Dark Matter - Theoretical and Observational Status
Lecture 1

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XXXIII Canary Islands Winter School of Astrophysics
Museo Ciencias y Cosmos, Tenerife
28 November 2022

Goals (Lecture 1)

- Explain the evidence for dark matter and what these searches imply for its properties
- Discuss the big picture of DM as new physics and some ways to think about the landscape of possible scenarios
- Outline/review the "thermal freezeout" scenario for DM production in the early universe
- Plan for later lectures:
  - Lecture 2: discuss viable parameter space, benchmark models, & features of cosmological history for particle DM (>1 eV), especially thermal relics
  - Lecture 3: terrestrial searches for particle DM (direct detection + accelerators)
  - Lecture 4: benchmark models & searches for wave-like DM (< 1 eV)

Historical overview

The missing mass

- Zwicky, 1933: estimated the mass in a galaxy cluster in two ways.

Method 1

Estimate mass from mass-to-light ratio, calibrated to local system.

- Count galaxies
- Add up total luminosity
- Convert to mass using mass-to-light ratio of ~3, calibrated from local Kapteyn stellar system.

Mass estimate 1

Method 2

Use virial theorem + measurements of galaxy velocities to estimate gravitational potential, and hence infer mass.

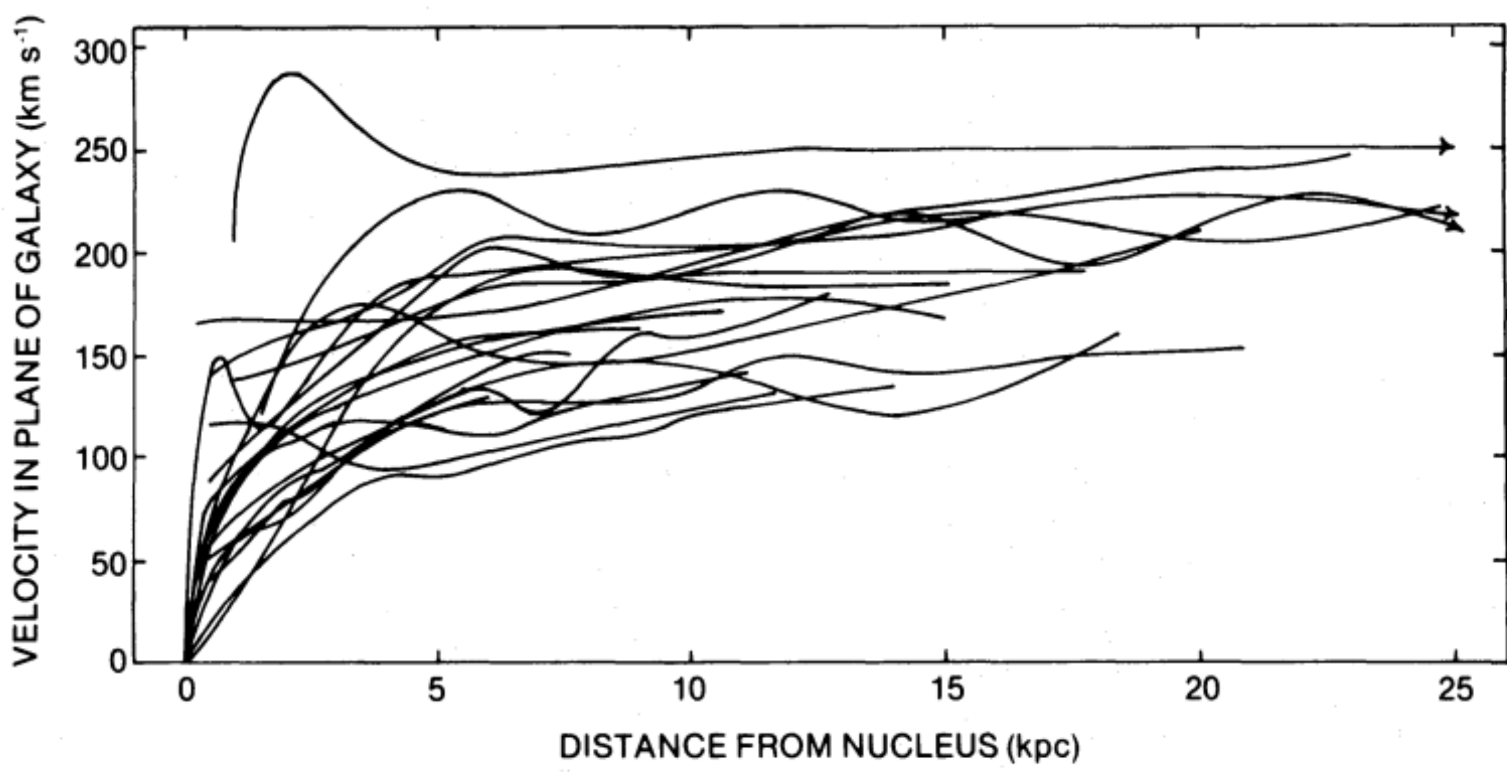
Galactic velocities measured by Doppler shifts

$$\mathrm{KE} = -\frac{1}{2}\mathrm{PE}$$ in equilibrium

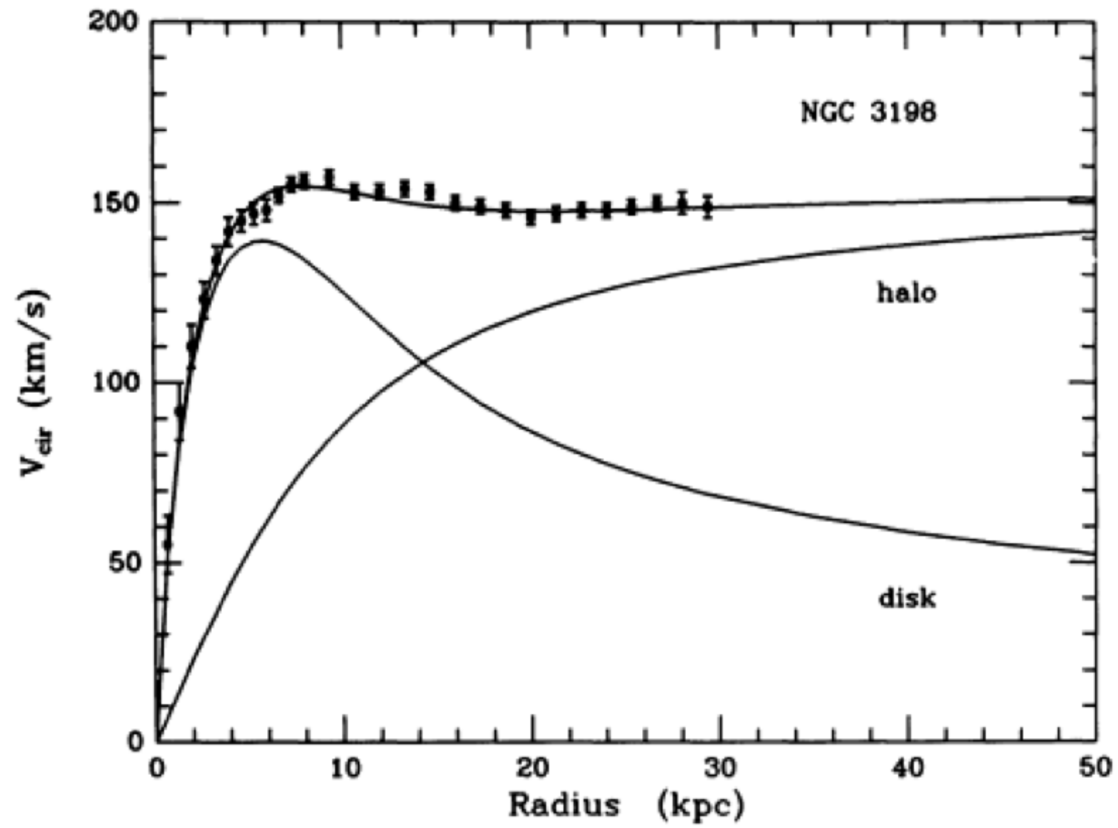
Mass estimate 2

- These numbers are different by 2+ orders of magnitude (second one is larger).
- One possibility: there is (lots of) gravitating non-luminous matter.

Rotation curves



Rubin, Ford & Thonnard, 1980



van Albada, T. S., Bahcall, J. N., Begeman, K., & Sancisi, R., 1985

- Rubin, Ford & Thonnard 1980 (following work in the 1970s): galactic rotation curves are flat, not falling as one would expect if mass was concentrated in the bulge at the Galactic center.
- Modified gravity? Or some "dark" unseen matter? If the latter, needs to extend to much larger radii than the observed Galactic disk - "dark halo".

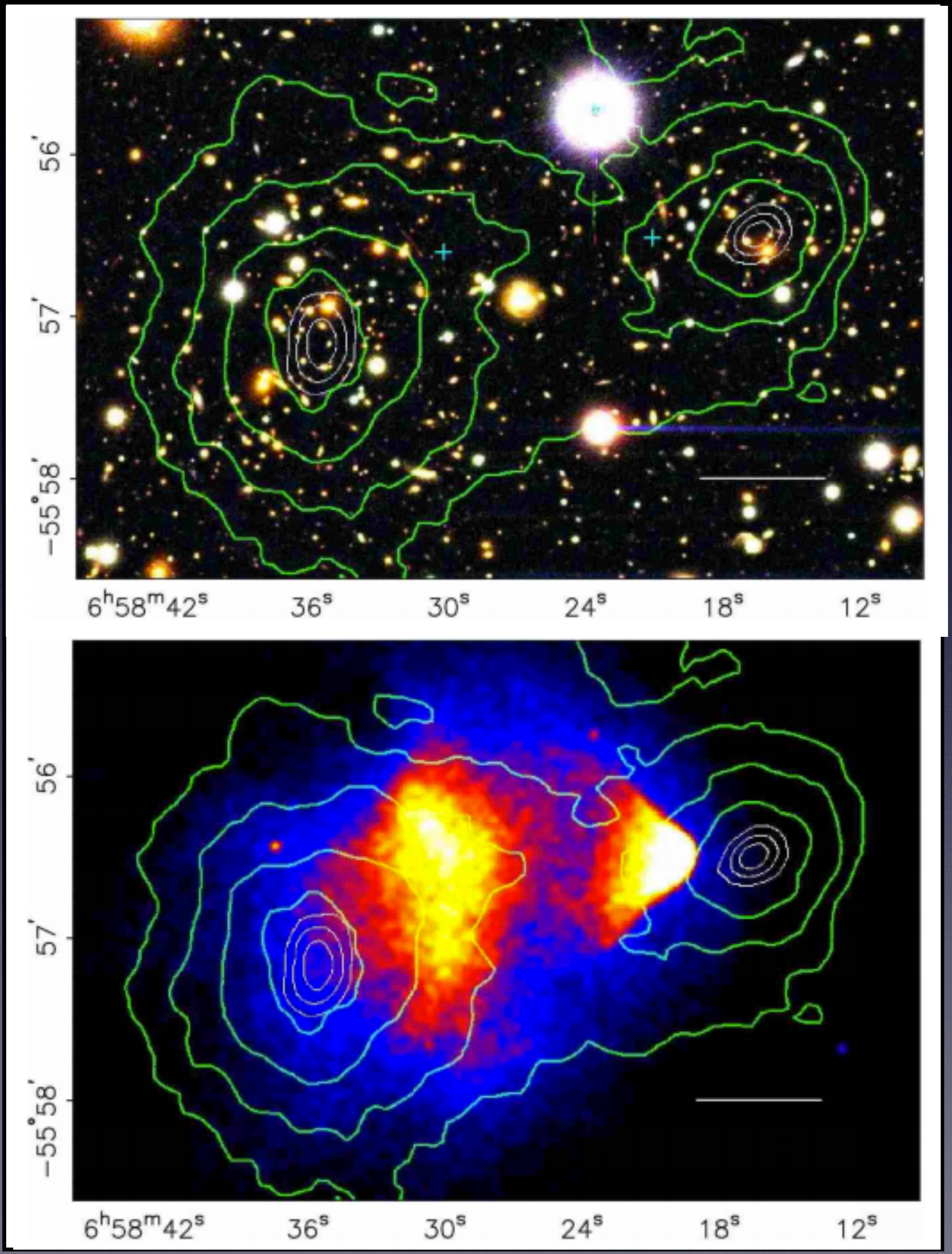
$$\frac{v^2}{r} = \frac{GM(r)}{r^2}$$

$$M(r) = M \Rightarrow v \propto \frac{1}{\sqrt{r}}$$

$$M(r) \propto r \Rightarrow v\,\mathrm{constant}$$

New matter or modified gravity?

