Shock Waves in (1+1 Dimensional) Curved Space-time (GR Shock Waves)

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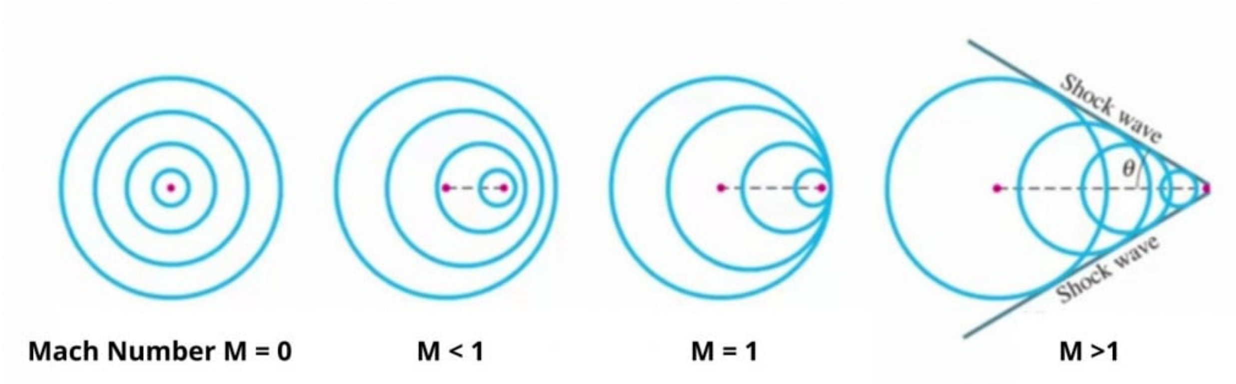
Workshop on the QCD equation of state in dense matter HIC and Astrophysics

Talk Outline

Introduction

- Shock wave $\Rightarrow$ Supersonic disturbances in the medium.
- In daily life we can encounter it when airplanes break through the sound barrier.
- Shock waves can also be formed in many known astrophysical events like supernova collapse, when the stellar wind encountering medium, etc.



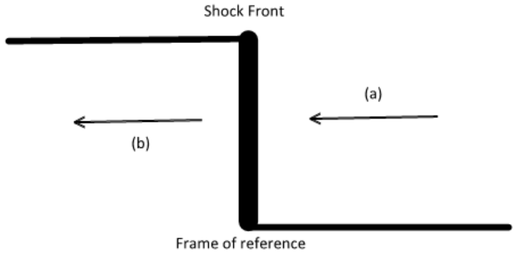
Source: scienceabc.com

- Across a shock front, there is always an abrupt change in the thermodynamic quantities.

Hydrodynamic Equations in Non-Relativity (Newtonian)

- Three basic conservation equations: Mass, Momentum, and Energy.
- General form of continuity or conservation equation:
  $\partial_0(density\ of\ the\ quantity) + \partial_1(flux\ of\ the\ quantity) = 0$

$$\partial_0\rho + \partial_1(\rho v) = 0$$
$$\partial_0(\rho v) + \partial_1(p + \rho v^2) = 0$$
$$\partial_0(\rho e + \frac{1}{2}\rho v^2) + \partial_1(v(\rho e + \frac{1}{2}\rho v^2 + p)) = 0$$



- From the shock frame of reference these equations become jump conditions also known as Rankine-Hugoniot equations.
- Using these jump conditions along with an equation of state we can study the NR shocks and their application.

Rankine-Hugoniot Equations to Study NR Shocks

- Conservation of particle Number or mass flux gives
  $\rho_a v_a = \rho_b v_b$
- Conservation of Momentum flux gives
  $\rho_a v_a^2 + p_a = \rho_b v_b^2 + p_b$
- Conservation of Energy flux gives
  $w_a + \frac{1}{2}v_a^2 = w_b + \frac{1}{2}v_b^2$
- Using these three equations we can derive a velocity-free equation known as the combustion adiabat equation.

$$w_b - w_a = \frac{1}{2}\left(\frac{1}{\rho_b} + \frac{1}{\rho_a}\right)(p_b - p_a)$$

- ***Velocity of upstream and downstream matter***
  - $v_a = \sqrt{\frac{(p_b - p_a)\rho_b}{(\rho_b - \rho_a)\rho_a}}$
  - $v_b = \sqrt{\frac{(p_b - p_a)\rho_a}{(\rho_b - \rho_a)\rho_b}}$

