1 Content of Lecture II

- Weak fields around black holes: the approximation
- Wave equation for integer spin fields
- A second look at no-hair properties
- Tidal properties of black holes
- Dynamics: the QNMs of black holes
- Resonant excitation of modes?
- Wave phenomena for massive fields
- Summary
- Open issues:

References:

1. T. Regge and J. A. Wheeler, Stability of a Schwarzschild Singularity, Physical Review 108 (1957)

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11. The Black Hole Perturbation Toolkit

2 Lecture 2

Let us now look at some dynamics. Start with a massive scalar,

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{k} - \frac{1}{2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} \mu^2 \Psi^2 \right) ,$$

Varying action, get equations of motion

$$\begin{aligned} \nabla_\mu \nabla^\mu \Psi &= \mu^2 \Psi \\ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= k \left(\frac{1}{2} \Psi^{,\mu} \Psi^{,\nu} - \frac{1}{4} g_{\mu\nu} \left(\Psi_{,\alpha} \Psi^{,\alpha} + \mu^2 \Psi^2 \right) \right) \end{aligned}$$

Not easy to solve! Work with $||\Psi|| \ll 1$:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 0 \\ \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi \right) &= \mu^2 \Psi \end{aligned}$$

Schwarzschild solves the first equation. We can solve the second in such a background. Decompose

$$\Psi = \sum_{\ell m} \frac{\Phi}{r} Y_{\ell m}(\theta, \phi) \,,$$

since $Y_{\ell m}$ are complete set (cf. notebook "scattering_scalars"). Find,

$$\frac{\partial^2 \Phi}{\partial r_*^2} - \frac{\partial^2 \Phi}{\partial t^2} - V\Phi = 0$$

$$\frac{dr_*}{dr} = \frac{1}{f} = \frac{1}{1-2M/r} \,, \qquad V = f\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

For vectors and tensors, need slightly different procedure. Take Maxwell theory

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{k} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \,, \quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

As before, solve $\nabla_\mu F^{\mu\nu} = 0$. A_μ is vector and separation needs care, components must transform like vector. Construct 3 vectors with help of scalar harmonics,

$$
\begin{aligned}
\nabla Y_{\ell m} &= (0, \partial_\theta Y_{\ell m}, \partial_\phi Y_{\ell m}) \\
\boldsymbol{L} Y_{\ell m} &= \left(0, \frac{i}{\sin\theta}\partial_\phi Y_{\ell m}, -i\sin\theta\partial_\theta Y_{\ell m}\right) \\
\boldsymbol{r}_r Y_{\ell m} &= (Y_{\ell m}, 0, 0)
\end{aligned}
$$

To include time, add extra independent component $\boldsymbol{e}_t Y_{\ell m}$,

$$
A_\mu = \sum_{\ell m} \left(\begin{bmatrix} 0 \\ 0 \\ \frac{a^{\ell m}(t,r)}{\sin\theta}\,\partial_\phi Y_{\ell m} \\ -a^{\ell m}(t,r)\sin\theta\,\partial_\theta Y_{\ell m} \end{bmatrix} + \begin{bmatrix} f^{\ell m}(t,r)\,Y_{\ell m} \\ h^{\ell m}(t,r)\,Y_{\ell m} \\ k^{\ell m}(t,r)\,\partial_\theta Y_{\ell m} \\ k^{\ell m}(t,r)\,\partial_\phi Y_{\ell m} \end{bmatrix} \right)
$$

Parity transformation: simultaneous inversion of all cartesian axes, corresponding to $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$. First term parity $(-1)^{\ell+1}$, second term parity $(-1)^{\ell}$. Inserting decomposition (cf. notebook “Perturbations”)

$$
\frac{\partial^2 \Psi}{\partial r_*^2} - \frac{\partial^2 \Psi}{\partial t^2} - V\Psi = 0
$$

$$
V = \left(1 - \frac{2M}{r}\right)\frac{\ell(\ell+1)}{r^2}
$$

where we chose

$$\Psi = \begin{cases} a(t,r) & \text{axial} \\ r^2(\partial_t h^{\ell m} - \partial_r f^{\ell m}) & \text{polar} \end{cases}$$

Repeat procedure for tensors. Upshot: for massless fields,

$$\frac{\partial^2 \Psi}{\partial r_*^2} - \frac{\partial^2 \Psi}{\partial t^2} - V\Psi = 0$$
$$V = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right)$$

Large ℓ, ω^2 recover null geodesics (cross section, etc).
Static solutions? $\Psi = Ar^{-\ell} + Br^{\ell+1}$ far away

Take scalar with $\ell = 0$. Solution:

$$\Psi = k_1 r + k_2 r \log(1 - 2M/r)$$

Take scalar with $\ell = 1$. Solution:

$$\Psi = k_1 r(r/M - 1) + k_2 r\left(1 + \frac{1}{2}(r/M - 1)\log(2M/r - 1)\right)$$

Regularity at horizon implies $k_2 = 0$...
No hair & Black holes don't polarize!

