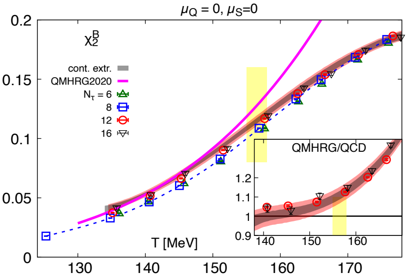
$\hat{\mu}_{BQS}$

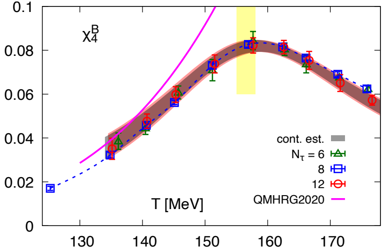
In heavy ion collisions,

$$n_S = 0 \qquad \text{and} \qquad \frac{n_Q}{n_B} = \frac{Z}{A} \approx 0.4$$

These can be ensured by tuning $\hat{\mu}_S$ and $\hat{\mu}_Q$.
On the other hand, things get simplified (with very little change in result) if one takes $\hat{\mu}_S = 0$ and $\hat{\mu}_Q = 0$ (isospin symmetric: $\frac{n_Q}{n_B} = 0.5$).
Continuum, high statistics results for χ^B have been obtained by HotQCD and Budapest-Wuppertal collaboration in recent years.
In next few slides, I will show some recent HotQCD results. They use HISQ (highly improved staggered quark) discretization, set $\hat{\mu}_Q$=0, and give results for both $n_S = 0$ and $\hat{\mu}_S$=0.

Quark number Susceptibilities

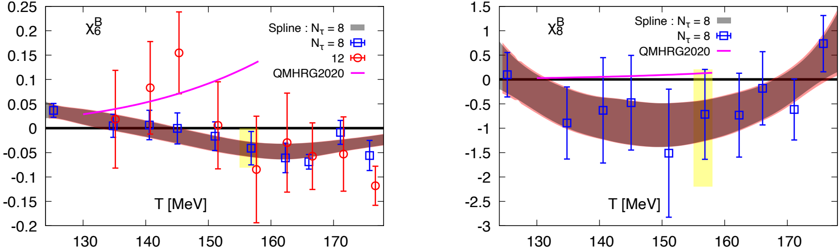




HotQCD (Bollweg, et al.), PRD 105 (2022) 074511.

- Calculation of higher order susceptibilities more difficult (look at the N_τ values).
- Deviation from HRG more prominent for higher order susceptibilities.
- For $T > 250$ MeV, deviation of $\chi^B_{2,4}$ from ideal gas value $\sim$ 10-15 % : explained by PT at $\mathcal{O}(g^2)$.

Quark number Susceptibilities



HotQCD (Bollweg, et al.), PRD 105 (2022) 074511.

- Calculation of higher order susceptibilities more difficult (look at the N_τ values).
- Deviation from HRG more prominent for higher order susceptibilities.
- $\chi^B_{6,8}$ vanish in free theory and also at $\mathcal{O}(g^2)$.

Equation of state at finite $\hat{\mu}_B$

From the susceptibilities and their derivatives, we can get the equations of state, e.g, pressure, number density

$$\frac{n_B}{T^3} = \frac{\partial\left(P/T^4\right)}{\partial\hat{\mu}_B}$$

and the energy density

$$\begin{aligned}\frac{\epsilon}{T^4} &= \left(T\frac{\partial}{\partial T} + 3\right)\frac{P}{T^4} \cdot \\ &= \sum_{lmn} \frac{3\,\chi^{BQS}_{lmn} + (\chi^{BQS}_{lmn})'}{l!\,m!\,n!}\,\hat{\mu}_B^l\,\hat{\mu}_Q^m\,\hat{\mu}_S^n, \qquad (\chi^{BQS}_{lmn})' = T\frac{d\chi^{BQS}_{lmn}}{dT}\end{aligned}$$

To get equation of state or susceptibilities at finite $\hat{\mu}_B$, we need to sum the series.
Entropy density:

$$Ts = \epsilon + P - \sum_{k=B,Q,S} \mu_k\, n_k$$

EoS at n_s=0

HotQCD has recently given detailed results on EoS at the more physical situation of n_s=0 (keeping $\hat{\mu}_Q = 0$).