- Clowe et al 2006: studied the Bullet Cluster, system of two colliding clusters.
  - X-ray maps from CHANDRA to study distribution of hot plasma (main baryonic component).
  - Weak gravitational lensing to study mass distribution.
- Result: a substantial displacement between the two.
- Attributed to a collisionless cold dark matter component. When the clusters collided, the dark matter halos passed through each other without slowing down - unlike the gas.

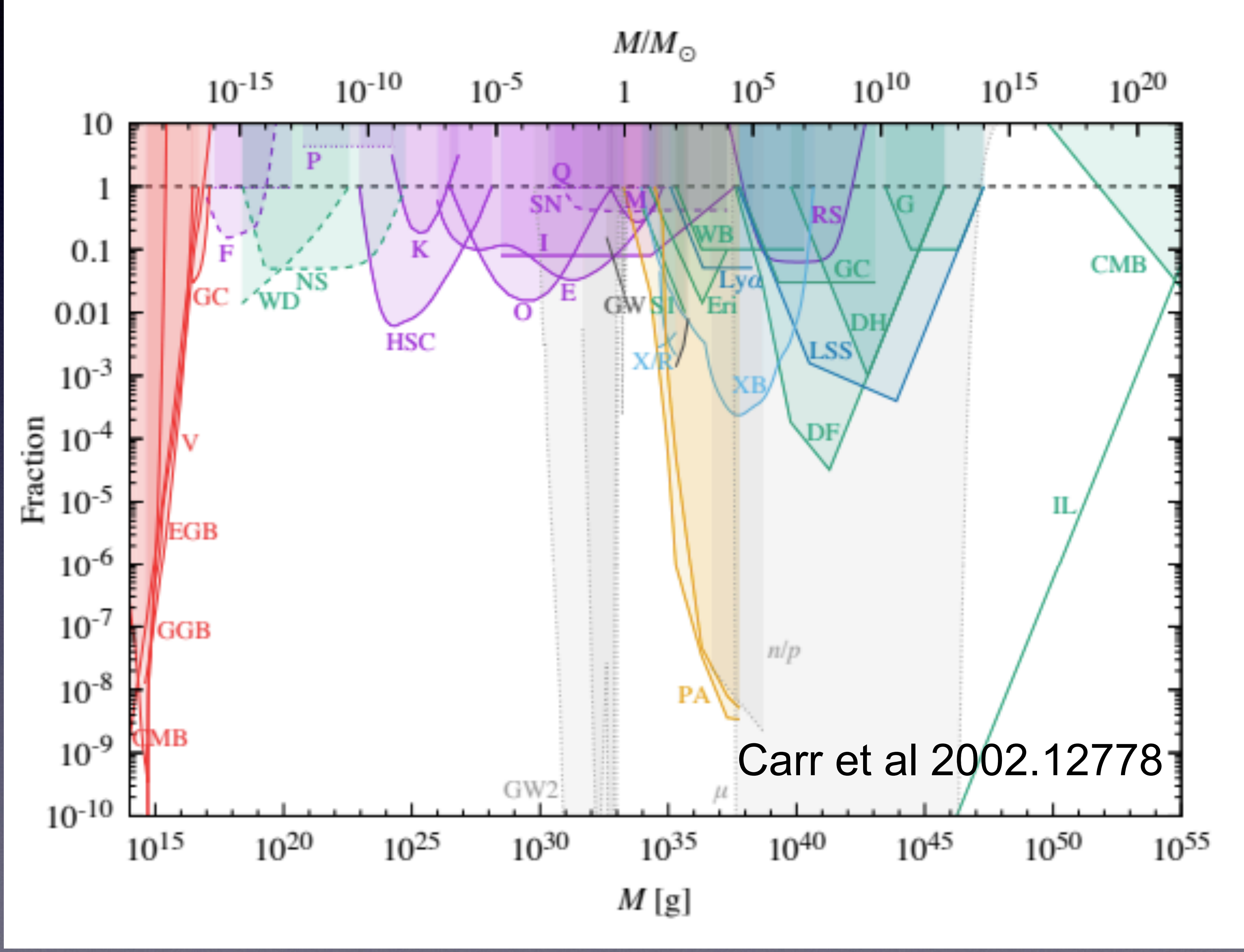


Particle DM or MACHOs?

- MACHOs = Massive Compact Halo Objects, e.g. brown dwarfs, primordial black holes. Effectively collisionless, and probably exist to some degree: can they be most of the dark matter?
- Most-studied example is primordial black holes
- Usual picture is that they are formed during inflation (although recent studies suggest lack of perturbative control of such scenarios [Inomata et al ’22, Kristiano et al ’22])
- May also be formed in subsequent phase transition [e.g. Kawana et al ’21]

Primordial black holes (PBHs) as dark matter

- Primordial black holes are a viable DM candidate but require new physics for their production
- There is an open window for all DM to be PBHs for PBH masses M~10^{17}-10^{23}g
- At the low end of this window, constraints come from non-observation of evaporation via Hawking radiation
- At the high end, gravitational lensing probes become constraining



Dashed lines = constraints have been proposed, but are not reliable or have been refuted

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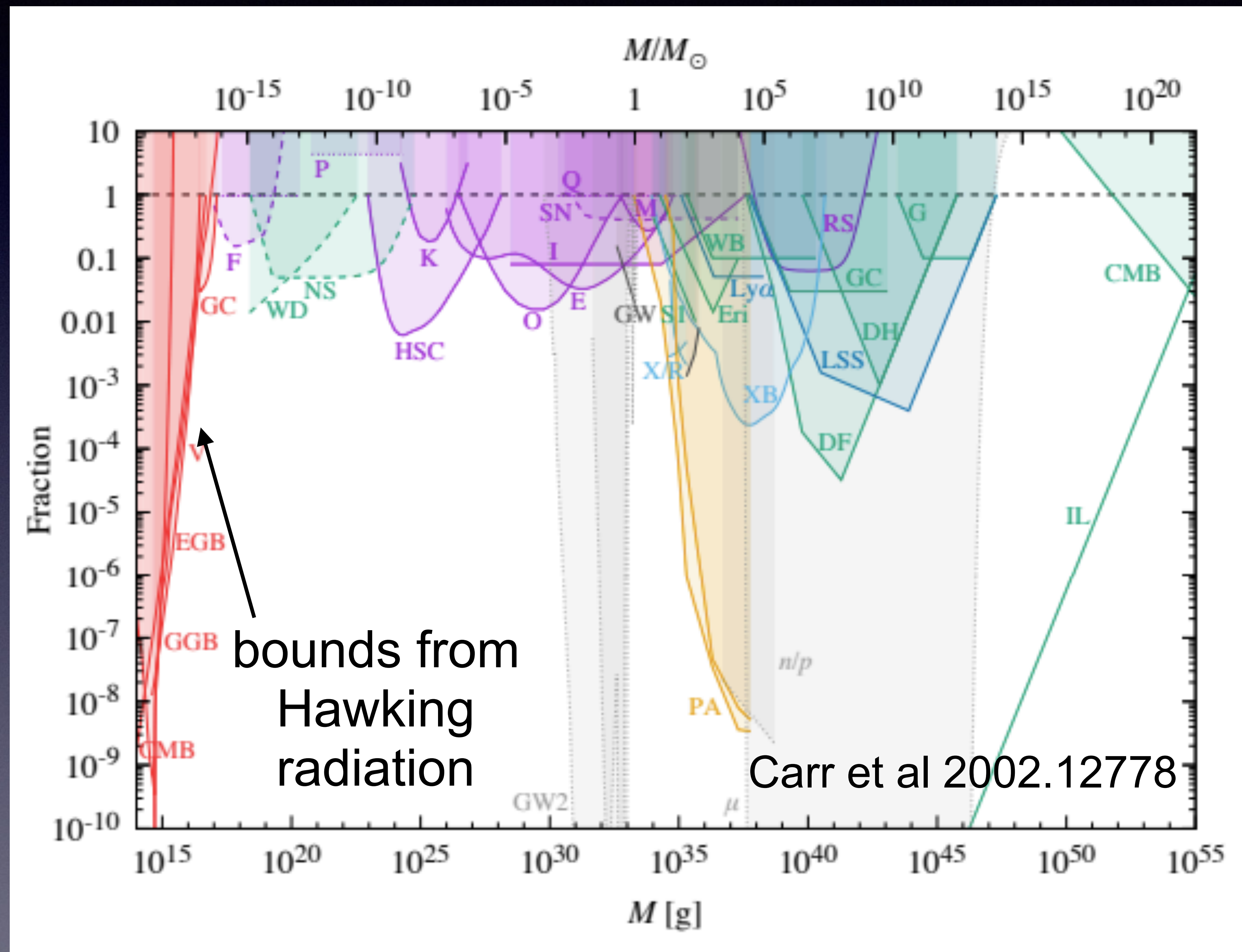

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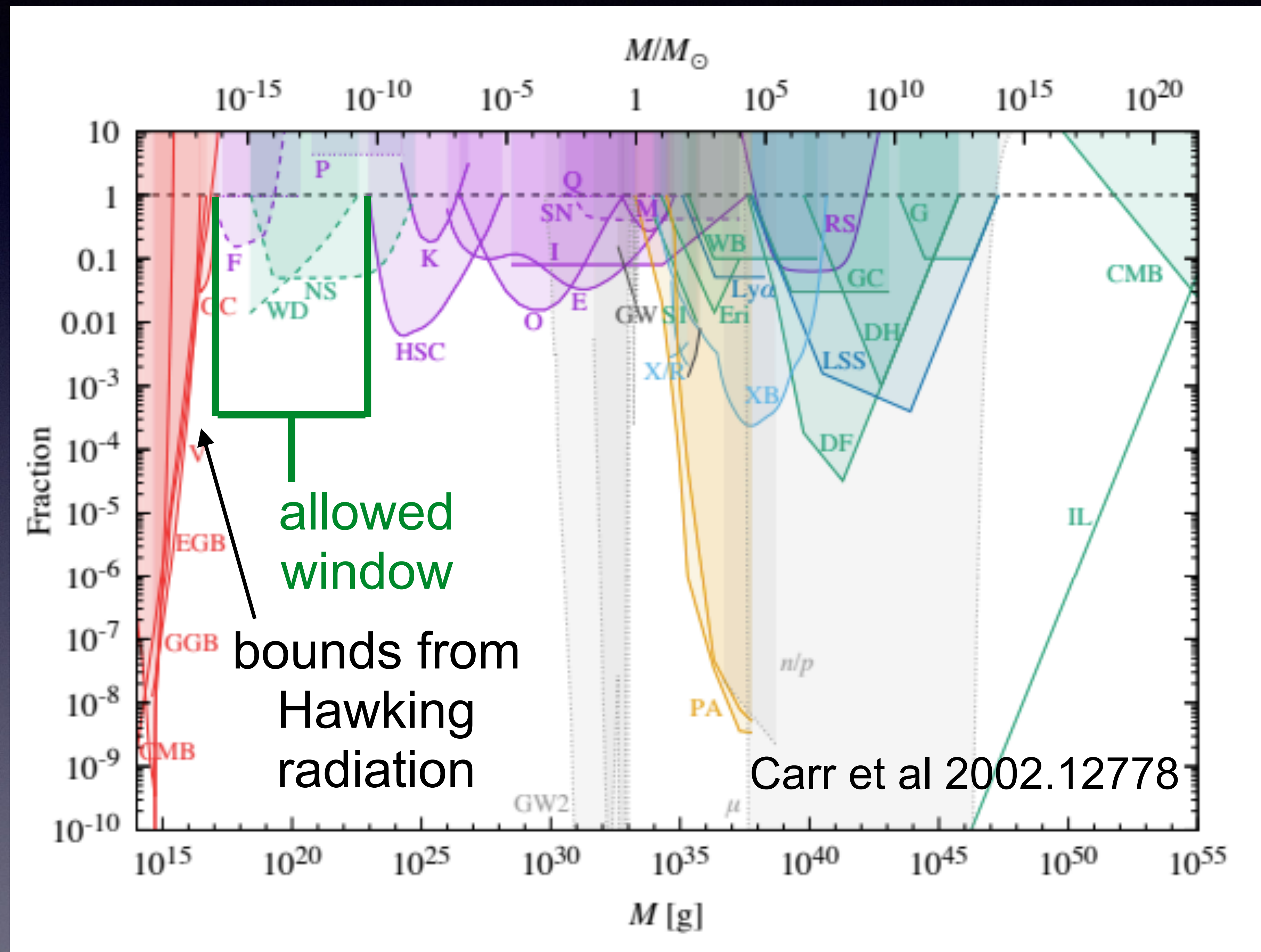