No-hair. Can generalize. For static fields,

$$\left(\left(1-\frac{2M}{r}\right)\Psi'\right)' - V\Psi = 0$$

Multiply by Ψ^* and integrate exterior,

$$\begin{aligned} &\int_{2M}^{\infty} \left(\left(1-\frac{2M}{r}\right)\Psi'\right)' \Psi^* - V|\Psi|^2\, dr = 0 \\ &\left[\left(1-\frac{2M}{r}\right)\Psi'\Psi^*\right]_{2M}^{\infty} \\ - &\int_{2M}^{\infty} dr \left(1-\frac{2M}{r}\right)|\Psi'|^2 - \int_{2M}^{\infty} dr\, V|\Psi|^2 = 0 \end{aligned}$$

At infinity $\Psi \sim r^{-\ell} + r^{\ell+1}$. At horizon $\Psi \sim k_1 k_2 \log(r - 2M)$. Since $V > 0$ it follows no hair, i..e, $\Psi = 0$. Exception 1: $l = 0$ EM for which $V = 0$ (corresponds to what?). Exception 2: $\ell = 0, 1$ gravitational, for which $V < 0$ (corresponds to what?).

Dynamics I. Look for monochromatic solutions, $\Psi \sim e^{-i\omega t} Z(r_*)$,

$$\frac{d^2 Z}{dr_*^2} + \left(\omega^2 - V\right) Z = 0$$

Solutions:

$$\Psi = \begin{cases} A_{\rm T} e^{-i\omega r_*} & r_* \to -\infty \\ A_{\rm in} e^{-i\omega r_*} + A_{\rm out} e^{i\omega r_*} & r_* \to +\infty \end{cases}$$

V is real, so Z^* also solution. $W = Z dZ^*/dr_* - Z^* dZ/dr_*$ is constant (easy to check), thus

$$\begin{aligned} W(-\infty) &= 2i\omega |A_{\rm T}|^2 \\ W(+\infty) &= 2i\omega \left(|A_{\rm in}|^2 - |A_{\rm out}|^2\right) \\ &\Rightarrow |A_{\rm in}|^2 - |A_{\rm out}|^2 = |A_{\rm T}|^2 \end{aligned}$$

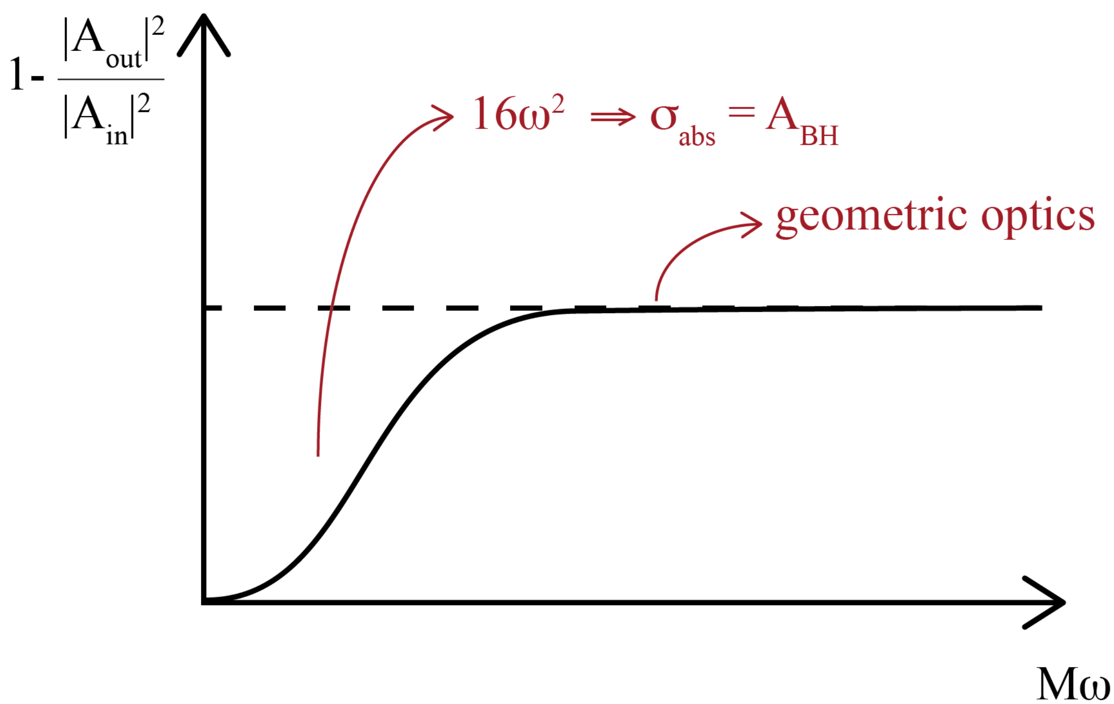

Figure 1: Absorption amplitudes from a Schwarzschild black hole.