Hydrodynamic Equations in General Relativity (Einstein's)

- In astrophysical scenarios, shock waves propagate at speed comparable to the speed of light.
- Generally curved space-time conservation equations take form like

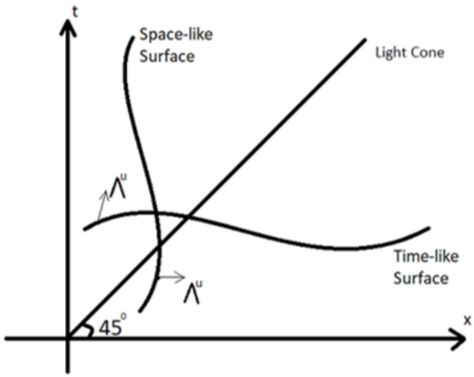
$$\nabla_\mu (n u^\mu) = 0 \tag{1}$$

$$\nabla_\nu T^{\mu\nu} = 0 \tag{2}$$

- 1st equation is particle number conservation.
- 2nd equation contains energy and momentum conservation.
- Using energy-momentum tensor and four-velocity, we find the hydrodynamic equations.

Special Relativistic

- In special relativistic case, $\nabla_\mu \rightarrow \partial_\mu$
- $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\Phi^2)$
- $u^\mu = (\gamma, \gamma v, 0, 0)$
  where, $\gamma = \frac{1}{\sqrt{1-v^2}}$
- $T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$
  where, $w = \epsilon + p$
- Using eqn 1 and 2, we can write dynamic equations as,



$$\partial_0(nu^0) + \partial_1(nu^1) = 0$$
$$\partial_0 T^{10} + \partial_1 T^{11} = 0$$
$$\partial_0 T^{00} + \partial_1 T^{01} = 0$$

$$\Lambda^\mu \Lambda_\mu = \begin{cases} -1, & \text{For SL } \Sigma \\ +1, & \text{For TL } \Sigma \end{cases}$$

$$\Sigma \equiv \text{Hypersurface}$$

Rankine-Hugoniot Equations to Study SR Shocks

- In relativity, space and time are on the same footing so in this case we get two kinds of shock waves (depending upon the frame of reference).

Space − Like

- Conservation of particle Number flux gives
  $n_a v_a \gamma_a = n_b v_b \gamma_b$
- Conservation of Momentum flux gives
  $w_a \gamma_a^2 v_a^2 + p_a = w_b \gamma_b^2 v_b^2 + p_b$
- Conservation of Energy flux gives
  $w_a \gamma_a^2 v_a = w_b \gamma_b^2 v_b$

Time − Like

- Conservation of particle Number density gives
  $n_a \gamma_a = n_b \gamma_b$
- Conservation of Momentum density gives
  $w_a \gamma_a^2 v_a = w_b \gamma_b^2 v_b$
- Conservation of Energy density gives
  $w_a \gamma_a^2 - p_a = w_b \gamma_b^2 - p_b$

- ***Velocity of upstream and downstream matter***

Space-like:

- $v_a = \sqrt{\frac{(p_b - p_a)(\epsilon_b + p_a)}{(\epsilon_b - \epsilon_a)(\epsilon_a + p_b)}}$
- $v_b = \sqrt{\frac{(p_b - p_a)(\epsilon_a + p_b)}{(\epsilon_b - \epsilon_a)(\epsilon_b + p_a)}}$

Time-like:

- $v_a = \sqrt{\frac{(\epsilon_b - \epsilon_a)(\epsilon_a + p_b)}{(p_b - p_a)(\epsilon_b + p_a)}}$
- $v_b = \sqrt{\frac{(\epsilon_b - \epsilon_a)(\epsilon_b + p_a)}{(p_b - p_a)(\epsilon_a + p_b)}}$

General Relativistic Hydrodynamic Equations

- $ds^2 = -e^{2\phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2\sin^2\theta d\varphi^2$
- $T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$
- $u^\mu = \gamma_g(1, v, 0, 0)$ where, $\gamma_g = \frac{1}{[e^{2\phi} - e^{2\Lambda}v^2]^{1/2}}$ and $v = \dfrac{dr}{dt}$.
- From eqn 1 and 2,

$$\frac{\partial_0(nu^0)}{(nu^1)} + \partial_1\Big(\log(nu^1 e^{\phi(r)+\Lambda(r)})\Big) = 0$$

$$\frac{\partial_0 T^{10}}{T^{11}} + \partial_1\Big(\log(T^{11}e^{2\Lambda(r)})\Big) = 0$$

$$\frac{\partial_0 T^{00}}{T^{01}} + \partial_1\Big(\log(T^{01}e^{3\phi(r)+\Lambda(r)})\Big) = 0$$

- Similarly, we can write Rankine-Hugoniot Equations in the GR case under similar considerations.