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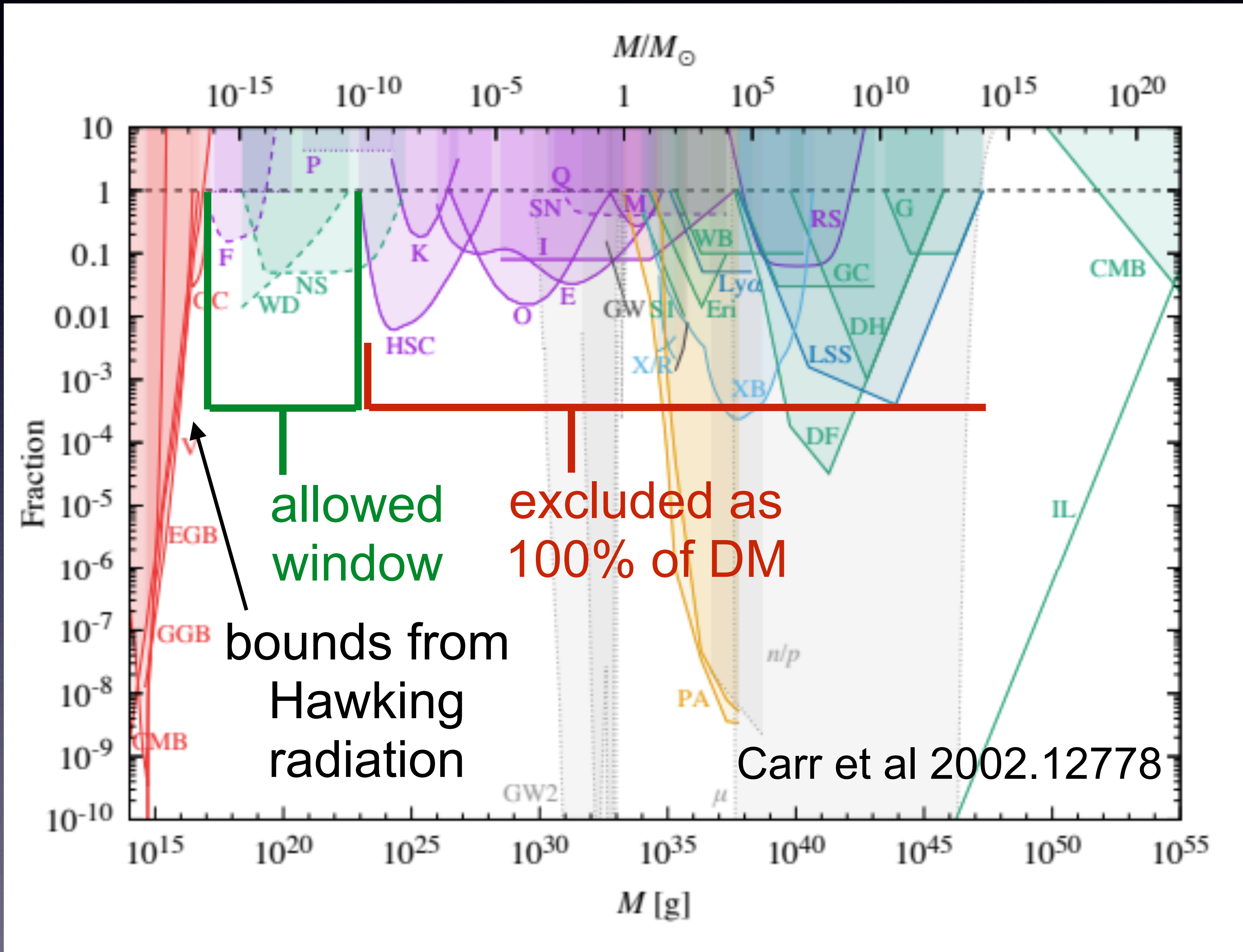

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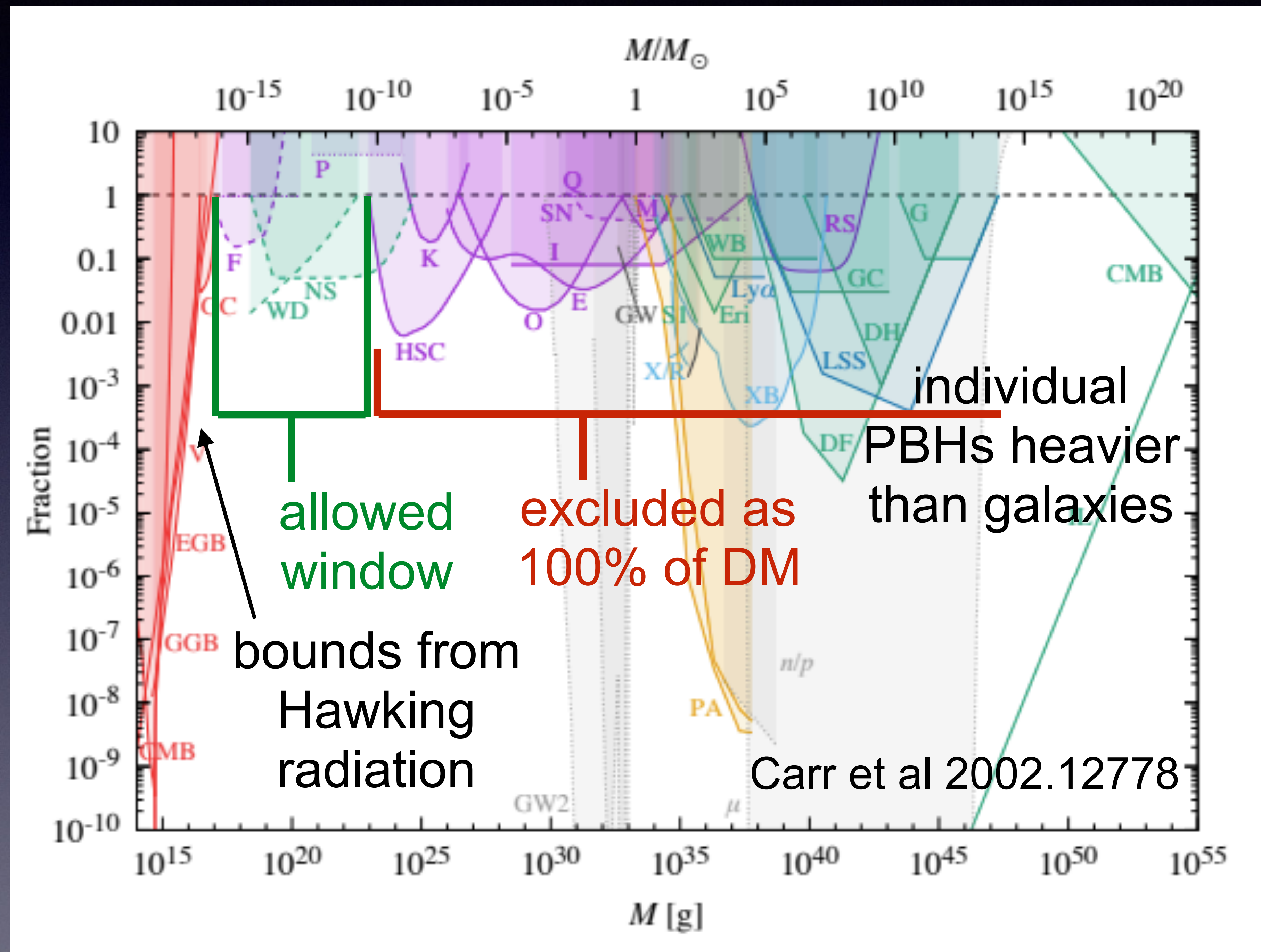

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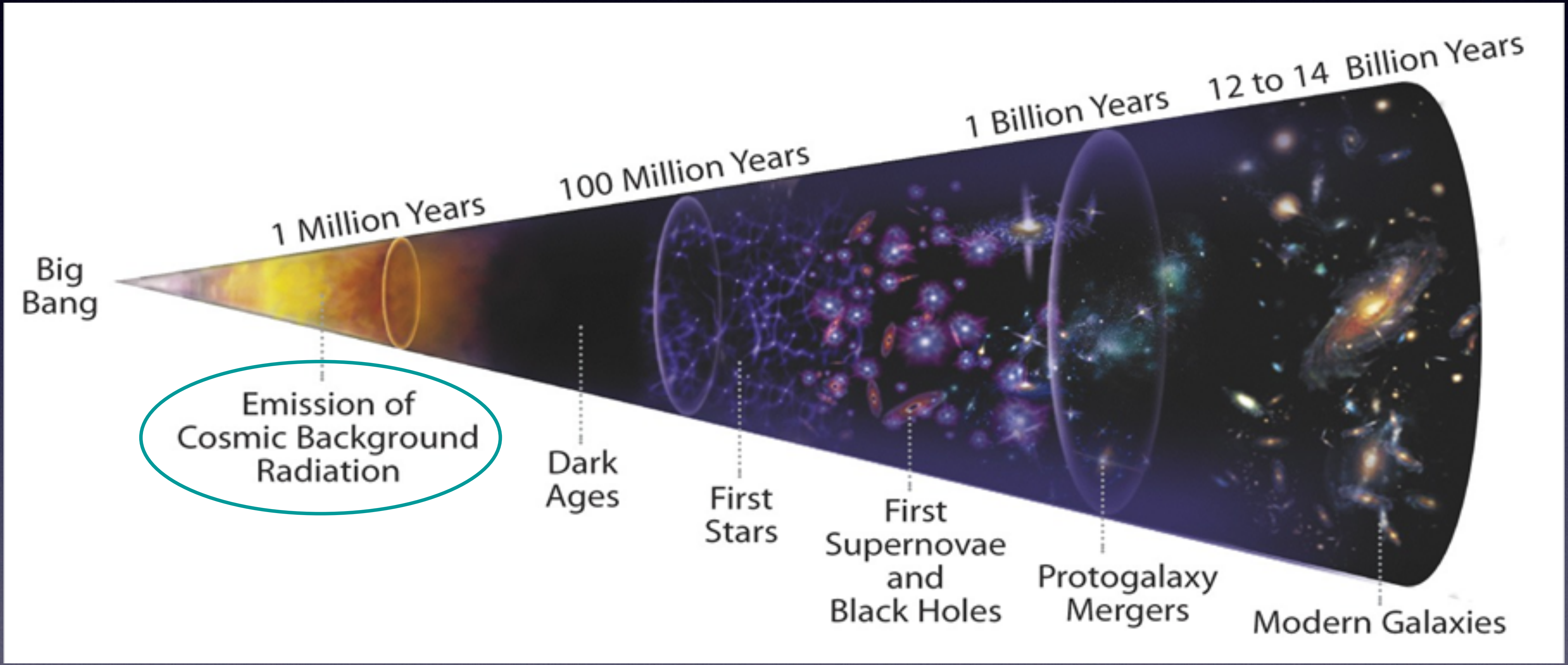
The cosmic microwave background



- When the universe was ~400 000 years old (redshift ~ 1000), H gas became largely neutral, universe transparent to microwave photons.
- Cosmic microwave background (CMB) radiation was last scattered at that time. We can measure that light now.
- Gives us a snapshot of the universe very early in its history.

