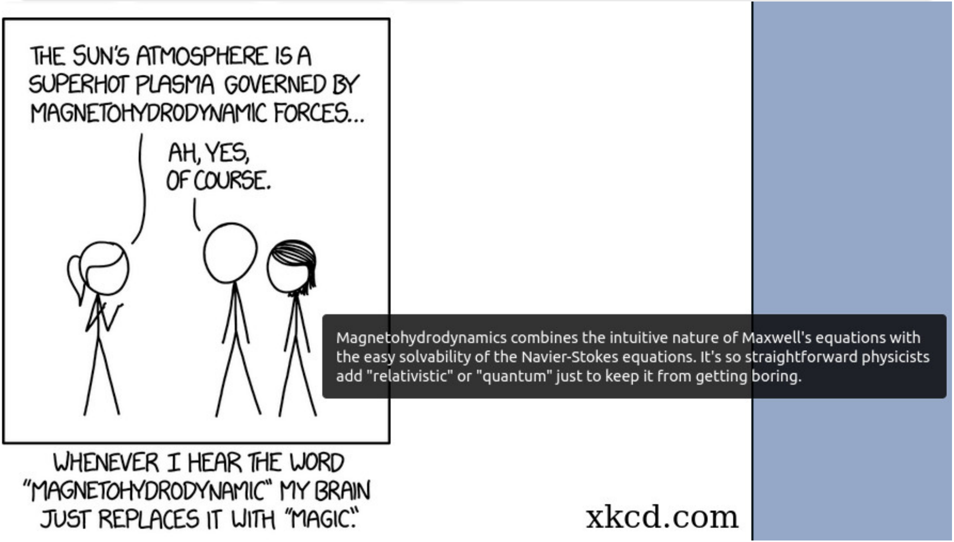
Relativistic spin-(magneto)hydrodynamics



Magnetohydrodynamics combines the intuitive nature of Maxwell's equations with the easy solvability of the Navier-Stokes equations. It's so straightforward physicists add "relativistic" or "quantum" just to keep it from getting boring.

xkcd.com

MAGIC23, Kovalam, Kerala

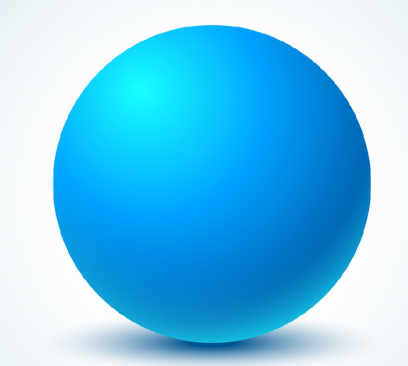
Hydrodynamics in nuclear astrophysics and HIC

- Relativistic magnetohydrodynamics applied in simulations of neutron star mergers.
- Mergers happen in presence of strong magnetic field ($10^8 - 10^{10}$ T) and rotation (~ 1000 Hz).
- Similar extreme conditions in relativistic heavy-ion collisions, ($10^{13} - 10^{14}$ T) and global angular momentum ($10^4 - 10^5 \hbar$).
- Relativistic hydrodynamics also applied in simulations of HIC.
- Magnetohydrodynamics incorporates magnetic field in hydrodynamics framework.
- Angular momentum conservation need to be included in the hydrodynamic formulation.
- Polarization measurements indicate angular momentum deposition in the fireball.