Dynamics II. Use Laplace transform $\tilde{\Psi} = \int_0^\infty e^{-st}\Psi dt$ for initial-value problems, and find ($s = -i\omega$)

$$\frac{d^2\tilde{\Psi}}{dr_*^2} + \left(\omega^2 - V\right)\tilde{\Psi} = -\frac{\partial\Psi(t=0)}{\partial t} + i\omega\Psi(t=0) \equiv I(r)$$

Describes also *sourced* equations. Define two independent homogeneous solutions $\tilde{\Psi}_L$, $\tilde{\Psi}_R$

$$\tilde{\Psi}_L = \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ A_{\rm in} e^{-i\omega r_*} + A_{\rm out} e^{i\omega r_*} & r_* \to +\infty \end{cases}$$

$$\tilde{\Psi}_R = e^{i\omega r_*}\,, \qquad r_* \to +\infty$$

General solution:

$$\begin{aligned} \tilde{\Psi} &= \tilde{\Psi}_R \int^{r_*} \frac{I\tilde{\Psi}_L}{W} dr_* + \tilde{\Psi}_L \int_{r_*} \frac{I\tilde{\Psi}_R}{W} dr_* + \tilde{\Psi}_{\rm hom} \\ W &= \tilde{\Psi}_L \tilde{\Psi}'_R - \tilde{\Psi}_R \tilde{\Psi}'_L = 2i\omega A_{\rm in} \end{aligned}$$

Impose BCs (cf. notebook "Scattering_scalar"):

$$\begin{aligned} \tilde{\Psi} &= \tilde{\Psi}_R \int_{-\infty}^{r_*} \frac{I\tilde{\Psi}_L}{W} dr_* + \tilde{\Psi}_L \int_{r_*}^{\infty} \frac{I\tilde{\Psi}_R}{W} dr_* \\ W &= \tilde{\Psi}_L \tilde{\Psi}'_R - \tilde{\Psi}_R \tilde{\Psi}'_L = 2i\omega A_{\rm in} \end{aligned}$$

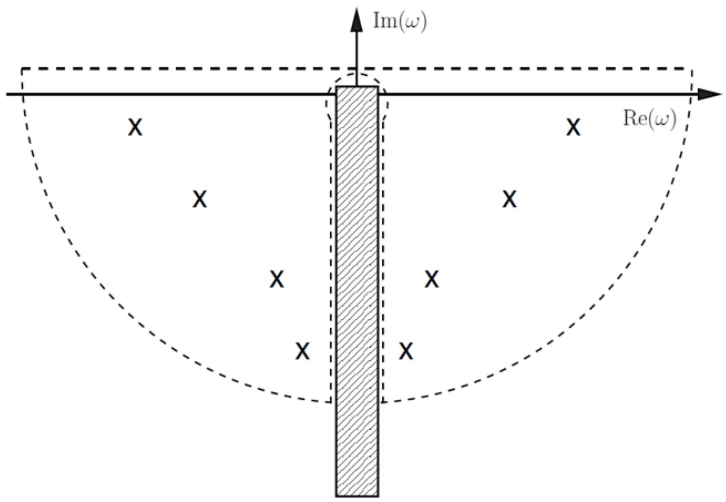

Figure 2: Contour integration in complex frequency plane. Crosses are poles of Green function: the quasinormal frequencies.

QNMs and tails. At large distances

$$\tilde{\Psi} = e^{i\omega r_*} \int_{-\infty}^{+\infty} \frac{I\tilde{\Psi}_L}{W} dr_*$$

Invert

$$\Psi = \frac{1}{2\pi} \int d\omega \, \tilde{\Psi} e^{-i\omega t} \,,$$

and perform ω integral closing contour. Branch-cut at $\omega = 0$ and poles inside contour: the quasinormal frequencies of the system. Correspond to $A_{\rm in} = 0$.