Rankine-Hugoniot Equations to Study GR Shocks

$Space - Like$

- Conservation of particle Number flux gives $n_a\gamma_{ga}v_a e^{(\phi_a+\Lambda_a)} = n_b\gamma_{gb}v_b e^{(\phi_b+\Lambda_b)}$
- Conservation of Momentum flux gives $w_a\gamma_{ga}^2 v_a^2 e^{2\Lambda_a} + p_a = w_b\gamma_{gb}^2 v_b^2 e^{2\Lambda_b} + p_b$
- Conservation of Energy flux gives $w_a\gamma_{ga}^2 v_a e^{(3\phi_a+\Lambda_a)} = w_b\gamma_{gb}^2 v_b e^{(3\phi_b+\Lambda_b)}$

$Time - Like$

- Conservation of particle Number density gives $n_a\gamma_{ga} = n_b\gamma_{gb}$
- Conservation of Momentum density gives $w_a\gamma_{ga}^2 v_{ra} = w_b\gamma_{gb}^2 v_{rb}$
- Conservation of Energy density gives $w_a\gamma_{ga}^2 - \frac{p_a}{e^{2\phi_a}} = w_b\gamma_{gb}^2 - \frac{p_b}{e^{2\phi_b}}$

- ***Velocity of upstream and downstream matter***

Defining,

$$A_1 = e^{\phi_a},\ A_2 = e^{\phi_b},$$

$$B_1 = e^{\Lambda_a},\ B_2 = e^{\Lambda_b}$$

$$w_a = (p_a + \epsilon_a),\ w_b = (p_b + \epsilon_b)$$

$$a_{11} = A_1^4 w_a^2 - A_2^4 (p_a(p_b + 2\epsilon_a - \epsilon_b) + 2p_b\epsilon_b - p_b\epsilon_a + \epsilon_a\epsilon_b)$$

$$b_{11} = \sqrt{w_b^2 A_2^8 - w_a^2 A_1^8 - 2A_1^4 a_{11}}$$

$$c_{11} = 2\frac{B_1^2}{A_1^2} A_2^4 (p_b + \epsilon_a)(\epsilon_a - \epsilon_b)$$

$$a_{21} = A_1^2 A_2^2 \left[B_2^2 w_a^2 - B_1^2 (\epsilon_b - p_b)(\epsilon_a - p_a)\right] - 2B_1 (A_2^4 p_a \epsilon_a + A_1^4 p_b \epsilon_b)$$

$$b_{21} = w_a A_1 A_2 \sqrt{A_1^2 A_2^2 (B_1^4 w_b^2 - B_2^4 w_a^2) + 2B_2^2 a_{21}}$$

$$c_{21} = B_1^4 (A_2^2 p_a + A_1^2 \epsilon_b)(A_2^2 p_a - A_1^2 p_b)$$

$\boldsymbol{Space - Like}$

- $v_a = \sqrt{\frac{a_{11} + w_a b_{11}}{c_{11}}}$
- $v_b = \frac{v_a B_1 (A_1^4 w_a + A_2^4 w_b - b_{11})}{2A_1^3 A_2 B_2 [p_a + \epsilon_b]}$

$\boldsymbol{Time - Like}$

- $v_a = \sqrt{\frac{A_1 (a_{21} - b_{21})}{2c_{21}}}$
- $v_b = \frac{v_a A_2 (B_2 w_a + B_1 w_b + \frac{b_{21}}{w_a A_1 A_2})}{2B_2 [A_1 p_b + A_2 \epsilon_a]}$

Special Relativistic

- $\left(\frac{w_a^2}{n_a^2} - \frac{w_b^2}{n_b^2}\right) + (p_b - p_a)\left(\frac{w_a}{n_a^2} + \frac{w_b}{n_b^2}\right) = 0$, Same for both SL and TL Shock.