Spin polarization of hadrons in heavy-ion collisions

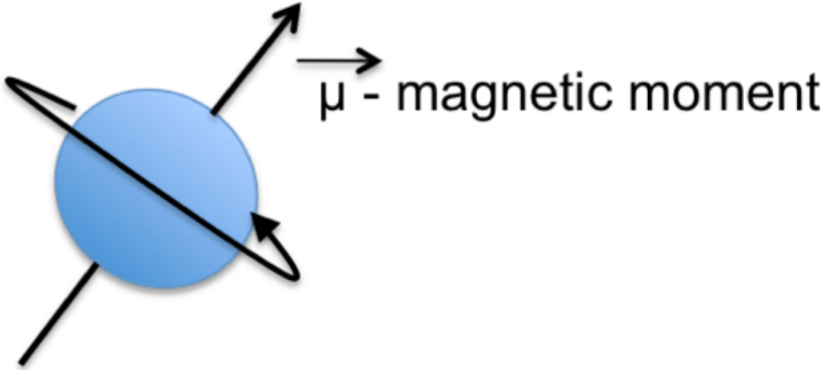
- Spin polarization is a relatively new topic in heavy ion collisions.
- Provides unique opportunity to probe QGP properties.
- Several measurements of spin polarization of hadrons.
- In baryon sector:
  - Λ (spin 1/2): STAR, Nature, 548, 62–65 (2017); HADES; ALICE.
  - Ω (spin 3/2): STAR, Phys. Rev. Lett. 126, 162301 (2021).
  - Ξ (spin 1/2): STAR, Phys. Rev. Lett. 126, 162301 (2021).
- In meson sector:
  - K^{*0} : ALICE, PRL 125, 012301 (2020); STAR, Nature, 614, 244-248 (2023).
  - ϕ : ALICE, PRL 125, 012301 (2020); STAR, Nature, 614, 244-248 (2023).
  - Heavy quarkonium, J/ψ and Υ(1S) : ALICE, PLB 815, 136146 (2021).
- Global and local polarization measurements.

Decay of scalar particles



No anisotropy in the rest frame: isotropic decay products.

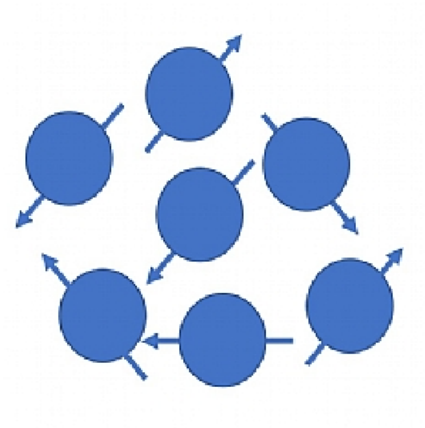
Decay of particles with spin



Preferred direction due to spin: anisotropic decay products

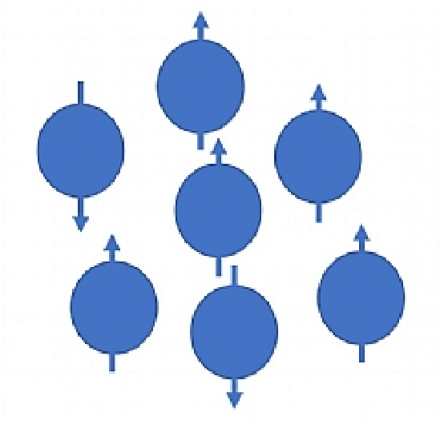
Basis for polarization observables.

Several random decays



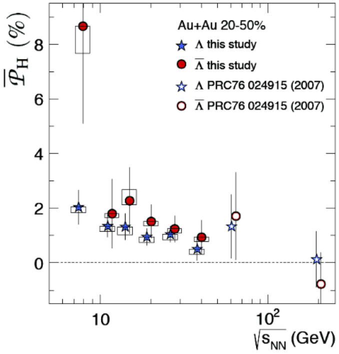
Averaging over random decays should lead to isotropic decay products.

Decay of spin polarized particles



Averaging over decay of spin-polarized particles should lead to anisotropic decay products.

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017



nature

THE INTERNATIONAL WEEKLY JOURNAL OF SCIENCE

First observation of fluid vortices formed by heavy-ion collisions
PAGES 34 & 62

SUBATOMIC SWIRLS

CLIMATE CHANGE
PARIS AGREEMENT
Time for nations to match words with deeds
PAGE 25