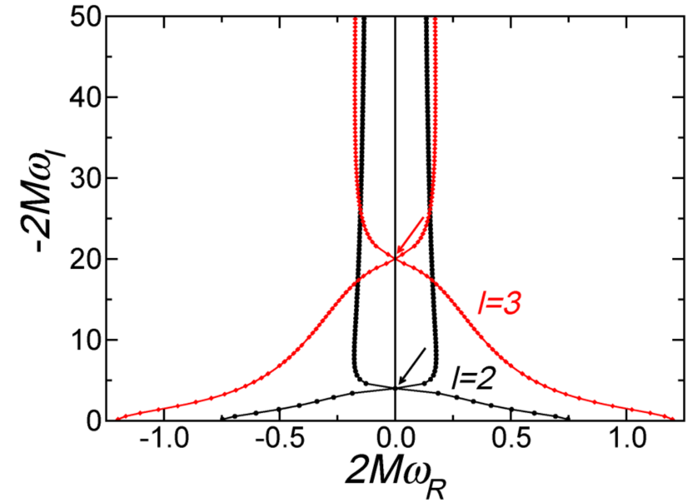

Figure 3: Gravitational QN frequencies of Schwarzschild black hole

Calculation of modes: see "QNMs"
Modes labeled by integers ℓ, n, from least damped (small $\Im(\omega)$) to large damping. At large n,

$$M\omega = \frac{\log 3}{8\pi} - i\frac{(2n+1)}{8}$$

At large ℓ,

$$M\omega = \frac{\ell}{3\sqrt{3}} - i\frac{n}{3\sqrt{3}} = \ell M\Omega - in\lambda$$