CMB anisotropies

- Universe at z~1000 was a hot, nearly perfectly homogeneous soup of light and atoms.
- Oscillations in temperature/density from competing radiation pressure and gravity.
- Photon temperature anisotropies today provide a "snapshot" of temperature/density inhomogeneities at recombination.
- Peaks occur at angular scales corresponding to a harmonic series based on the sound horizon at recombination.

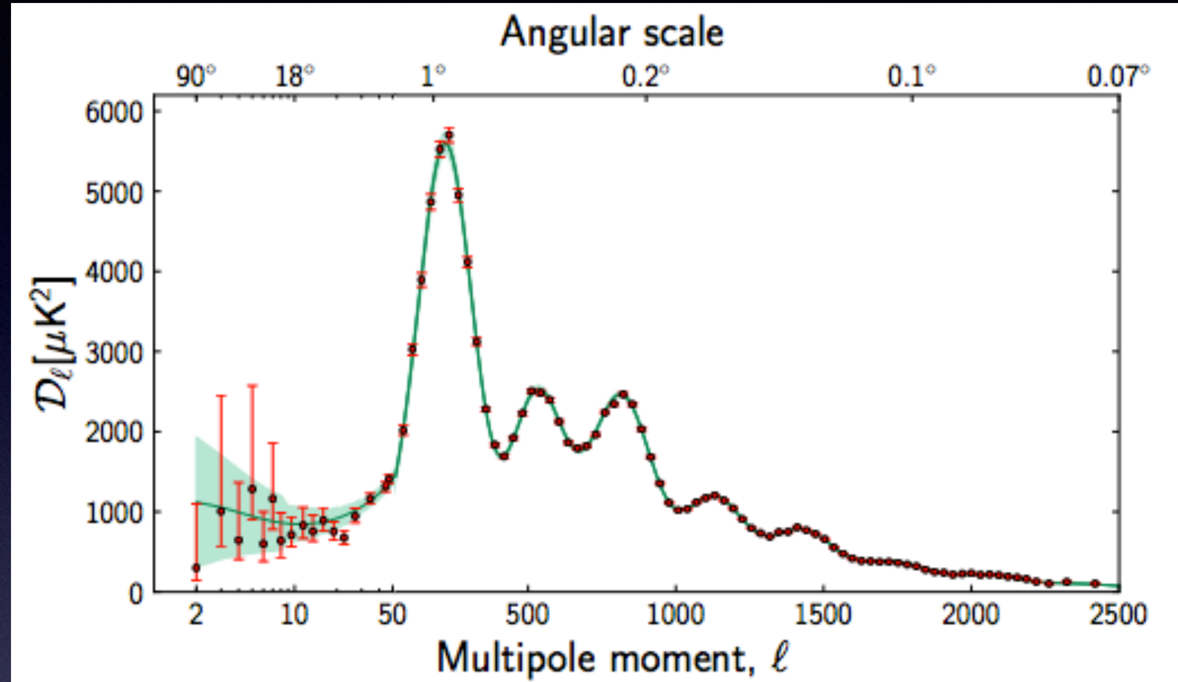


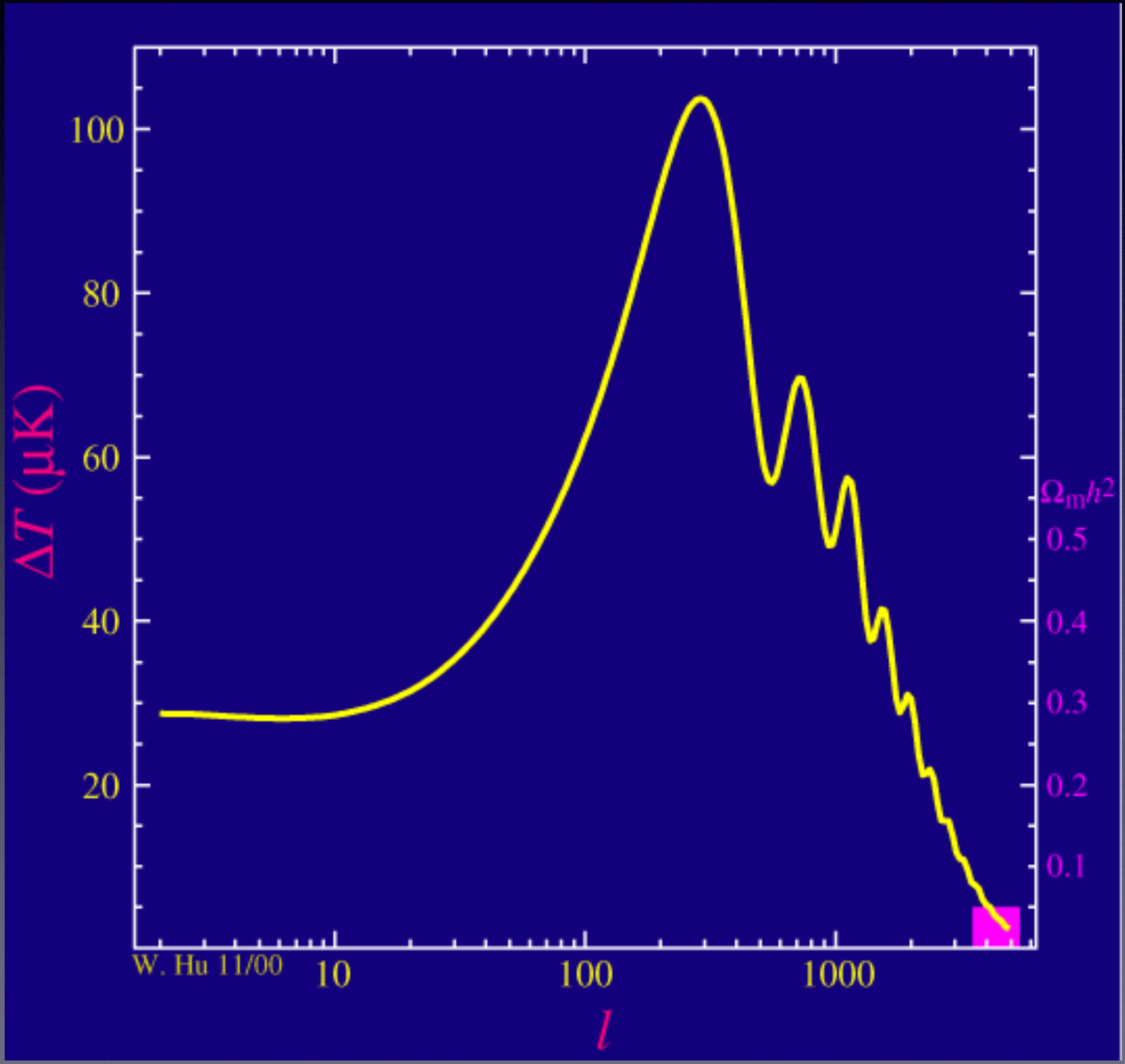
Figure 37. The 2013 *Planck* CMB temperature angular power spectrum. The error bars include cosmic variance, whose magnitude is indicated by the green shaded area around the best fit model. The low-ℓ values are plotted at 2, 3, 4, 5, 6, 7, 8, 9.5, 11.5, 13.5, 16, 19, 22.5, 27, 34.5, and 44.5.

Table 8. Constraints on the basic six-parameter ΛCDM model using *Planck* data. The top section contains constraints on the six primary parameters included directly in the estimation process, and the bottom section contains constraints on derived parameters.

	Planck		*Planck*+WP	
Parameter	Best fit	68% limits	Best fit	68% limits
$\Omega_b h^2$	0.022068	0.02207 ± 0.00033	0.022032	0.02205 ± 0.00028
$\Omega_c h^2$	0.12029	0.1196 ± 0.0031	0.12038	0.1199 ± 0.0027
$100\theta_{MC}$	1.04122	1.04132 ± 0.00068	1.04119	1.04131 ± 0.00063
τ	0.0925	0.097 ± 0.038	0.0925	$0.089^{+0.012}_{-0.014}$
n_s	0.9624	0.9616 ± 0.0094	0.9619	0.9603 ± 0.0073
$\ln(10^{10} A_s)$	3.098	3.103 ± 0.072	3.0980	$3.089^{+0.024}_{-0.027}$
Ω_Λ	0.6825	0.686 ± 0.020	0.6817	$0.685^{+0.018}_{-0.016}$
Ω_m	0.3175	0.314 ± 0.020	0.3183	$0.315^{+0.016}_{-0.018}$
σ_8	0.8344	0.834 ± 0.027	0.8347	0.829 ± 0.012
z_{re}	11.35	$11.4^{+4.0}_{-2.8}$	11.37	11.1 ± 1.1
H_0	67.11	67.4 ± 1.4	67.04	67.3 ± 1.2
$10^9 A_s$	2.215	2.23 ± 0.16	2.215	$2.196^{+0.051}_{-0.060}$
$\Omega_m h^2$	0.14300	0.1423 ± 0.0029	0.14305	0.1426 ± 0.0025
Age/Gyr	13.819	13.813 ± 0.058	13.8242	13.817 ± 0.048
z_*	1090.43	1090.37 ± 0.65	1090.48	1090.43 ± 0.54
$100\theta_*$	1.04139	1.04148 ± 0.00066	1.04136	1.04147 ± 0.00062
z_{eq}	3402	3386 ± 69	3403	3391 ± 60

Measuring dark matter from the CMB

- Model universe as photon bath + coupled baryonic matter fluid + decoupled "dark" matter component (+ "dark" radiation, i.e. neutrinos).
- Dark component: does not experience radiation pressure, effects on oscillation can be separated from that of baryons.
- Result: this simple model fits the data well with a dark matter component about 5x more abundant than baryonic matter (total matter content is ~0.3 x critical density).