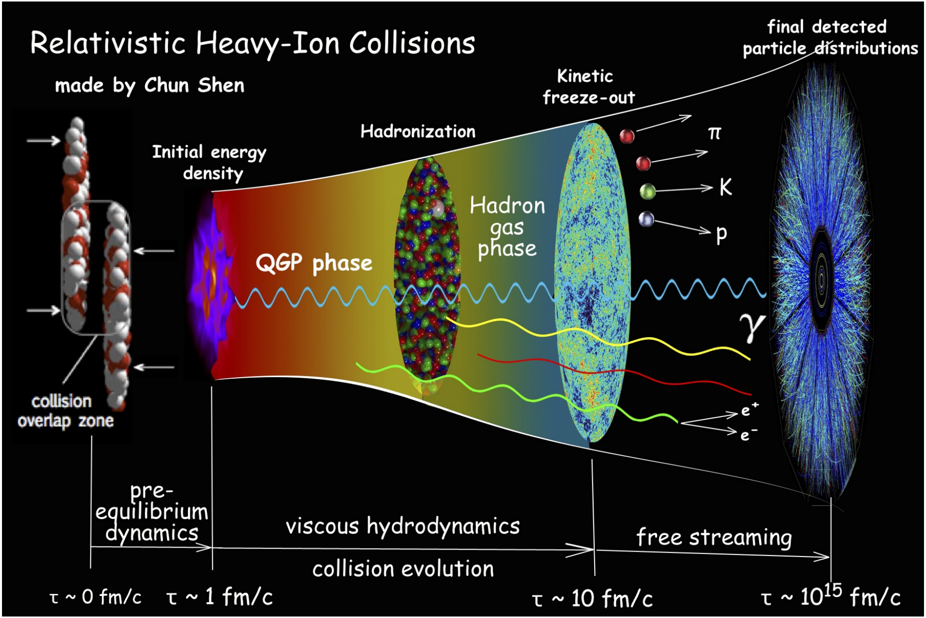
BOOKS
SUMMER SELECTION
Recommended reading for the holiday season
PAGE 20

STEM CELLS
YOUTHFUL SECRETS
How the hypothalamus helps to control the ageing process
PAGE 52

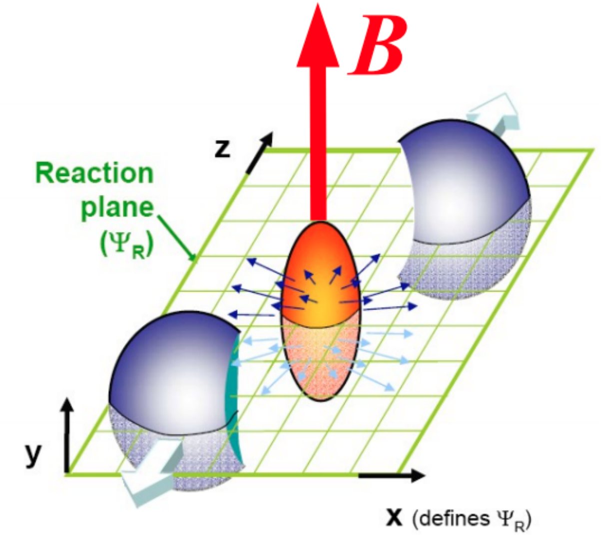
NATURE.COM/NATURE
3 August 2017
Vol. 548, No. 7665

First evidence of a quantum effect in (relativistic) hydrodynamics

Adapted from F. Becattini
'Subatomic Vortices'

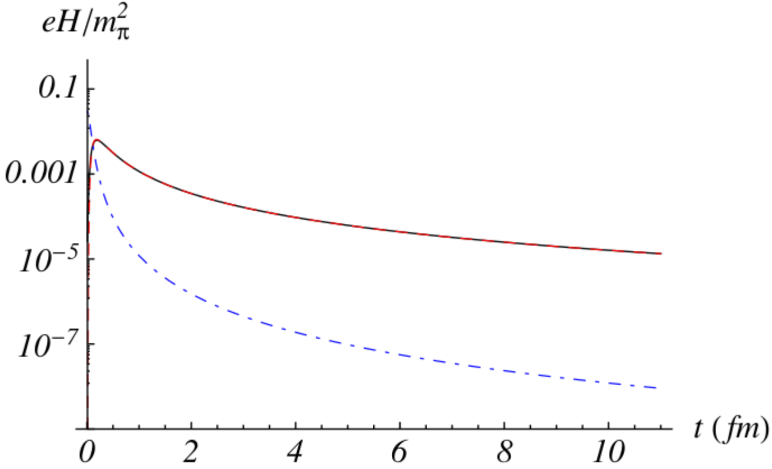

Relativistic Heavy-Ion Collisions
made by Chun Shen
final detected particle distributions
Kinetic freeze-out
Hadronization
Initial energy density
π
K
p
Hadron gas phase
QGP phase
γ
e^+
e^-
collision overlap zone
pre-equilibrium dynamics
viscous hydrodynamics
free streaming
collision evolution
$\tau \sim 0$ fm/c
$\tau \sim 1$ fm/c
$\tau \sim 10$ fm/c
$\tau \sim 10^{15}$ fm/c