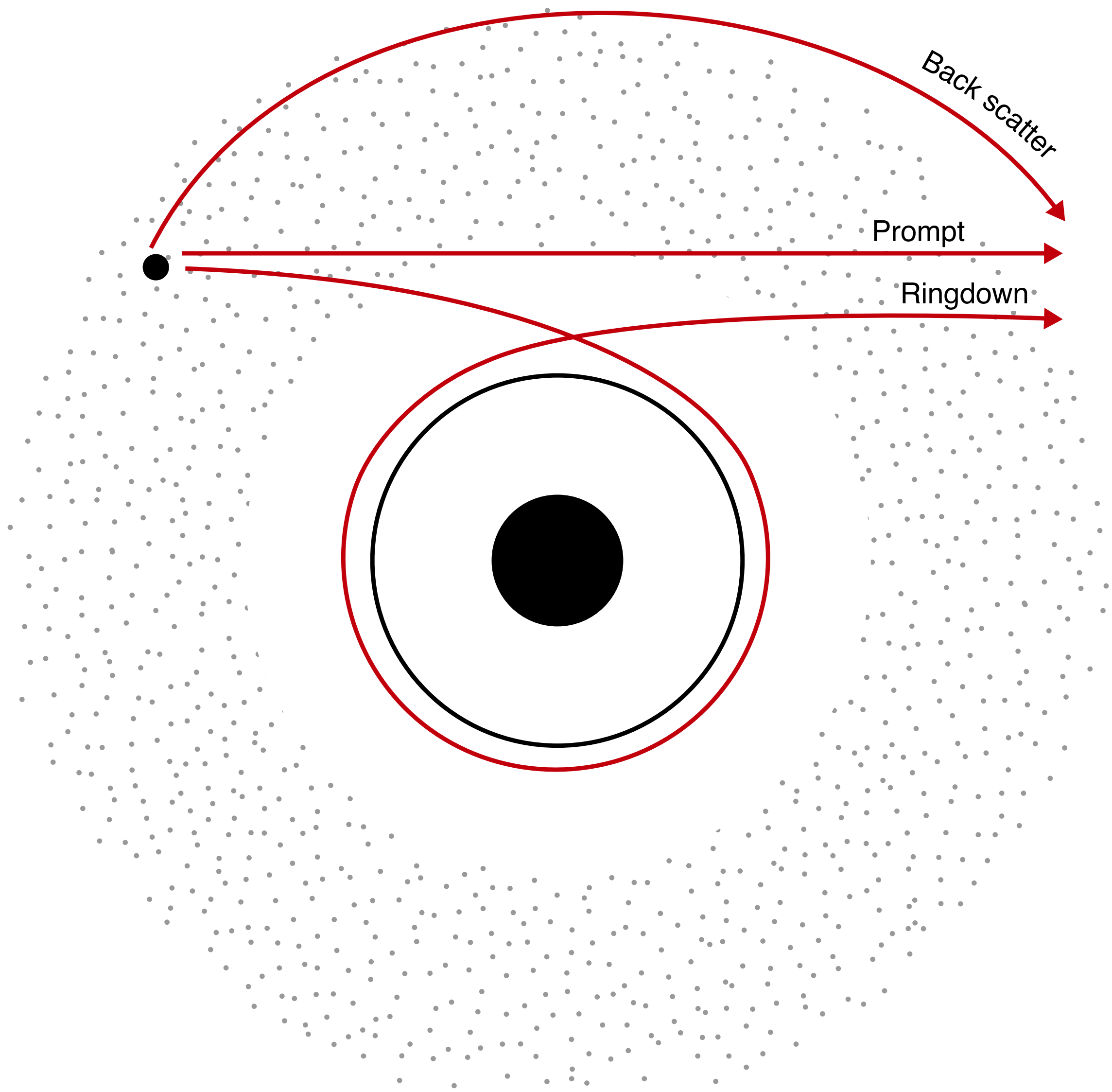

Figure 4: Cartoon of wave propagation on black hole background

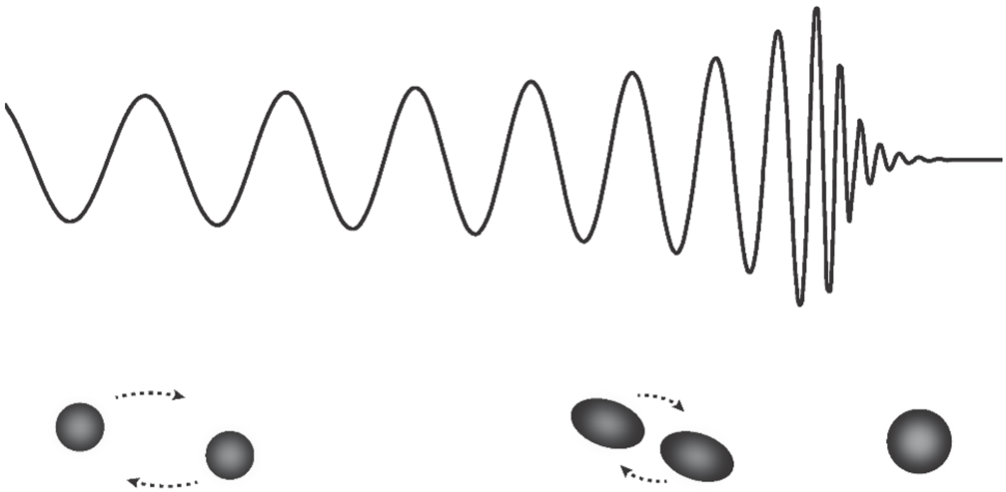

Figure 5: The ringdown stage of black holes

$$
\begin{aligned}
f &= \frac{\Re\omega}{2\pi} = 1.207\left(\frac{10M_\odot}{M}\right)\ \mathrm{kHz} \\
\tau &= \frac{1}{|\Im(\omega)|} = 0.5537\left(\frac{10M_\odot}{M}\right)\ \mathrm{ms}
\end{aligned}
$$