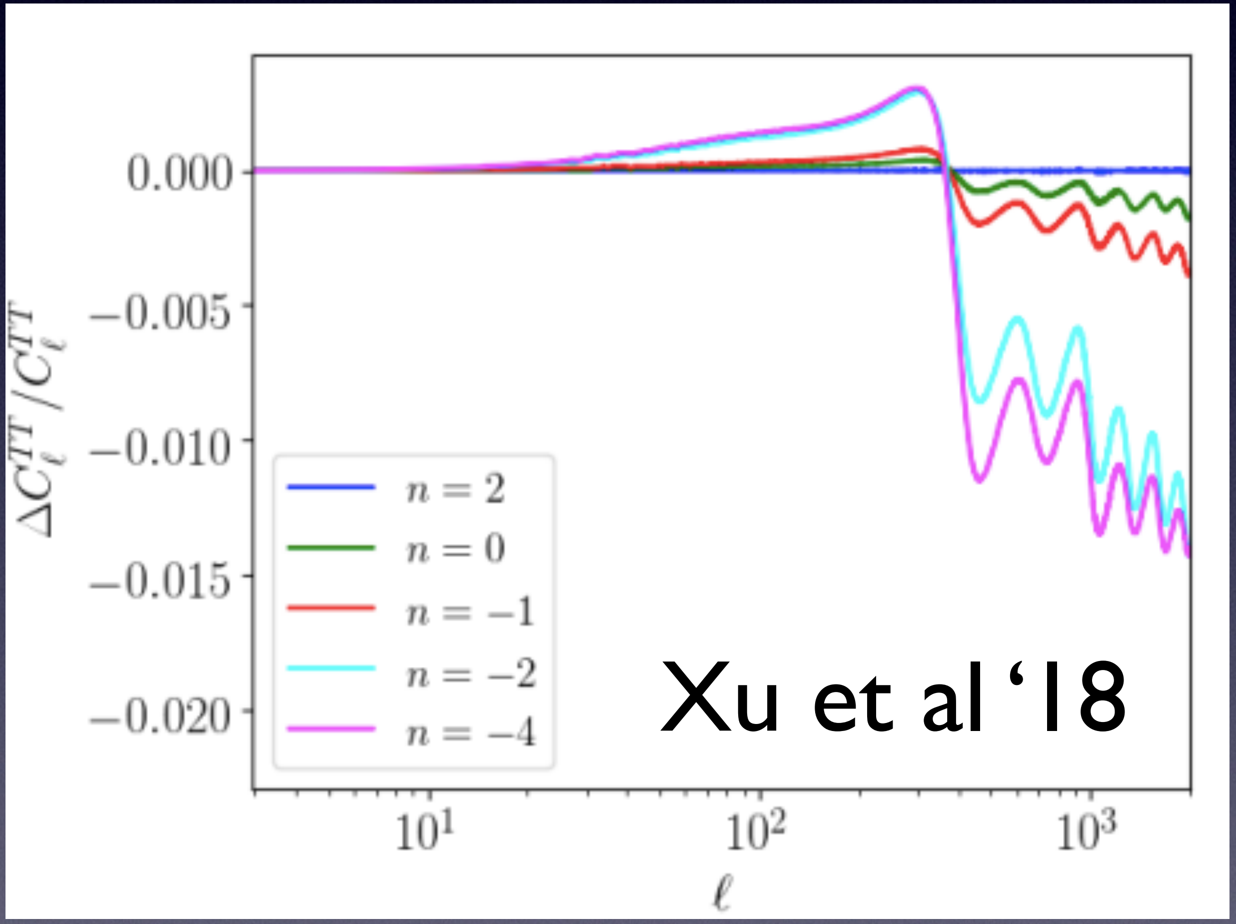


Wayne Hu, http://background.uchicago.edu/~whu/

CMB constraints on DM-baryon interactions

- Non-negligible scattering between the DM and baryons would dampen the growth of density fluctuations in the DM fluid + modify the relative velocity of the DM and baryon fluids

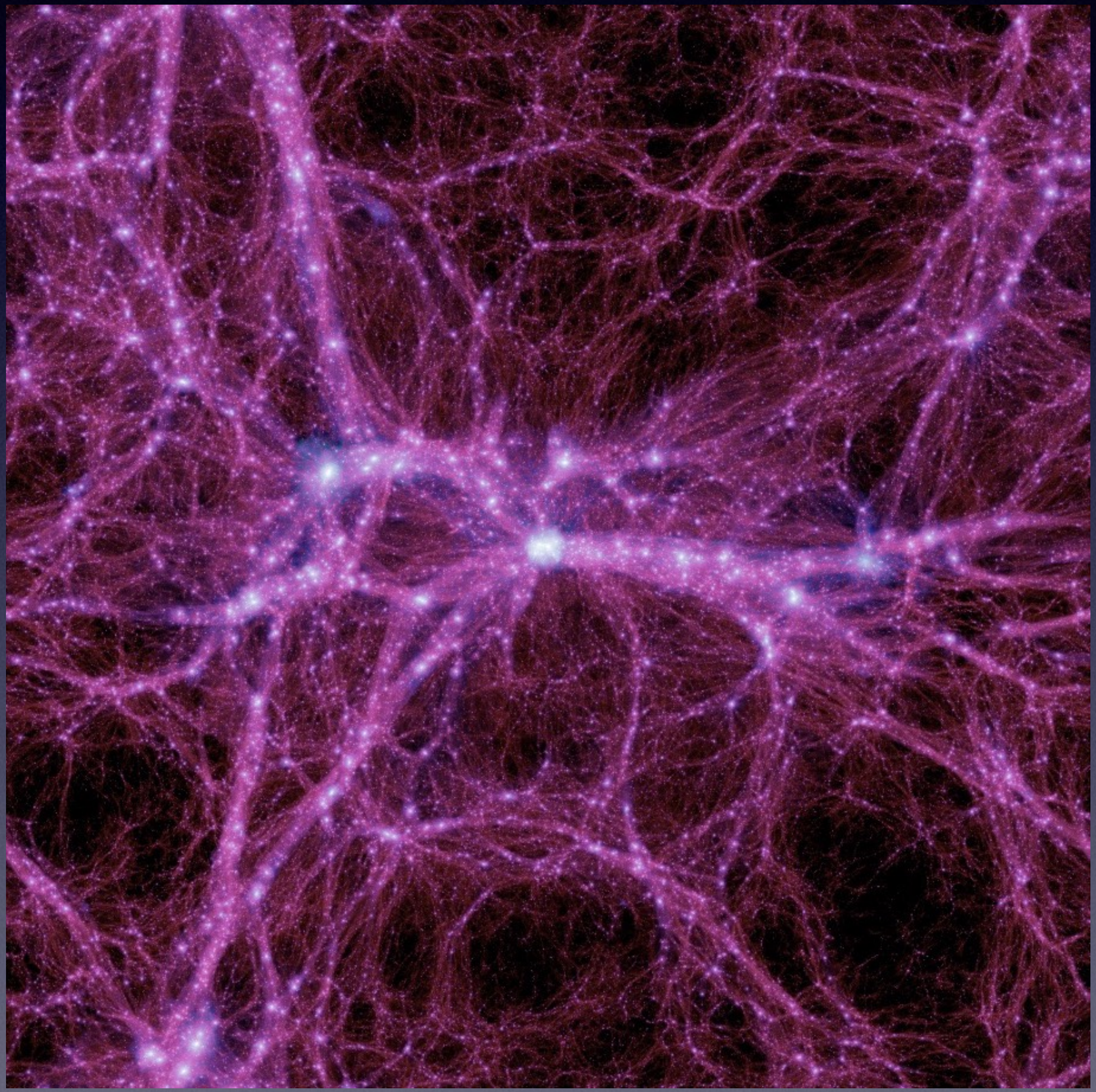
- Leads to a suppression of power on small scales + a shift in the acoustic peaks [e.g. Xu et al '18]



- CMB measurements thus constrain DM-baryon scattering (by the bulk of DM - can evade bounds if only a sub-percent fraction scatters, e.g. Kovetz et al '18). See also lectures by Prof. Blum.

Structure formation

- CMB also maps out initial conditions for cosmic structure formation.
- After the photons decouple from the baryons, overdensities continue to grow under gravity, eventually collapsing into virialized structures.



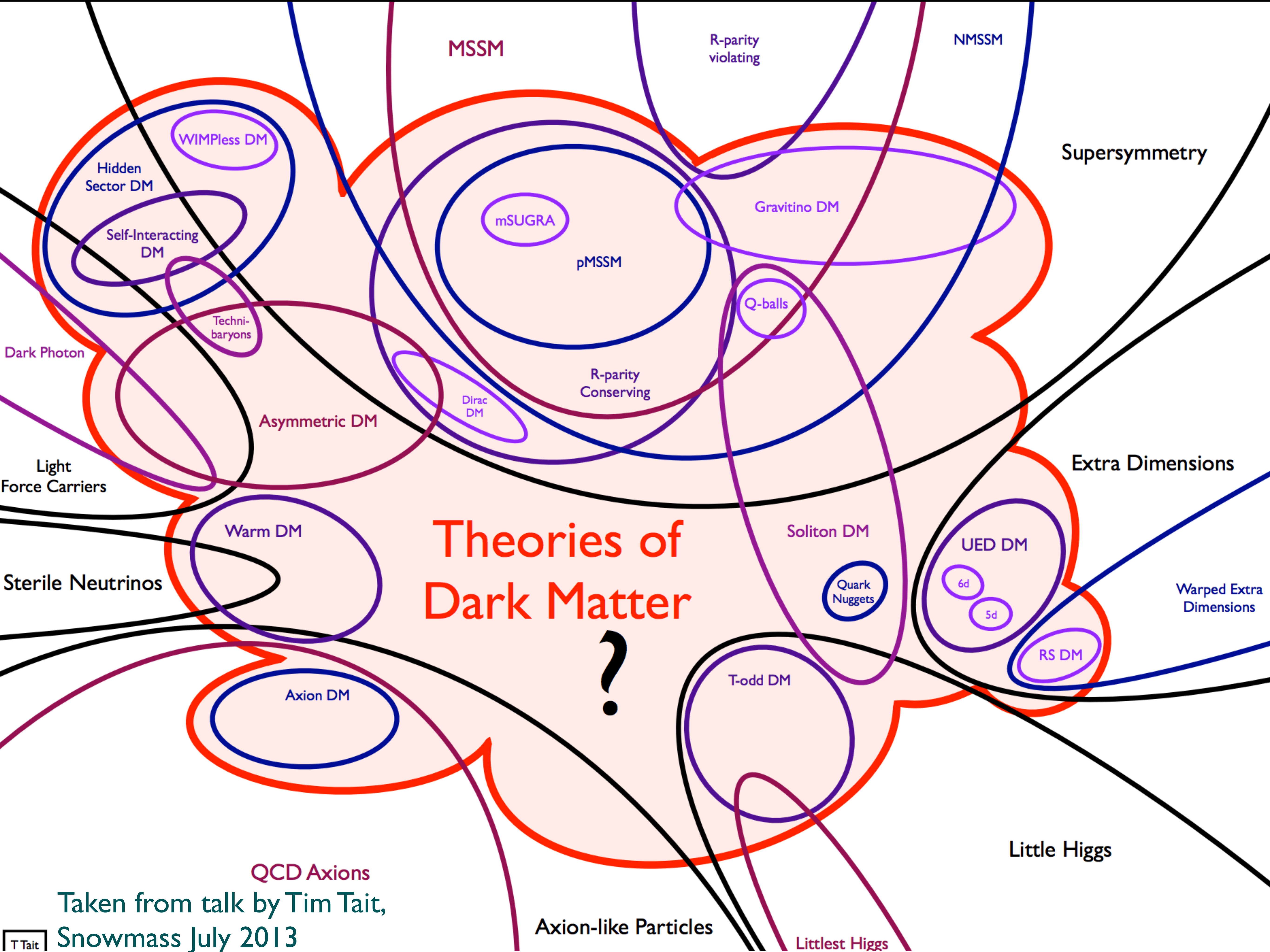
Hot or cold? (or warm)

- Structure formation varies markedly according to the kinematics of the dark matter, in particular whether it can free-stream during the growth of perturbations.
- If most DM is "hot" (relativistic during the early phases of structure formation), free-streaming erases structures on small scales. Large structures form first, then fragment.
- If most DM is "cold" (non-relativistic throughout this epoch), small clumps of DM form first, then accrete together to form larger structures.
- The relative ages of galaxies and clusters tell us that the bulk of DM must be cold - if dark matter was hot, galaxies would not have formed by the present day.
- Equivalently, hot dark matter predicts a low-mass cutoff in the matter power spectrum, that is not observed.
- Neutrinos <u>are</u> hot dark matter - but cannot be all the DM.

DM as new physics

- Standard Model (SM) of particle physics has been spectacularly successful - but no particle dark matter candidate to our current knowledge. We need something:
  - Stable on cosmological timescales
  - Near-collisionless, i.e. electrically neutral
  - "Cold" or "warm" rather than "hot" - not highly relativistic when the modes corresponding to the size of Galactic dark matter halos first enter the horizon (around $z \sim 10^6$, temperature of the universe around 300 eV).
- Only stable uncharged particles are neutrinos, and they would be hot dark matter.

- DM is one of the most powerful pieces of evidence for physics beyond the SM.
- Everything we have learned so far has come from studying the gravitational effects of dark matter, or from its inferred distribution.