Generation of magnetic field in heavy ion collisions



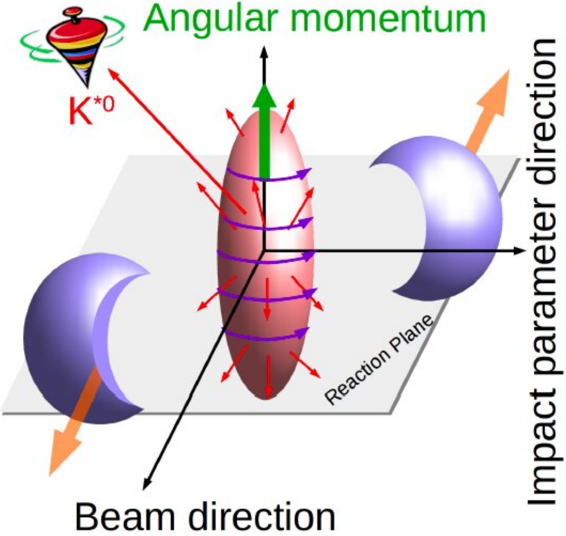
[Adapted from D. Kharzeev @ CPOD 2013.]

Magnetic field time evolution



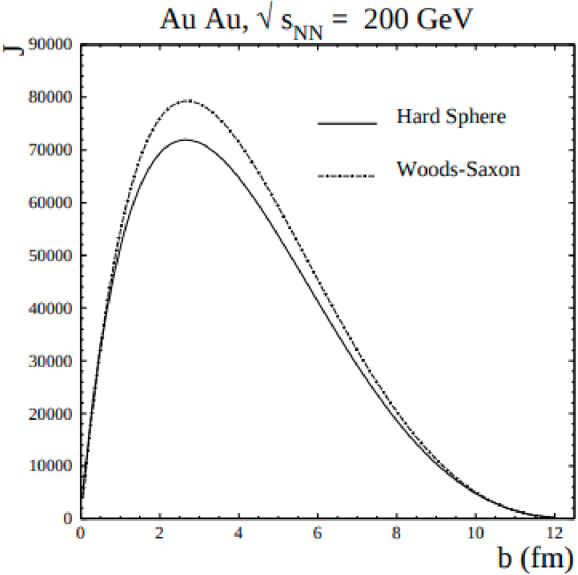
[K. Tuchin, Int. J. Mod. Phys. E23, 1430001 (2014).]

Global angular momentum in heavy ion collisions



[B. Mohanty, ICTS News 6, 18-20 (2020).]

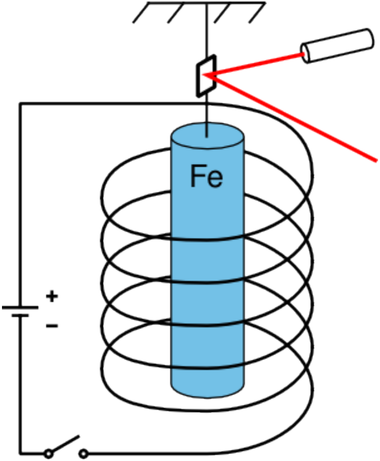
Angular momentum generation in non-central collisions



[F. Becattini, et al., Phys. Rev. C77, 024906 (2008).]

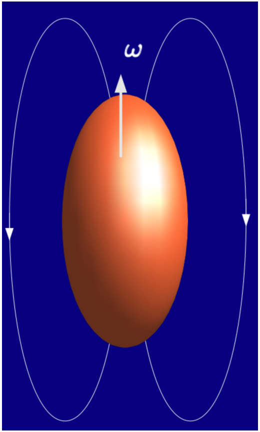
Other effects related to magnetic field and spin polarization.

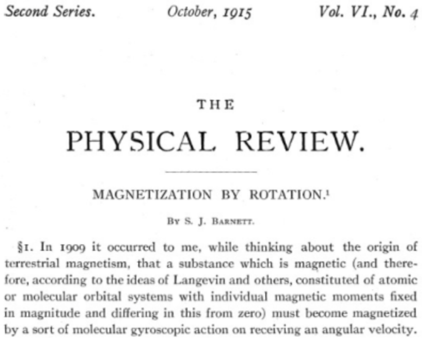
Einstein-de Haas effect



Electron spins get aligned in external magnetic field which is compensated by rotation of the ferromagnetic material.