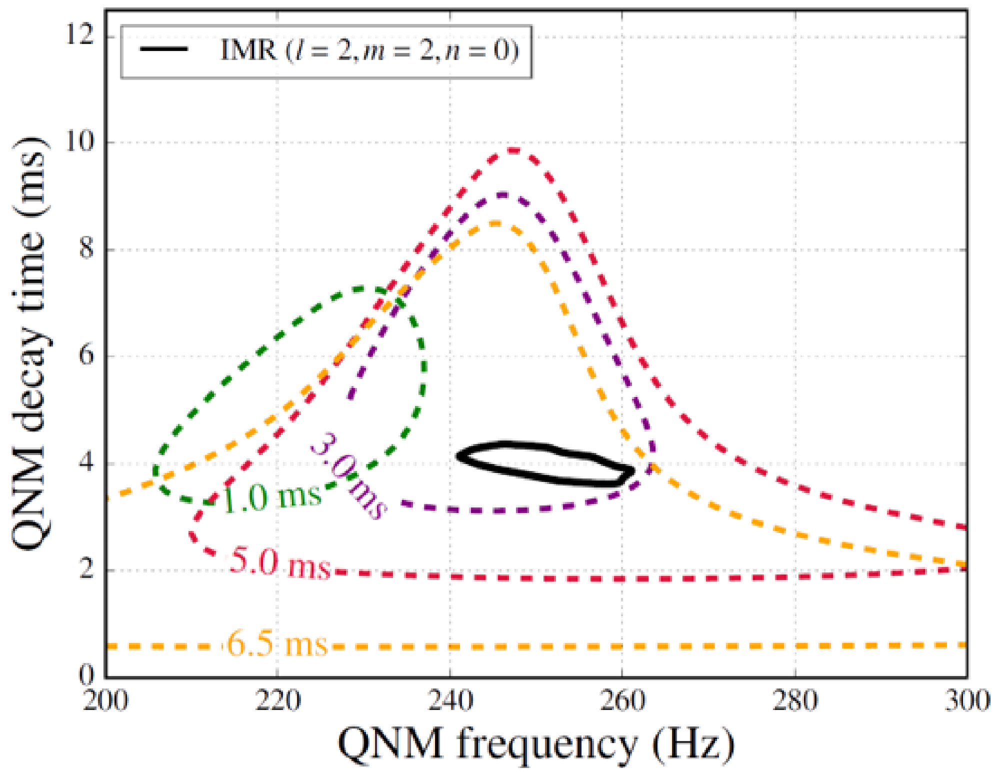

Figure 6: **Black hole spectroscopy with LIGO**. Shown 90% posterior distributions. Black solid is 90% posterior of QNM as derived from the posterior mass and spin of remnant. See LSC PRL116:221101 (2016); arXiv:2010.14529; Cotesta+ arXiv:2201.00822

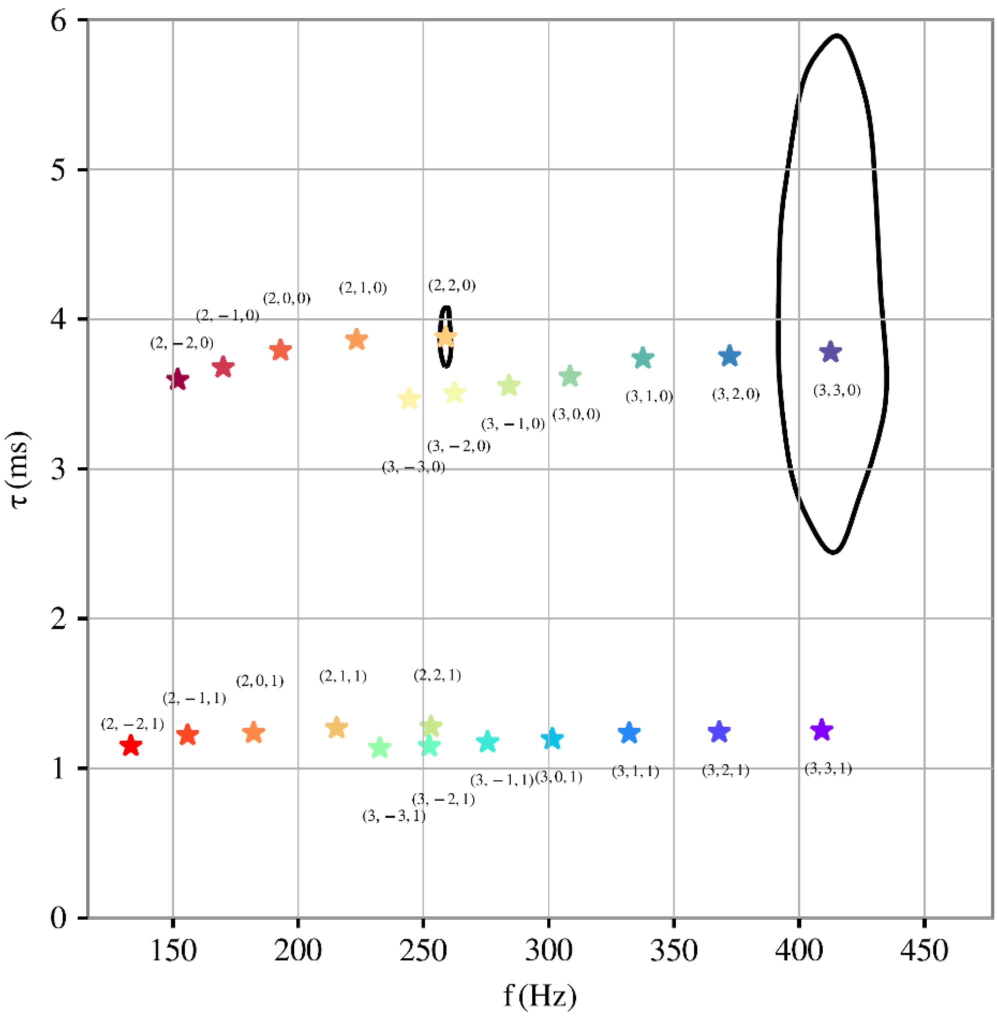

Figure 7: **Black hole spectroscopy in the near-future**, assuming SNR of 40. LISA will see SNR of thousands.

Spectral stability. Add a small bump to potential.