Taken from talk by Tim Tait, Snowmass July 2013

What is the DM mass?

neutrinos | electrons | mesons, hadrons | Higgs, gauge bosons

~eV ~keV ~MeV ~GeV ~TeV ~100 TeV

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Down to 10^{-21} eV
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bosonic (Tremaine-Gunn bound)

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Traditional
WIMP window

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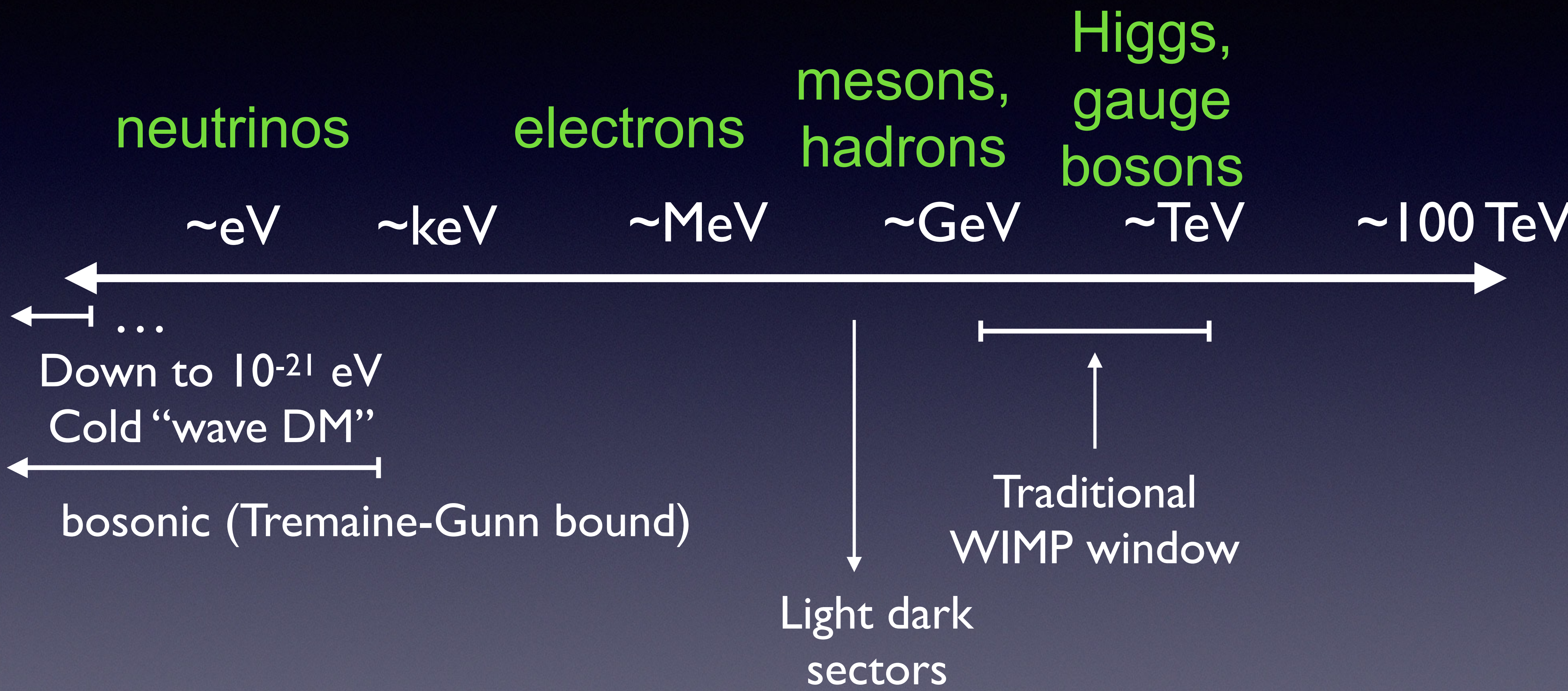

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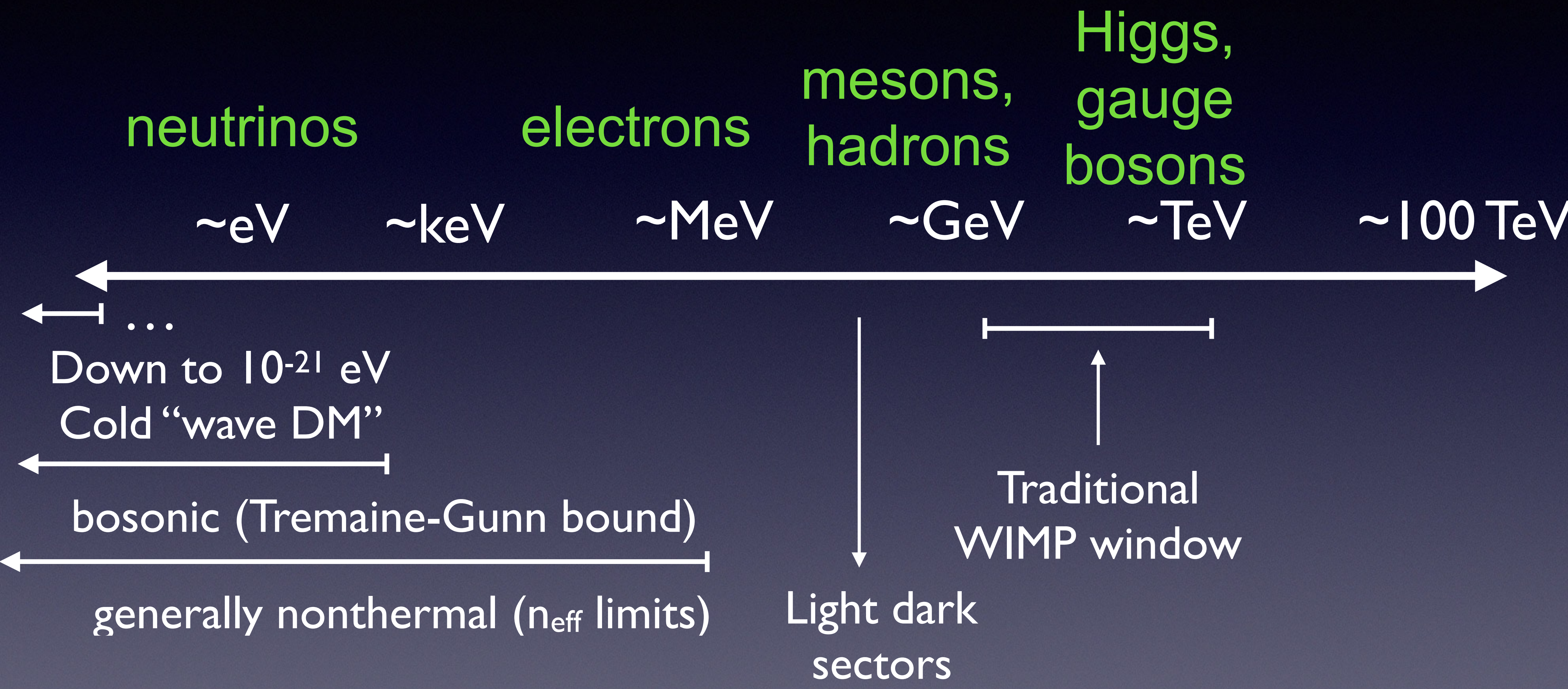

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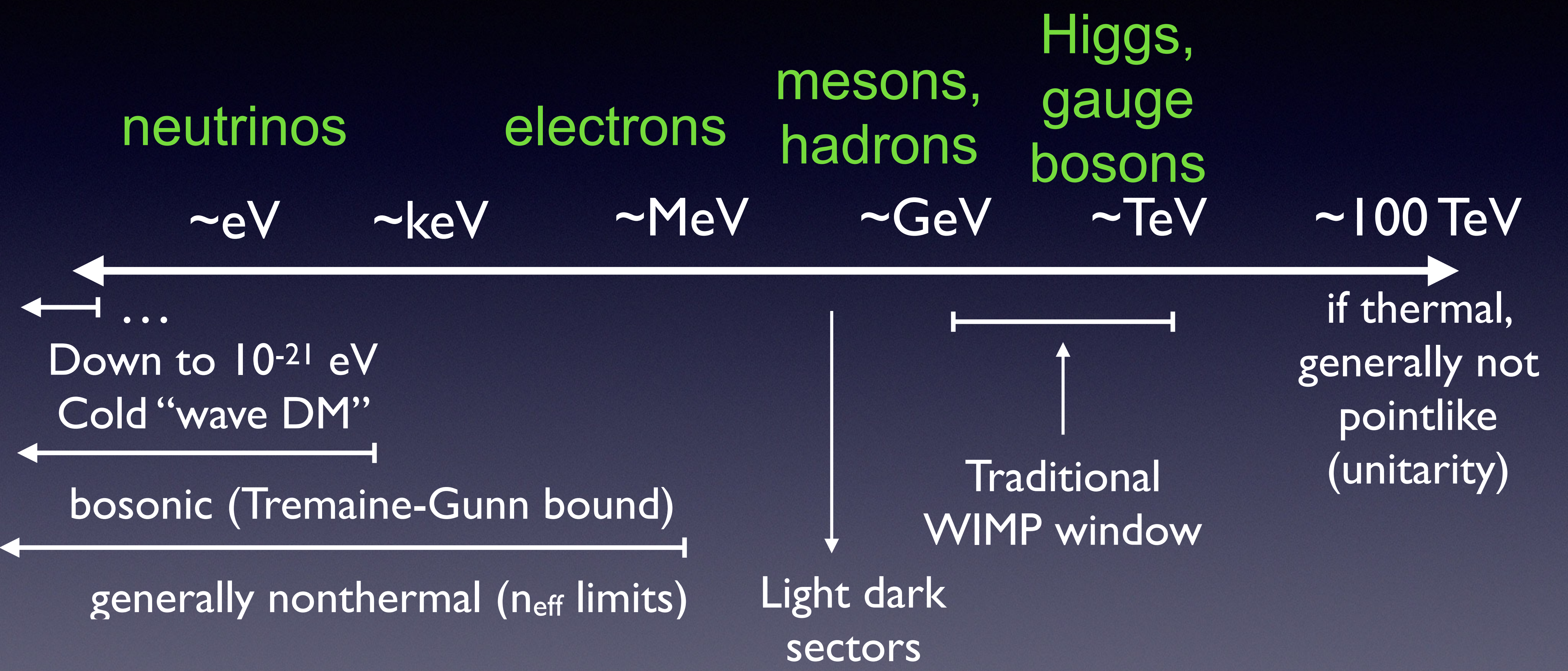