Converse: Barnett effect





Second Series. *October, 1915* *Vol. VI., No. 4*

THE

PHYSICAL REVIEW.

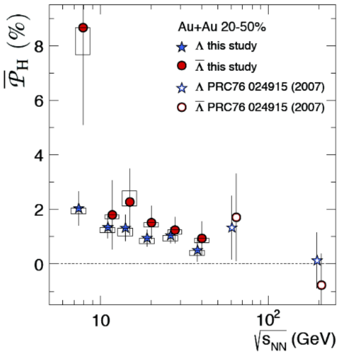
MAGNETIZATION BY ROTATION.[1]

By S. J. Barnett.

§1. In 1909 it occurred to me, while thinking about the origin of terrestrial magnetism, that a substance which is magnetic (and therefore, according to the ideas of Langevin and others, constituted of atomic or molecular orbital systems with individual magnetic moments fixed in magnitude and differing in this from zero) must become magnetized by a sort of molecular gyroscopic action on receiving an angular velocity.

Spontaneous magnetization when spun around. Transformation of orbital angular momentum into spin alignment. Angular velocity decreases with appearance of magnetic field. Explanation appeals to spin-orbit coupling.

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017



First evidence of a quantum effect in (relativistic) hydrodynamics

Adapted from F. Becattini
'Subatomic Vortices'

Angular momentum conservation: particles

- Angular momentum of a particle with momentum $\vec{p}$:

$$\vec{L} = \vec{x} \times \vec{p} \quad \Rightarrow \quad L_i = \varepsilon_{ijk}\, x_i\, p_j$$

- One can obtain the dual tensor:

$$L_{ij} \equiv \varepsilon_{ijk}\, L_k \quad \Rightarrow \quad L_{ij} = x_i\, p_j - x_j\, p_i$$

- We know that both definitions are equivalent.
- In absence of external torque, $\dfrac{d\vec{L}}{dt} = 0$, we also have: $\partial_i L_{ij} = 0$.
- Relativistic generalization: $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $\partial_\mu L^{\mu\nu} = 0$.
- This treatment valid for point particles.
- For fluids, particle momenta → "generalized fluid momenta"

The energy-momentum tensor

Angular momentum conservation: fluid

- The orbital angular momentum for relativistic fluids is defined as

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

- Keeping in mind the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$:

$$\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$$

- Obviously, for symmetric $T^{\mu\nu}$, orbital angular momentum is automatically conserved. Classically $T^{\mu\nu}$ symmetric.
- For medium constituent with intrinsic spin, different story

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- Ensure total angular momentum conservation: $\partial_\lambda J^{\lambda,\mu\nu} = 0$.
- Basis for formulation of spin Hydrodynamics.

  [Florkowski et. al., Prog.Part.Nucl.Phys. 108 (2019) 103709; Bhadury et. al., Eur.Phys.J.ST 230 (2021) 3, 655-672]

Pseudo-gauge transformations

- Total angular momentum is given by

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- With $\partial_\mu T^{\mu\nu} = 0$, and $\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$,

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \quad \Longrightarrow \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Hence the final hydrodynamic equations can be written as

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Also holds with the following redefinition

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} - \Phi^{\mu,\lambda\nu} - \Phi^{\nu,\lambda\mu}\right)$$

$$\tilde{S}^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu}$$

- Freedom due to space-time symmetry; including torsion fixes this.

[Gallegos et. al., SciPost Phys. 11, 041 (2021); Hongo et. al., JHEP 11 (2021) 150]

Our Formulation: Relativistic Kinetic Theory

Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.
- Let us consider a system of relativistic particles of rest mass m with momenta $\mathbf{p}$ and energy p^0

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}$$

- For large no. of particles, $f(x,p)$ gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.
- $f(x,p)\Delta^3 x\,\Delta^3 p$ gives average no. of particles in the volume element $\Delta^3 x$ at point x with momenta in the range $(\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})$.
- Statistical assumptions:
  - No. of particles contained in $\Delta^3 x$ is large ($N \gg 1$).
  - $\Delta^3 x$ is small compared to macroscopic volume ($\Delta^3 x/V \ll 1$).
- The equilibrium distribution: $f_{eq}(x,p,s) = \left[\exp\left(\beta \cdot p - \xi\right) \pm 1\right]^{-1}$

Extended phase-space for spin degrees of freedom

- The phase-space for single particle distribution function gets extended $f(x, p, s)$.