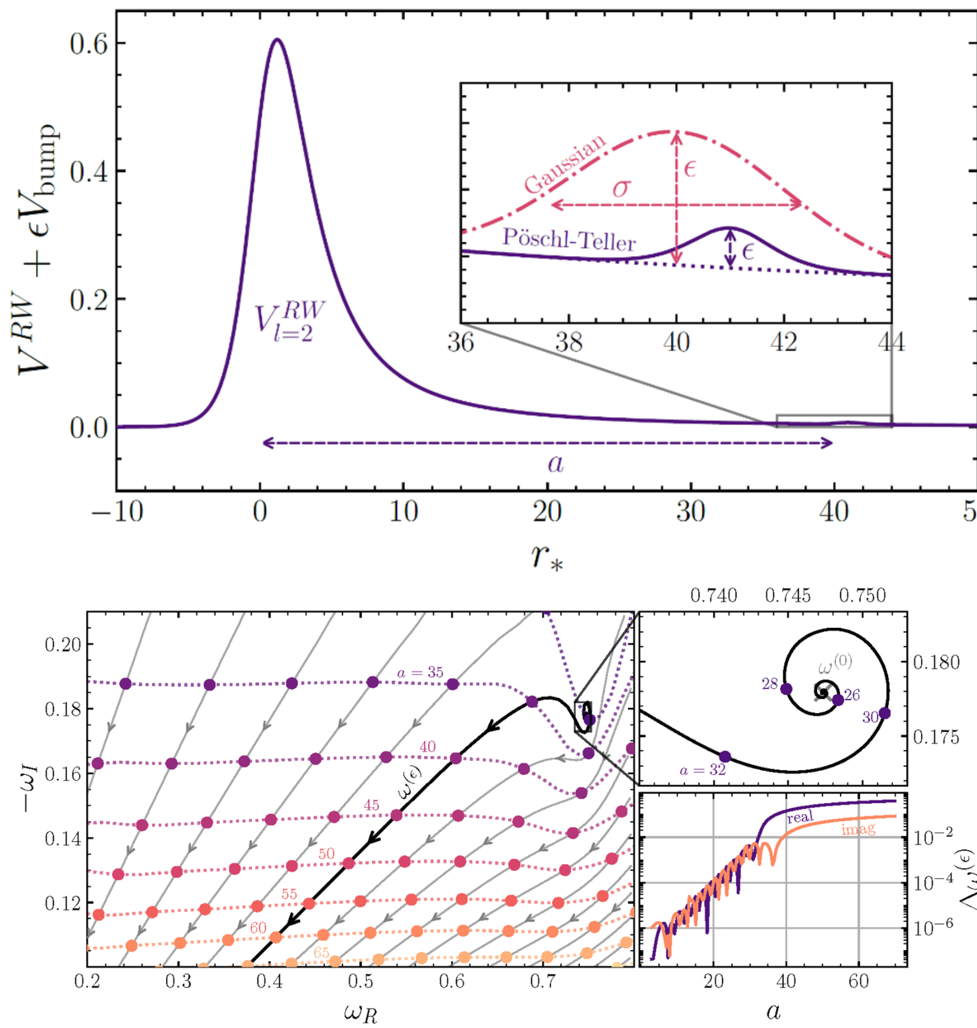

Figure 8: **Spectral instability of black holes**. See Cheung+ PRL128:111103 (2022); also arXiv:2205.08547.

Temporal stability.

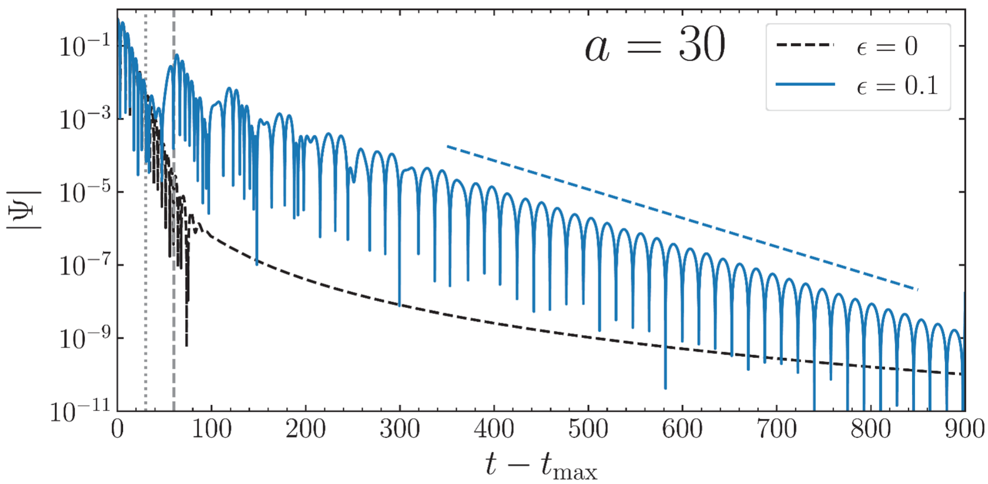

Figure 9: **Spectral instability of black holes**. see arXiv:2205.08547.