General Relativistic

- ***Space − Like TA***

$$(p_b - p_a)\left[\frac{w_b^2 e^{2(\Lambda_b+\phi_b)}}{n_b^4} - \frac{w_a^2 e^{2(\Lambda_a+\phi_a)}}{n_a^4}\right] -$$
$$\left(\frac{w_b e^{\Lambda_b}}{n_b^2 e^{\phi_b}} - \frac{w_a e^{\Lambda_a}}{n_a^2 e^{\phi_a}}\right)\left[\frac{w_b^2 e^{(3\phi_b+\Lambda_b)}}{n_b^4} - \frac{w_a^2 e^{(3\phi_a+\Lambda_a)}}{n_a^4}\right] = 0$$

- ***Time − Like TA***
$\left[\frac{p_b}{e^{2\phi_b}} - \frac{p_a}{e^{2\phi_a}}\right]\left[\frac{w_a^2 e^{2\phi_a}}{n_a^4 e^{2\Lambda_a}} - \frac{w_b^2 e^{2\phi_b}}{n_b^4 e^{2\Lambda_b}}\right] - \left(\frac{w_b}{n_b^2} - \frac{w_a}{n_a^2}\right)\left[\frac{w_a^2}{n_a^2 e^{2\Lambda_a}} - \frac{w_b^2}{n_b^2 e^{2\Lambda_b}}\right] = 0$
- On putting $\Lambda = \phi = 0 \Rightarrow \Lambda_a = \phi_a = \Lambda_b = \phi_b = 0$, the general relativistic line element reduces to relativistic line element and we recover all the results of special relativistic shocks.

Application-1: The Schwarzschild Metric

- $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$
- $T^{\mu\nu} = wu^\mu u^\nu + pg^{\mu\nu}$
- $e^{2\phi} = \left(1 - \frac{2M}{r}\right)$
- $e^{2\Lambda} = \left(1 - \frac{2M}{r}\right)^{-1}$
- $u^\mu = \gamma_g(1, v, 0, 0)$ where, $\gamma_g = \sqrt{\frac{\left(1 - \frac{2M}{r}\right)}{\left(1 - \frac{2M}{r}\right)^2 - v^2}}$ and $v = \frac{dr}{dt}$
.
- Here, r and M will be the same for both upstream and downstream of the shock since the thickness of the shock wave is considered to be negligible.

Jump Conditions and Velocities

$\boldsymbol{Space - Like}$

- Conservation of particle Number flux gives
  $n_a v_a \gamma_{ga} = n_b v_b \gamma_{gb}$
- Conservation of Momentum flux gives
  $p_a + \frac{w_a \gamma_{ga}^2 v_a^2}{\left(1 - \frac{2M}{r}\right)} = p_b + \frac{w_b \gamma_{gb}^2 v_b^2}{\left(1 - \frac{2M}{r}\right)}$
- Conservation of Energy flux gives
  $w_a \gamma_{ga}^2 v_a = w_b \gamma_{gb}^2 v_b$
- $v_a^2 = \frac{\left(1 - \frac{2M}{r}\right)^2 (p_b - p_a)(p_a + \epsilon_b)}{(\epsilon_b - \epsilon_a)(p_b + \epsilon_a)}$
- $v_b^2 = \frac{\left(1 - \frac{2M}{r}\right)^2 (p_b - p_a)(p_b + \epsilon_a)}{(\epsilon_b - \epsilon_a)(p_a + \epsilon_b)}$

$\boldsymbol{Time - Like}$

- Conservation of particle Number density gives
  $n_a \gamma_{ga} = n_b \gamma_{gb}$
- Conservation of Momentum density gives
  $w_a \gamma_{ga}^2 v_a = w_b \gamma_{gb}^2 v_b$
- Conservation of Energy density gives
  $w_a \gamma_{ga}^2 - \frac{p_a}{\left(1 - \frac{2M}{r}\right)} = w_b \gamma_{gb}^2 - \frac{p_b}{\left(1 - \frac{2M}{r}\right)}$
- $v_a^2 = \frac{\left(1 - \frac{2M}{r}\right)^2 (\epsilon_b - \epsilon_a)(p_b + \epsilon_a)}{(p_b - p_a)(p_a + \epsilon_b)}$
- $v_b^2 = \frac{\left(1 - \frac{2M}{r}\right)^2 (\epsilon_b - \epsilon_a)(p_a + \epsilon_b)}{(p_b - p_a)(p_b + \epsilon_a)}$