- The equilibrium distribution for Fermions is given by

$$f_{eq}(x,p,s) = \frac{1}{\exp\left[\beta\cdot p - \xi - \frac{1}{2}\,\omega : s\right] + 1} \qquad \begin{cases} \beta\cdot p \equiv \beta_\mu p^\mu \\ \omega : s \equiv \omega_{\mu\nu} s^{\mu\nu} \end{cases}$$

- Quantities $\beta^\mu = u^\mu/T,\ \xi = \mu/T,\ \omega_{\mu\nu}$ are functions of x.

- $\xi,\ \beta^\mu,\ \omega^{\mu\nu}$: Lagrange multipliers for conserved quantities.

- $s^{\mu\nu}$: Particle spin, similar to particle momenta p^μ.

- Hydrodynamics: average over particle momenta and spin.

- Classical treatment of spin.

Bhadury et. al., PLB 814, 136096 (2021); PRD 103, 01430 (2021).

Conserved currents and spin-hydrodynamics

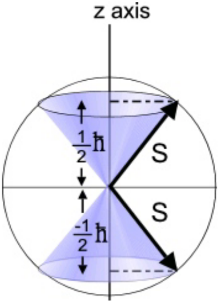
- Express hydrodynamic quantities in terms of $f(x,p,s)$.

$$T^{\mu\nu}(x) = \int dPdS\; p^\mu p^\nu \left[f(x,p,s) + \bar{f}(x,p,s)\right]$$

$$N^{\mu}(x) = \int dPdS\; p^\mu \left[f(x,p,s) - \bar{f}(x,p,s)\right]$$

$$S^{\lambda,\mu\nu}(x) = \int dPdS\; p^\lambda s^{\mu\nu} \left[f(x,p,s) + \bar{f}(x,p,s)\right]$$

$$dP \equiv \frac{d^3p}{E_p(2\pi)^3}, \quad dS \equiv m\frac{d^4s}{\pi\,\mathfrak{s}}\,\delta(s\cdot s + \mathfrak{s}^2)\,\delta(p\cdot s)$$



$$\int dS = 2; \quad \mathfrak{s}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}; \quad s^\mu \equiv \frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu s_{\alpha\beta}$$

- Classical treatment of spin: internal angular momentum.
- Equations of motion: $\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = 0$.
- Non-dissipative spin hydrodynamics: $f(x,p,s) = f_{eq}(x,p,s)$.

Dissipative spin-hydrodynamics Bhadury et. al., PLB 814, 136096 (2021)

- Introduce out-of-equilibrium distribution function $f(x, p, s)$.
- Use Boltzmann equation for evolution of $f = f_{eq} + \delta f$.

$$p^\mu \partial_\mu f = C[f]$$

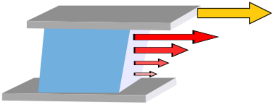
- Employ relaxation-time approximation for collision kernel.

$$C[f] = -(u \cdot p) \frac{f - f_{eq}}{\tau_{eq}}$$

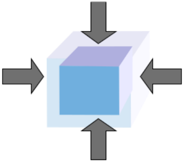
- Solve assuming small departure from equilibrium, $\delta f / f_{eq} \ll 1$.
- First order dissipative spin hydrodynamics for $\delta f = \delta f_1$.
- Relativistic Navier-Stokes analog of spin-hydrodynamics.

Dissipative effects

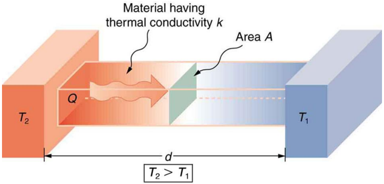
- Shear viscosity: fluid's resistance to shear forces



- Bulk viscosity: fluid's resistance to compression



- Charge/heat conductivity: fluid's resistance to flow of charge/heat.
- Dissipation to spin current new: Kubo formalism necessary.