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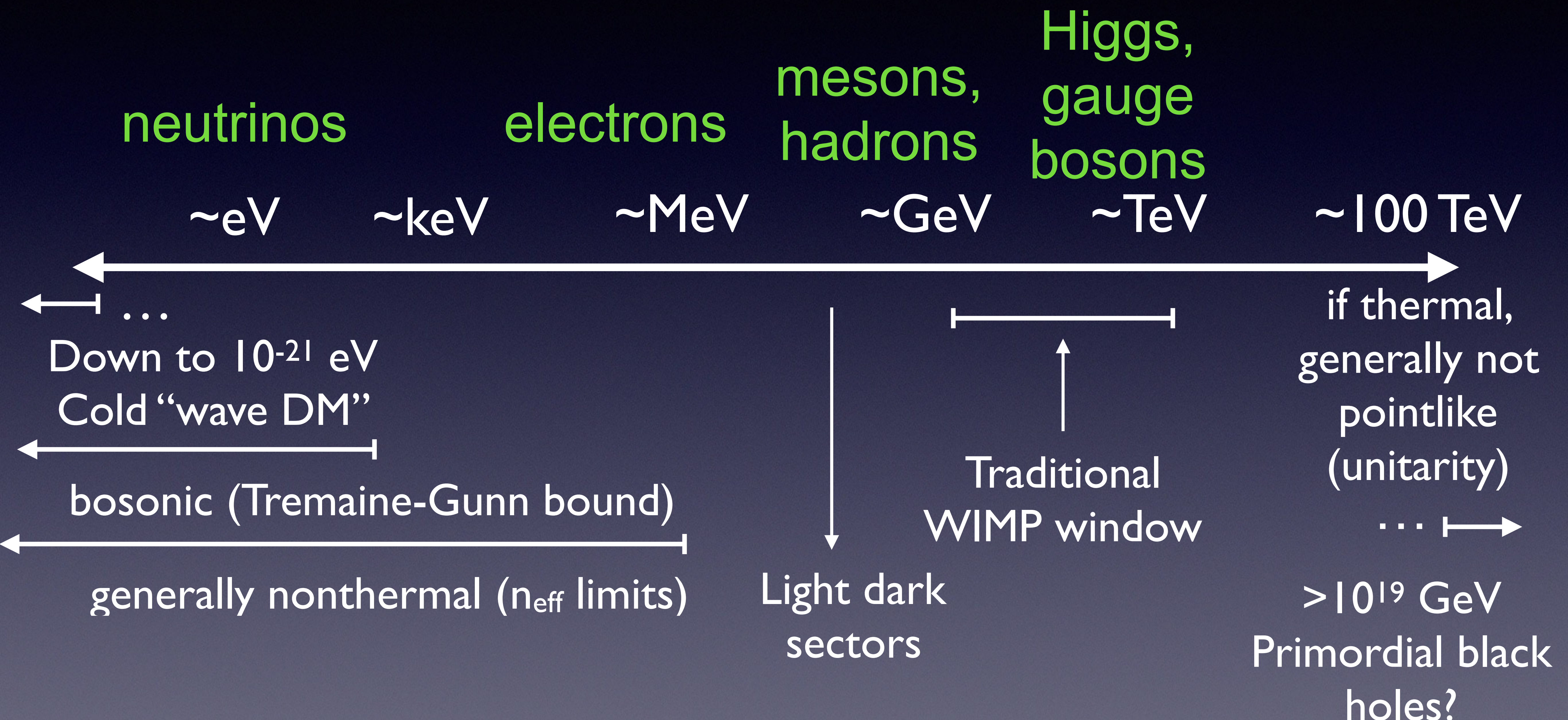
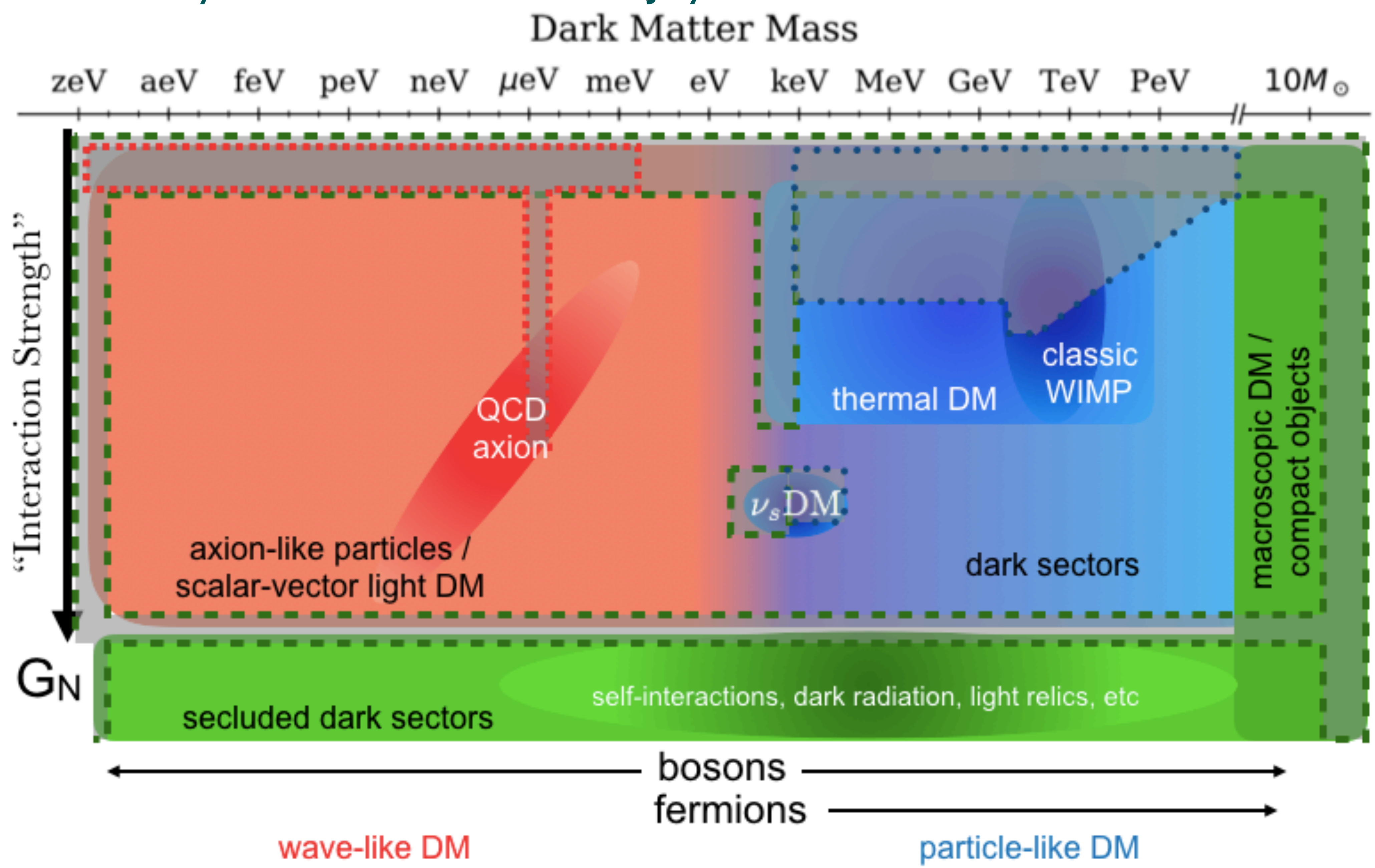


The Cosmic Frontier in the next 10 years

Taken from talk by Aaron Chou, Snowmass July 2022



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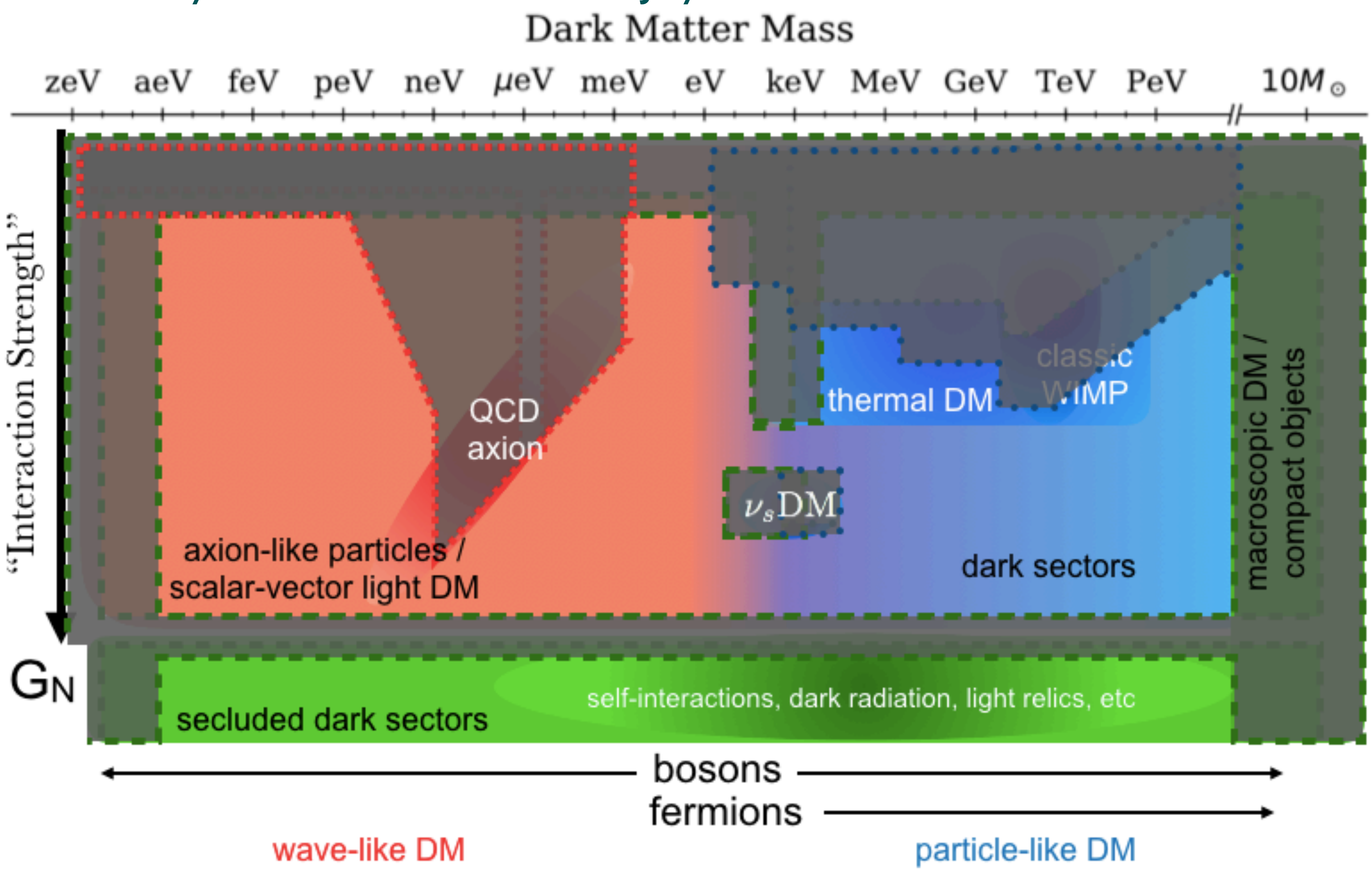
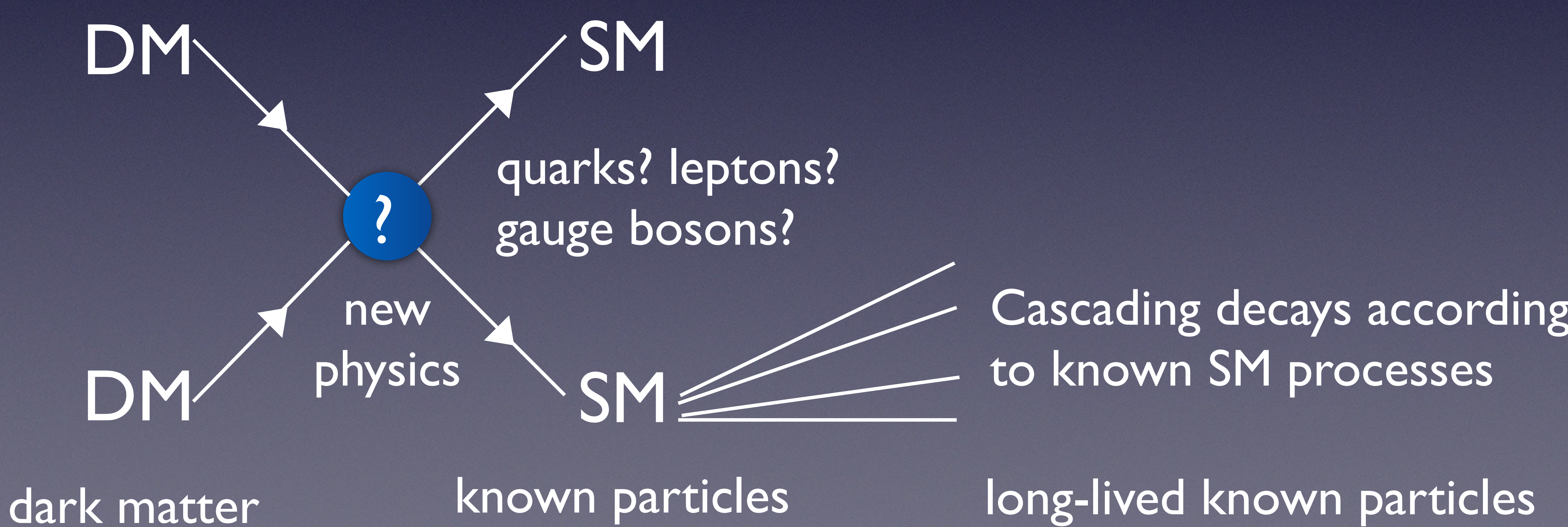


Thermal targets

- Much of DM model-building is informed by the question of whether the DM was ever in thermal contact with the SM after inflation
- DM models with this property often have abundances determined through their SM interactions - provide predictive targets
- Typically these models populate masses at the keV+ scale to avoid warm dark matter bounds
- Classic/simplest scenario: <u>thermal freezeout</u> (and its many variations)