- **Combustion/Taub Adiabat for both SL and TL Shock:**

$$\left(\frac{w_a^2}{n_a^2} - \frac{w_b^2}{n_b^2}\right) + (p_b - p_a)\left(\frac{w_a}{n_a^2} + \frac{w_b}{n_b^2}\right) = 0.$$

Application-2: Inside a Neutron Star

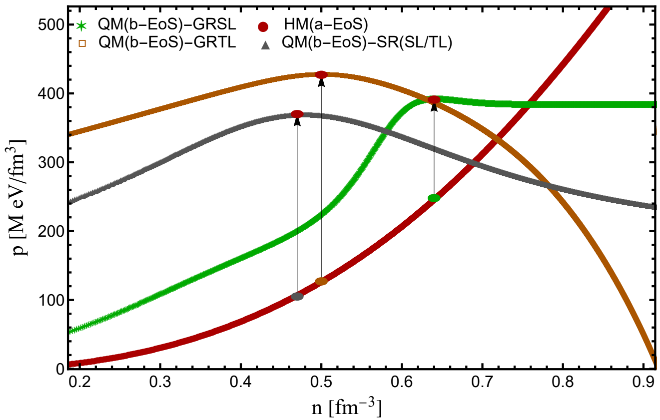
- If a shock wave produce and propagate from the center to the surface.
- A combustion process can happen inside an NS due to shock propagation.
- We will use upstream matter as hadronic and downstream as quark.

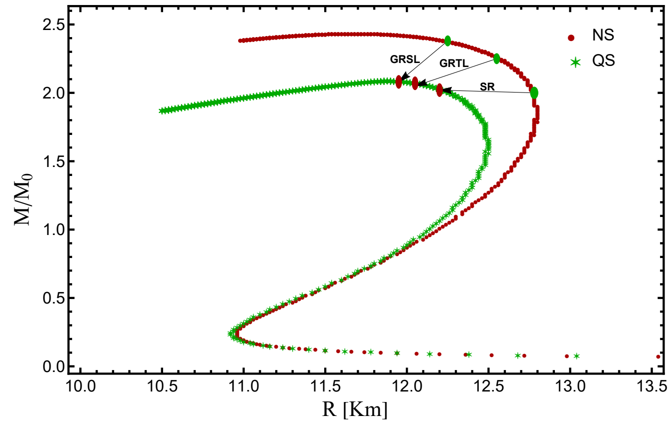
Shock-induced combustion inside a NS

- In the relativistic case, things were easy, we need only equation of states to solve the TA/CA.
- But in the general relativistic case, we also need a variation of the metric potentials along with the equation of states.
- So, for the variation of metric potentials we will use ***TOV Equations*** to model a spherically symmetric, non-rotating, static neutron star.
  - $\frac{dp(r)}{dr} = -\frac{[p(r)+\epsilon(r)]\left[M(r)+4\pi r^3 p(r)\right]}{r[r-2M(r)]}$
  - $\frac{dM(r)}{dr} = 4\pi\epsilon(r)r^2$
  - $\frac{d\Lambda}{dr} = \frac{1}{2r}[(8\pi\epsilon r^2 - 1)e^{2\Lambda} + 1]$
  - $\frac{d\phi}{dr} = \frac{1}{2r}[(8\pi p r^2 + 1)e^{2\Lambda} - 1]$