Spin Magnetohydrodynamics Bhadury et. al., PRL 129, 192301 (2022)

- The particle four-current and its conservation is given by

$$N^\mu = n u^\mu + n^\mu, \qquad \partial_\mu N^\mu = 0$$

- Total stress-energy tensor of the system: $T^{\mu\nu} = T_{\rm f}^{\mu\nu} + T_{\rm int}^{\mu\nu} + T_{\rm em}^{\mu\nu}$

$$T_{\rm f}^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\,\Delta^{\mu\nu} + \pi^{\mu\nu},$$

$$T_{\rm int}^{\mu\nu} = -\Pi^\mu u^\nu - F^\mu{}_\alpha M^{\nu\alpha}$$

$$T_{\rm em}^{\mu\nu} = -F^{\mu\alpha} F^\nu{}_\alpha + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

- Maxwell's equation: $\partial_\mu H^{\mu\nu} = J^\nu$ and $H^{\mu\nu} = F^{\mu\nu} + M^{\mu\nu}$,

$$\partial_\mu T_{\rm em}^{\mu\nu} = F^\nu{}_\alpha J^\alpha$$

- Current generating external field, $J^\mu = J_{\rm f}^\mu + J_{\rm ext}^\mu$ where $J_{\rm f}^\mu = \mathfrak{q} N^\mu$,

$$\partial_\mu T^{\mu\nu} = -f_{\rm ext}^\nu, \qquad f_{\rm ext}^\nu = F^\nu{}_\alpha J_{\rm ext}^\alpha$$

Equations of motion

- Divergence of matter part of energy-momentum tensor,

$$\partial_\nu T_{\rm f}^{\mu\nu} = F^{\mu}{}_{\alpha} J_{\rm f}^{\alpha} + \frac{1}{2}\left(\partial^{\mu} F^{\nu\alpha}\right) M_{\nu\alpha}$$

- Next, consider total angular momentum conservation:

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\mu} + S^{\lambda,\mu\nu}$$

- In presence of external torque its divergence leads to,

$$\partial_\lambda J^{\lambda,\mu\nu} = -\tau_{\rm ext}^{\mu\nu}, \qquad \tau_{\rm ext}^{\mu\nu} = x^{\mu} f_{\rm ext}^{\nu} - x^{\nu} f_{\rm ext}^{\mu}$$

- Torque due to moment of external force; "pure" torque ignored.
- The orbital part of angular momentum and its divergence is

$$L^{\lambda,\mu\nu} = x^{\mu} T^{\lambda\nu} - x^{\nu} T^{\lambda\mu}, \qquad \partial_\lambda L^{\lambda,\mu\nu} = -\tau_{\rm ext}^{\mu\nu}$$

- Spin part of the total angular momentum is conserved

$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

- Along with particle four-current conservation, $\partial_\mu N^{\mu} = 0$.

Boltzmann equation

- Boltzmann equation (BE) in relaxation-time approximation (RTA)

$$\left(p^\alpha \frac{\partial}{\partial x^\alpha} + m\,\mathcal{F}^\alpha \frac{\partial}{\partial p^\alpha} + m\,\mathcal{S}^{\alpha\beta} \frac{\partial}{\partial s^{\alpha\beta}}\right) f = C[f] = -\,(u\cdot p)\,\frac{f - f_{\rm eq}}{\tau_{\rm eq}}$$

- The force term is:

$$\mathcal{F}^\alpha = \frac{\mathfrak{q}}{m}\,F^{\alpha\beta}p_\beta + \frac{1}{2}\left(\partial^\alpha F^{\beta\gamma}\right) m_{\beta\gamma}, \qquad m^{\alpha\beta} = \chi s^{\alpha\beta}$$

- There is a "pure" torque term:

$$\mathcal{S}^{\alpha\beta} = 2\,F^{\gamma[\alpha}\,m^{\beta]}{}_\gamma - \frac{2}{m^2}\left(\chi - \frac{\mathfrak{q}}{m}\right) F_{\phi\gamma}\,s^{\phi[\alpha}\,p^{\beta]}p^\gamma$$

- We ignore this "pure" torque term for now.
- Employ the Boltzmann equation to obtain $\delta f = \delta f_1$.
- Evolution equations for spin-magnetohydrodynamics.