HotQCD (Bollweg, et al.), 2212.09043

$$n_S = \frac{\partial P}{\partial \hat{\mu}_S} = 0 \Rrightarrow \hat{\mu}_S = c_1(T)\,\hat{\mu}_B + c_2(T)\,\hat{\mu}_B^3 + \ldots$$

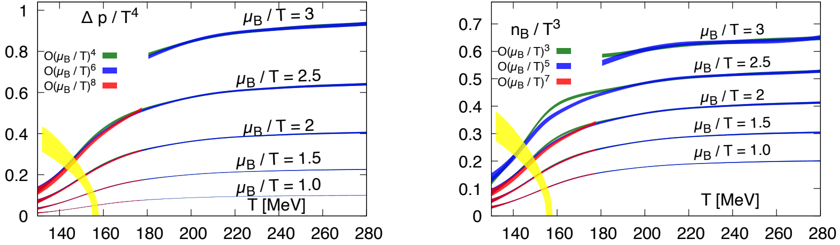
Using this, we can rewrite the Taylor expansion of P as

$$\frac{\Delta P}{T^4} = \sum_k \mathcal{P}_{2k}\,\hat{\mu}_B^{2k}$$

Behavior of $\mathcal{P}_{2k}$ very similar to those of χ^B_{2k}.
This simplifies the calculation: n_B, energy density, entropy can be calculated from the $\mathcal{P}_{2k}$ and their temperature derivatives.

$$\frac{n_B}{T^3} = \sum_k 2k\,\mathcal{P}_{2k}\,\hat{\mu}_B^{2k}, \qquad \frac{\Delta\epsilon}{T^4} = \sum_k \left(3\mathcal{P}_{2k} + T\frac{d\mathcal{P}_{2k}}{dT}\right)\hat{\mu}_B^{2k}$$

Convergence of Taylor series



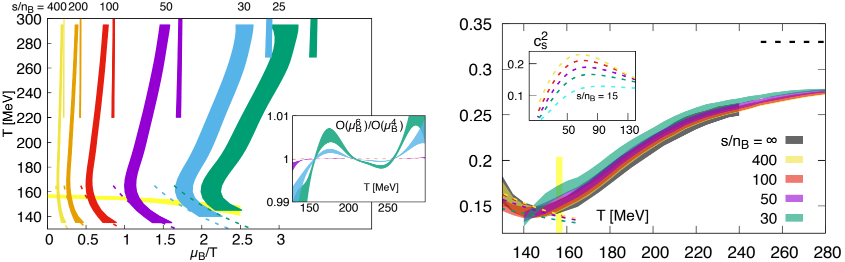
HotQCD, 2212.09043

- For $T > 200$ GeV, reliable results upto $\hat{\mu}_B \sim 3T$. $\mathcal{P}_{6,8}$ negligible.
- Lower temperature more difficult. Can get pressure for $\hat{\mu}_B \lesssim 2.5T$ but n_B only for $\hat{\mu}_B \lesssim 2T$.
- One can invert the Taylor series to get $\hat{\mu}_B(n_B)$ and therefore, $p(n_B)$: convergence set by the convergence of n_B.

Isentropic trajectory, speed of sound

In heavy ion collisions, we expect the system to evolve along isentropic trajectories. One can get them from $s(n_B)$.
Another important quantity is the speed of sound,

$$c_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_{s/n_B, n_S} = \left.\frac{\partial p/\partial T}{\partial \epsilon/\partial T}\right|_{s/n_B, n_S}$$



HotQCD, 2212.09043

Summary

- Nonperturbative information about equation of state at finite $\hat{\mu}_B$ difficult, due to the sign problem.
- Various approximation methods, however, have been devised.
- Using a Taylor expansion in $\hat{\mu}_B$, one can get reliable results for various thermodynamic quantities for $\hat{\mu}_B \lesssim 2T$.
- Interesting results have been obtained at large $\hat{\mu}_I$. In some region of the pion condensate phase, the speed of sound c_s^2 has been found to exceed $1/3$.

Brandt, Cuteri & Endrodi, arXiv:2212.14016

- Based on their Taylor expansion results, HotQCD do not find evidence of a critical point in $(T, \hat{\mu}_B)$ plane below $\hat{\mu}_B = 2.5T$.
- Multipoint Pade from imaginary chemical potential seems to be a new tool for looking at the critical point.

Dimopoulos, et al., 2110.15933. Schmidt, talk in ICPAQGP