Shock-induced combustion inside a NS

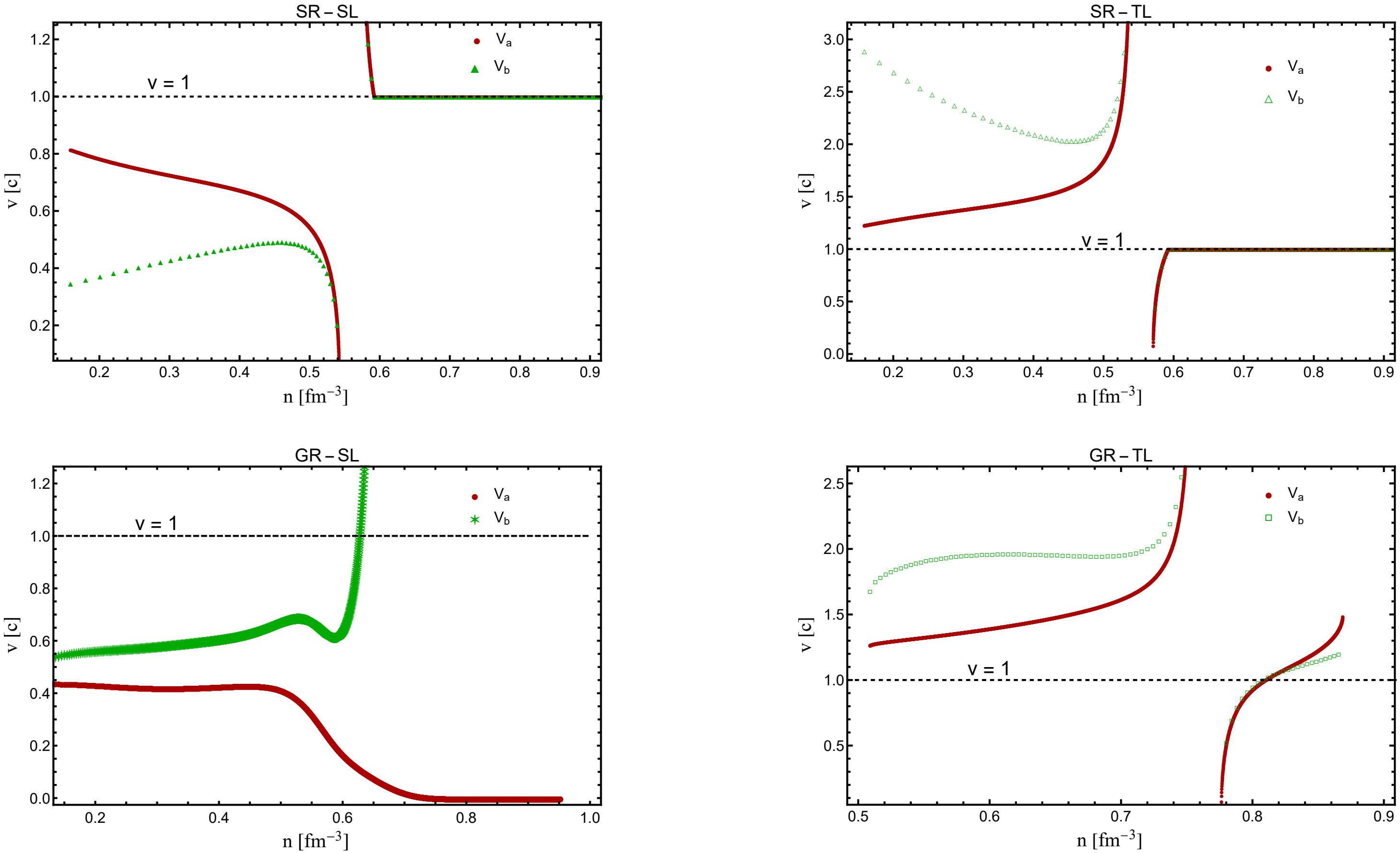
- Actual maximum mass of NS and QS corresponding to their EoSs are 2.42 $M_\odot$ and 2.10 $M_\odot$ respectively.





- Using relativistic TA/CA equation, the mass of phase transitioned QS comes out to be 1.95 $M_\odot$.
- While, using GR space-like and time-like TA/CA, it comes out to be 2.02 $M_\odot$ and 2.03 $M_\odot$ respectively, which agrees with the M-R curve of QS.
- Using velocity values we can also guess the type of combustion process.
- $v_a > v_b \Rightarrow$ Deflagration & $v_a < v_b \Rightarrow$ Detonation

Velocities of Upstream and downstream matter



- In the velocity curve, there are some points where the velocity of upstream or downstream or both upstream and downstream matter become luminous.
- Considering a fixed upstream pressure and energy density value we can find the possible downstream pressure and energy density values at which velocity become luminous.