Hydrodynamic equations from kinetic theory

- Impose Landau frame and extended matching conditions

$$u_\mu T^{\mu\nu} = \epsilon u^\nu, \quad \epsilon = \epsilon_{\rm eq}, \quad n = n_{\rm eq}, \quad u_\lambda \delta s^{\lambda,\mu\nu} = 0$$

- Zeroth, first and "spin" moment of the RTA collision vanishes

$$\int dP dS\, C[f] = \int dP dS\, p^\mu\, C[f] = \int dP dS\, s^{\mu\nu} C[f] = 0$$

- Using definitions of hydro quantities, these moments of BE gives

$$\partial_\mu N^\mu = 0, \quad \partial_\nu T^{\mu\nu}_{\rm f} = F^\mu{}_\alpha J^\alpha_{\rm f} + \frac{1}{2}\left(\partial^\mu F^{\nu\alpha}\right) M_{\nu\alpha}, \quad \partial_\lambda S^{\lambda,\mu\nu} = 0$$

- Same equations as obtained from macroscopic arguments.
- Polarization/magnetization emerge naturally at gradient order.
- Boltzmann equation → dissipative spin-magnetohydodynamics.

Einstein-de Haas and Barnett effects

- One can define the polarization-magnetization tensor as

$$M^{\mu\nu} = m \int dP dS \, m^{\mu\nu} \left(f - \bar{f}\right)$$

- The equilibrium polarization-magnetization tensor is

$$M^{\mu\nu}_{eq} = m \int dP dS \, m^{\mu\nu} \left(f_{eq} - \bar{f}_{eq}\right)$$

- Magnetic dipole moment $m^{\mu\nu} = \chi s^{\mu\nu}$.
- χ: resembles the gyromagnetic ratio.
- Integrating over the momentum and spin degrees of freedom,

$$M^{\mu\nu}_{eq} = a_1 \, \omega^{\mu\nu} + a_2 \, u^{[\mu} u_\gamma \omega^{\nu]\gamma}$$

- In global equilibrium, $\omega^{\mu\nu}$ corresponds to rotation of the fluid.
- Rotation produces magnetization (Barnett effect) and vice versa (Einstein-de Hass effect).

Our work in this direction within kinetic theory

- Ideal spin-hydrodynamics:
  - W. Flokowski, B. Friman, A. Jaiswal and E. Speranza, Physical Review C 97, 041901 (2018).
  - W. Flokowski, B. Friman, A. Jaiswal, R. Ryblewski and E. Speranza, Physical Review D 97, 116017 (2018).
- Dissipative spin-hydrodynamics:
  - S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physics Letters B 814, 136096 (2021).
  - S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physical Review D 103, 014030 (2021).

- Relativistic Spin Magnetohydrodynamics: S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. Lett., 129, 192301 (2022).

Ongoing work from field theory and geometry

- Starting from the symmetries of the Lagrangian of a given theory, one can construct conserved currents using Noether's theorem.
- Energy-momentum tensor-variation of Lagrangian with metric $g^{\mu\nu}$: conservation is a consequence of diffeomorphism invariance.
- Conserved charge current-variation with gauge field A^{μ}: consequence of local gauge symmetry.
- Spin-current can be constructed similarly.
- Price to pay: introduce torsion in metric, non-Riemannian geometry.
- Spin current: variation w.r.t torsion.
- Angular momentum conservation: consequence of local Lorentz invariance.

- Kubo relations for dissipation in spin current.
- Pseudogauge freedom and the SO(3) algebra of spin operators.

Other relevant works

- Other parallel approaches from Wigner function [N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang and D. Rischke, PRD 100 (2019) 056018].
- Appraoch based on chiral kinetic theory [S. Shi, C. Gale and S. Jeon, PRC 103 (2021) 044906].
- Appraoch based on Lagrangian method [D. Montenegro and G. Torrieri, PRD 100 (2019) 056011].
- Formulation with torsion in metric [A. D. Gallegos, U. Gürsoy and A. Yarom, SciPost Phys. 11, 041 (2021); M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov, H.-U. Yee, JHEP 11 (2021) 150].
- Useful reviews on spin hydro: [W. Florkowski, R. Ryblewski and A. Kumar, Prog.Part.Nucl.Phys. 108 (2019) 103709; S. Bhadury, J. Bhatt, A. Jaiswal and A. Kumar, Eur.Phys.J.ST 230 (2021) 3, 655-672].
- Relativistic spin-magnetohydrodynamics: unexplored area.
- Much work needed in this direction.

Thank you!