Particles around black holes. Assume point-particles,

$$T^{\mu\nu} = m_p \int v^\mu v^\nu \frac{\delta^{(4)}(x^\beta - y_p^\beta)}{\sqrt{-g}} d\tau$$

and find source to equation

$$\frac{d^2\tilde{\Psi}}{dr_*^2} + \left(\omega^2 - V\right)\tilde{\Psi} = \mathcal{S}(r)$$

source depends on motion. Compute Ψ and gravitational-wave amplitude at large distances. Find energy flux, wave amplitude, etc. For low-velocities, equation can be solved, and makes contact with quadrupole results:

$$\begin{aligned} h_+ &= -\frac{2Gm_p}{c^2 r}\left(\frac{GM\Omega}{c^3}\right)^{2/3}(1+\cos^2\theta)\cos 2\psi \\ h_\times &= -\frac{4Gm_p}{c^2 r}\left(\frac{GM\Omega}{c^3}\right)^{2/3}\cos\theta\sin 2\psi \\ \psi &= \Omega(t-r) - \phi \end{aligned}$$

$$\frac{dE}{dt} = \frac{32}{5}\frac{G}{c^5}m_p^2 L^4 \Omega^6 = \frac{32}{5}\frac{c^5}{G}\frac{m_p^2}{M^2}\left(\frac{GM\Omega}{c^3}\right)^{10/3}$$

- Equivalent to Einstein's quadrupole formula
- Relativistic systems: c^5/G Thorne-Dyson conjecture
- Quantum system if $\dot{E}/\Omega \lesssim \hbar\Omega$
- Why circular? Because

$$\frac{de}{dt} = -e\frac{304}{15}\frac{G^3}{c^5}\frac{M^2 m_p}{a^4(1-e^2)^{5/2}}\left(1 + \frac{121}{304}e^2\right)$$

orbits evolve, under GW emission, to circular.

semi-major axis decreases and plunge

Summary: Black holes don't polarize, but vibrate. Prospect for black hole spectroscopy. Their dynamical response is governed, not by the horizon, but mostly by the photon-sphere, the light ring. Can use astrophysical mergers to test uniqueness properties using ringdown stage, or inspiral stage.

Open issues:

- Resonant excitation of ringdown?
- Detection of overtones?
- Environmental effects
- Spectral stability
- Energy extraction from higher-spin fields
- Is there an upper bound on amplification?
- Quantization of superradiance?
- Open issues: why do LIGO BHs spin so slowly?

Exercises

- **a)** Calculate the scalar and electromagnetic energy fluxes expressed in terms of the master variables.
- **b)** Show that there is no static solution to the massless scalar equation in a Schwarzschild background. What about the Maxwell equations? What is the allowed solution, and what does it mean?
- **c)** I am waving my hand at a friend, producing a time-varying quadrupole moment. Do I emit gravitational waves? The flux in gravitational waves is given by quadrupole formular in flat space,

$$\dot{E} = \frac{G}{c^5}\frac{1}{5}\left\langle \frac{d^3Q_{ij}}{dt^3}\frac{d^3Q_{ij}}{dt^3}\right\rangle$$

$$Q_{ij} = \int d^3x\rho(x^i)\left(x_ix_j - \frac{1}{3}r^2\delta_{ij}\right)$$

For a waving hand of mass $M \sim 1\,\mathrm{Kg}$ and doing motion of amplitude $A = 1\,m$ and period $T = 1\,\mathrm{sec}$, in cartesian coordinates we can model this as simply $x = A\cos 2\pi t/T$, and we find,

$$\dot{E} = \frac{4096GA^4M^2\pi^6}{45c^5T^6}$$

Thus, the energy emitted over one period is

$$\frac{\dot{E}T}{\hbar\omega} = \frac{2048 G A^4 M^2 \pi^5}{45 \hbar c^5 T^4} \sim 10^{-15}\,.$$

This is a quantum process, no graviton is emitted in a single period!

- **d)** Calculate the quasinormal frequencies of a scalar field around non-rotating black hole

- **e)** Consider the system, mimicking the dynamical response of a black hole

$$\frac{d^2\tilde{\Psi}}{dx^2} + \left(\omega^2 - 2V_0\delta(x)\right)\tilde{\Psi} = i\omega\delta(x - x_0)$$

Calculate the quasinormal frequencies of this system, and solve analytically for $\Psi(t)$.