Condition For v=1

- ***In Relativistic Case***

$$v_a^2 = 1 = v_b^2 \Rightarrow p_b = \epsilon_b + p_a - \epsilon_a$$

- ***In General Relativistic Case***

$$\boldsymbol{Space - Like}$$

- $v_a^2 \approx 1 \Rightarrow p_b = \frac{h_1 + h_2 + h_3 + h_4}{A_2^4(A_1^2 + B_1^2)(A_1^2(p_a + \epsilon_b) + B_1^2(\epsilon_b - \epsilon_a))}$

Condition For v=1

- $v_b^2 \approx 1 \Rightarrow p_b = \frac{k((A_1^4 B_1 B_2 w_a + A_2^4 B_1 B_2 \epsilon_b) - (p_a + \epsilon_b) A_1^3 B_2^2 A_2 - A_1 B_1^2 A_2^3 \epsilon_a)}{A_2^3 B_1 (A_1 B_1 - k A_2 B_2)}$

where,

$$h_1 = A_1^6 B_1^2 w_a^2, k = 1.00001$$
$$h_2 = A_2^4 B_1^4 \epsilon_a (\epsilon_a - \epsilon_b)$$
$$h_3 = A_1^4 A_2^4 p_a (p_a + \epsilon_b)$$
$$h_4 = A_1^2 A_2^4 B_1^2 (p_a (\epsilon_b - 2\epsilon_a) - \epsilon_a \epsilon_b)$$

$$\boldsymbol{Time - Like}$$

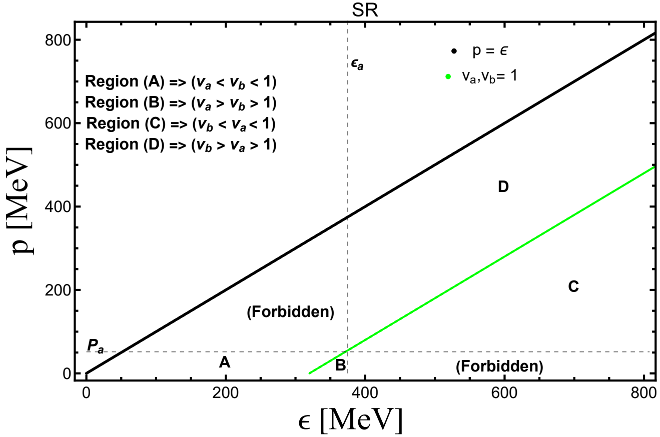
- $v_a^2 \approx 1 \Rightarrow p_b = \frac{A_2(A_2 G1 + A_1 G3)}{A_1({A_1}^3 \epsilon_b + {A_1}^2 G4 + A_1 B_1 G5 + A_2 {B_1}^2 p_a)}$

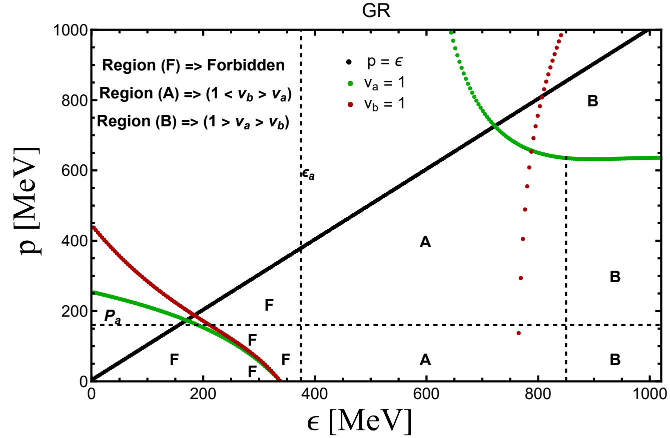
- $v_b^2 \approx 1 \Rightarrow p_b = \frac{h A_2 [A_1 (B_1 \epsilon_b + B_2 p_a - B_2 \epsilon_a) - A_2 B_1 p_a]}{A_1 (A_1 B_2 - A_2 B_1)}$

where,

$$G1 = ({A_1}^2 {\epsilon_a}^2 + 2 A_1 B_1 p_a \epsilon_a + {B_1}^2 {p_a}^2)$$
$$G2 = (-B_1 p_a \epsilon_b + B_1 \epsilon_a \epsilon_b - B_2 {p_a}^2 - 2 B_2 p_a \epsilon_a - B_2 {\epsilon_a}^2)$$
$$G3 = -{A_1}^2 \epsilon_a \epsilon_b + A_1 G2 + {B_1}^2 p_a \epsilon_b$$

$$G4 = -(A_2\epsilon_a - 2B_1\epsilon_b)$$
$$G5 = (-A_2p_a + A_2\epsilon_a + B_1\epsilon_b)$$
$$h = Constant.$$





- Plots will change with the change of initial conditions (p_a & ϵ_a).