Thermal abundance

- Suppose dark matter:
  - can annihilate to Standard Model particles
  - was at some point kept in <u>full thermal equilibrium</u> with the Standard Model by annihilation



Thermal freezeout

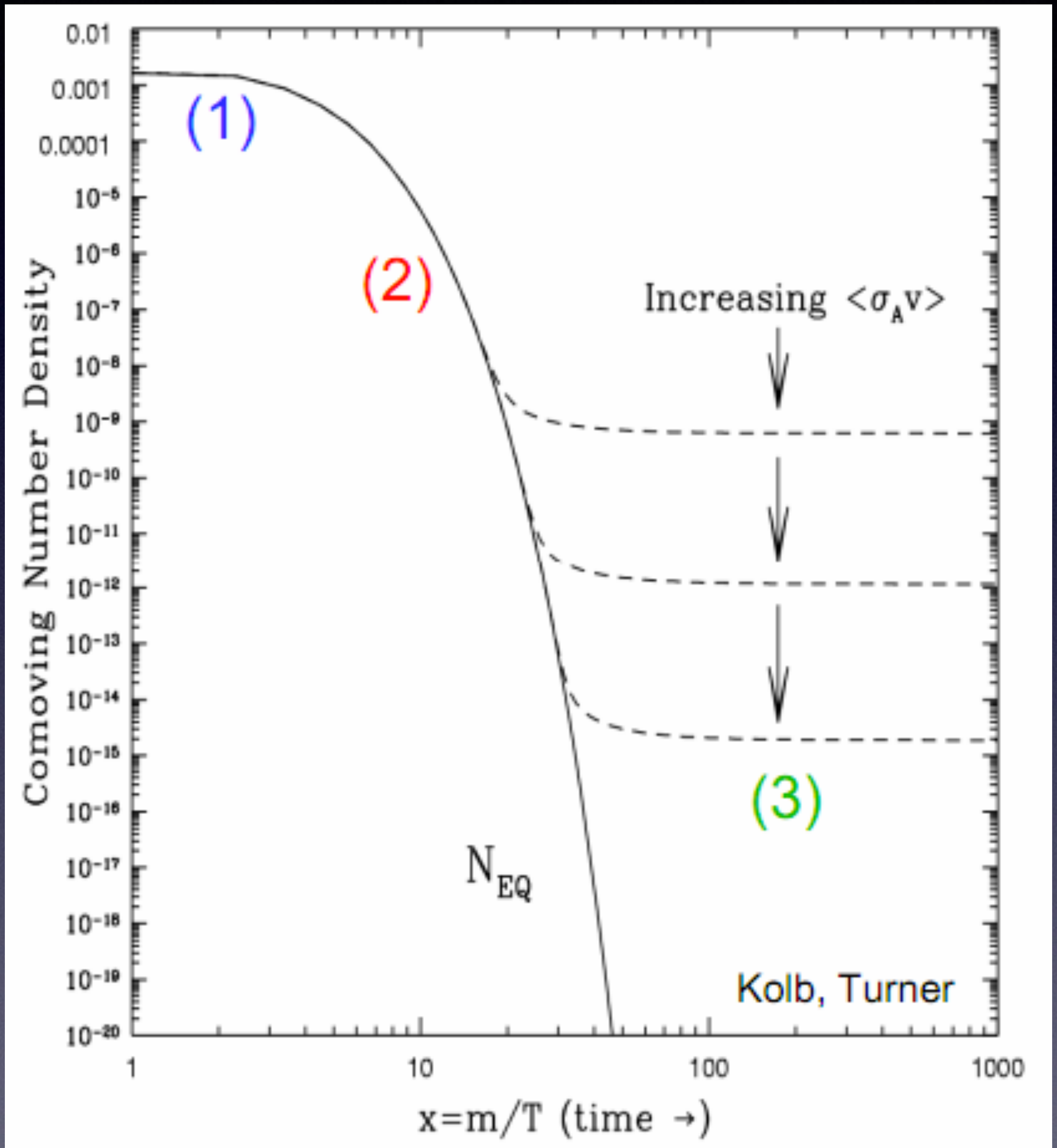
- In the early universe, let the DM particle be thermally coupled to the SM. Can annihilate to SM particles, or SM particles can collide and produce it.

$$\chi\chi \leftrightarrow \mathrm{SM}\,\mathrm{SM} \quad (1)$$

- Temperature(universe) < particle mass => can still annihilate, but can't be produced.

$$\chi\chi \rightarrow \mathrm{SM}\,\mathrm{SM}$$
$$\chi\chi \not\leftarrow \mathrm{SM}\,\mathrm{SM} \quad (2)$$

- Abundance falls exponentially, cut off when timescale for annihilation ~ Hubble time. The *comoving* dark matter density then freezes out.



So (known) late-time density is set by annihilation rate.

$$\langle\sigma v\rangle \sim 3\times 10^{-26}\mathrm{cm}^3/\mathrm{s} \sim \pi\alpha^2/(100\,\mathrm{GeV})^2 \quad (3)$$

Outline of calculation

- Ingredients: annihilation rate for identical particles given by

$$\text{annihilations / dt / dV} = n^2\langle\sigma v\rangle/2$$

- Boltzmann equation:

$$\frac{dn}{dt} + 3Hn = -\langle\sigma v\rangle\left[n^2 - n_{\rm eq}^2\right]$$

- Equilibrium density (Boltzmann distribution):

$$n_{\rm eq} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$$

- Temperature of universe (assume radiation domination):

$$H^2 \propto \rho \propto T^4 \Rightarrow T \propto \sqrt{H} \propto t^{-1/2}$$

Estimating freezeout

- For precision solution, can solve this differential equation numerically (see Steigman et al '12 for results)
- But we can get a simple estimate of important quantities analytically.
  - Freezeout occurs when expansion timescale ~ collision timescale:

    $$H \sim n\langle\sigma v\rangle$$

  - Up to freezeout, $n \sim n_{eq}$, so we require

    $$H \sim g(mT/2\pi)^{3/2} e^{-m/T} \langle\sigma v\rangle$$

- After freezeout, we expect the DM density to scale as a^{-3} as the universe expands
- We see that the condition for freezeout is exponentially sensitive to the ratio m/T; thus we expect freezeout to occur for m/T ~ 1

Estimating the freezeout temperature

Estimating the freezeout temperature

- Slightly more precisely: we know that the original # density of DM is similar to that of radiation. Today the mass of DM per photon can be estimated as:
  5 GeV (DM mass per baryon) x 6 x 10^{-10} (baryon-to-photon ratio)
  ~ 3 eV

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- Thus number density of DM is suppressed by a factor of O(m/eV) relative to radiation
- Expect $e^{-m/T}$ ~ eV/m ⇒ m/T ~ ln(m/eV) ~ 20 for GeV-scale DM, and this ratio scales logarithmically with mass

Estimating the required cross section

- To estimate the required cross-section, we can use the condition $H \sim n\langle\sigma v\rangle$ and $H \sim T_f^2/m_{Pl}$
- After freezeout n is just diluted with the expansion of the universe, and at matter-radiation equality (MRE) we should have $m n_{MRE} \sim T_{MRE}^4$
- T is proportional to 1/a (ignoring heating of the thermal bath), so we can write:

$$\langle\sigma v\rangle \sim H_f/n_f \sim \frac{T_f^2}{m_{\mathrm{Pl}}} \times \frac{1}{n_{\mathrm{MRE}}(T_f/T_{\mathrm{MRE}})^3} \sim \frac{1}{T_f m_{\mathrm{Pl}}} \times \frac{T_{\mathrm{MRE}}^3}{T_{\mathrm{MRE}}^4/m}$$

$$\Rightarrow \langle\sigma v\rangle \sim \frac{m/T_f}{m_{\mathrm{Pl}} T_{\mathrm{MRE}}}$$

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O(10) number as discussed earlier

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$$\Rightarrow \langle\sigma v\rangle \sim \boxed{\frac{m/T_f}{m_{\rm Pl} T_{\rm MRE}}}$$

$\sim 1/(10^{19}$ GeV x 1 eV$) \sim 1/(100$ TeV$)^2$

Caveat: hot/warm DM

- Since the mass of DM per photon is O(eV), if we have eV-scale DM then there is no need for a Boltzmann suppression: DM can freeze out while relativistic, no need for $T_f \sim m$
- If 100% of the DM is in this category, violates constraints from structure formation as discussed earlier (OK for ~1% of the DM to be hot [e.g. Archidiacono et al '13])
- Modestly heavier DM can freeze out while still having appreciable velocity: "warm DM", leaves imprints on DM structure (see lectures by Prof. Blum)

Characteristic mass scale for cold DM

- If the DM is weakly-coupled and annihilates at tree-level, and there are no large hierarchies in the problem, we might expect $\sigma v_{rel} \sim \alpha^2/m^2$ for some coupling α
- Equating this with our estimate for the thermal-relic xsec, we find: $\alpha^2/m^2 \sim 20/(m_{Pl}\, T_{MRE})$
  $\Rightarrow m \sim \alpha/\sqrt{20}\ (m_{Pl}\, T_{MRE})^{1/2} \sim (\alpha/\sqrt{20})$ 100 TeV
- This suggests that for $\alpha \sim 10^{-2}$ (weak-scale), we find m~200 GeV (weak-scale!). This is sometimes called the "WIMP miracle". (WIMP = Weakly Interacting Massive Particle.)
- Also suggests we may run into trouble for m ~ 100 TeV or larger (requires strong coupling, tree-level estimate breaks down): we will return to this next lecture

Multi-body annihilation

- What if there are 2+ DM particles in the initial state?
- Such N-body annihilations (N > 2) are suppressed at low density, but can dominate freezeout if the 2-body annihilations are suppressed via symmetries or kinematics
- Now freezeout will be even more sensitive to exponentially falling DM density; approximate $T_f \sim m$ for simplicity (we can also estimate T_f/m as for the 2-body case)
- Instead of a cross section we have a rate coefficient R such that the Boltzmann equation looks like:

$$\frac{dn}{dt} + 3Hn = -R\left[n^N - n_{\mathrm{eq}}^N\right]$$

Characteristic mass scale for multi-body annihilation

- Now freezeout occurs at $H_f \sim R\, n^{N-1}$
- As before we estimate $T \propto 1/a, mn_{\mathrm{MRE}} \sim T^4_{\mathrm{MRE}}$
- Then following our previous estimate:

$$R \sim \frac{H_f}{n_f^{N-1}} \sim \frac{T_f^2}{m_{\mathrm{Pl}}} \frac{1}{(n_{\mathrm{MRE}}(T_f/T_{\mathrm{MRE}})^3)^{N-1}} \sim \frac{m^{2(2-N)}}{m_{\mathrm{Pl}} T_{\mathrm{MRE}}^{N-1}}$$

- If we furthermore estimate (by dimensional analysis), $R \sim \alpha^N/m^{3N-4}$ for some coupling α, we find:

$$\alpha^N = m^N/m_{\mathrm{Pl}} T_{\mathrm{MRE}}^{N-1}$$
$$\Rightarrow m \sim \alpha (m_{\mathrm{Pl}} T_{\mathrm{MRE}}^{N-1})^{1/N}$$

Maximum mass scale:

$$m \sim (10^{28}\mathrm{eV} \times 1\mathrm{eV})^{1/2}$$
$$\sim 100\mathrm{TeV}(N=2)$$

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Maximum mass scale:

$$m \sim (10^{28}\text{eV} \times 1\text{eV}^2)^{1/3}$$
$$\sim 1\text{GeV} (N = 3)$$

Characteristic mass scale for multi-body annihilation

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- As before we estimate $T \propto 1/a, mn_{\mathrm{MRE}} \sim T^4_{\mathrm{MRE}}$
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$$\Rightarrow m \sim \alpha (m_{\mathrm{Pl}} T_{\mathrm{MRE}}^{N-1})^{1/N}$$

Maximum mass scale:

$$m \sim (10^{28}\mathrm{eV} \times 1\mathrm{eV}^3)^{1/4}$$
$$\sim 10\mathrm{MeV} (N=4)$$