Other Important Results - 2: Entropy Across Front

- For Special Relativistic SL/TL Shocks:
$$s_b - s_a = \frac{1}{12\mu_a T_a}\frac{\partial^2(\mu V)}{\partial p^2}(p_b - p_a)^3 + O[(p_b - p_a)^4] + ...$$
- For General Relativistic SL Shocks:
$$s_b - s_a = \frac{1}{12\mu_a T_a}\frac{\partial^2(\mu V)}{\partial p^2}(p_b - p_a)^3 + O[(p_b - p_a)^4] + ...$$
- For General Relativistic TL Shocks:
$$s_b - s_a = \frac{A\left(2\mu_a V_a + (p2 - p1)\frac{\partial(\mu V)}{\partial p}\right) + B\frac{\partial^2(\mu V)}{\partial p^2}}{6\left(2 f_b^2 g_a^2 T_a \mu_a + C\frac{\partial(\mu V)}{\partial s}\right) + D\frac{\partial^2(\mu V)}{\partial p\,\partial s}} + ...$$

where,

$g_a = \frac{1}{e^{\Lambda_a}}$, $g_b = \frac{1}{e^{\Lambda_b}}$, $f_a = \frac{e^{\phi_a}}{e^{\Lambda_a}}$, $f_b = \frac{e^{\phi_b}}{e^{\Lambda_b}}$.

$A = 6p_b(f_b g_a - f_a g_b)(f_b g_a + f_a g_b)$

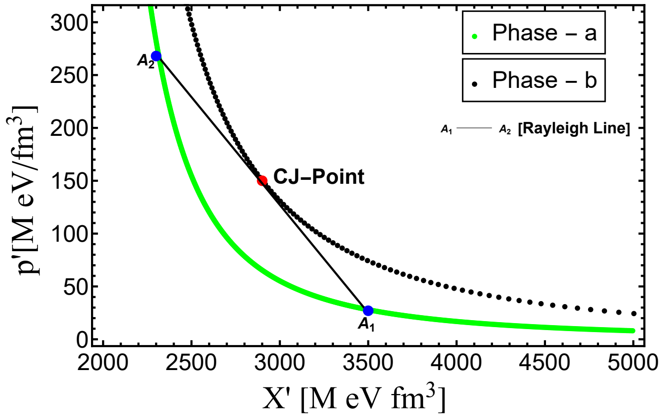
$B = (p_b - p_a)^2(f_b^2 g_a^2(2p_b + p_a) - 3f_a^2 g_b^2 p_b)$

$C = (f_b^2 g_a^2(2p_b - p_a) - f_a^2 g_b^2 p_b)$

$D = p_b(p_b - p_a)(f_b g_a - f_a g_b)(f_b g_a + f_a g_b)$

Other Important Results - 3: The CJ Point

The Chapman–Jouguet point is a point at which chord (Rayleigh Line) ≡ tangent to the combustion adiabat.



- For Special Relativistic SL Shock:
  $u_b^2 = u_{sb}^2$
- For Special Relativistic TL Shock:
  $u_b^2 = 1 - u_{sb}^2$
- For General Relativistic SL Shock:
  $u_b^2 = \frac{u_{sb}^2 e^{\phi_b}}{[e^{3\Lambda_b}(1 - u_{sb}^2(e^{2\phi_b} - e^{2\Lambda_b}))]}$
- For General Relativistic TL Shock:
  $u_b^2 = \frac{u_{sb}^2 e^{2\phi_b} - 1}{[e^{2\phi_b}(u_{sb}^2(e^{2\phi_b} - e^{2\Lambda_b}) - 1)]}$

Summary

- We have derived the Rankine-Hugoniot Equations in a general relativistic background.
- Velocities of upstream and downstream matter are also shown.
- We have got different Taub adiabat/Combustion adiabat equations in the GR case.
- We have studied an application of shock wave in an isolated Neutron star using GR and SR shock formulations.
- We have also discussed several other important results.

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Acknowledgement

Thank You for Your Attention ...