

# Living on a **Supermanifold**

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*Based on:* K Finn, S Karamitsos, AP, PRD102 (2020) 045014 [arXiv:1910.06661]  
EPJC81 (2021) 572 [arXiv:2006.05831]  
PRD103 (2021) 065004  
PRD98 (2018) 016015

S Karamitsos, AP, NPB907 (2016) 785; NPB927 (2018) 219

Viola Gattus, AP, *to appear in May 2023*

# Outline:

- From Geometrizing the Cosmos to Micro-Cosmos
- Brief History on Covariant Methods in QFT
- Grand Covariance in Quantum Gravity
- The Fermion problem: Living on a Supermanifold?
- Grand Covariant Effective Action with Fermions
- Conclusions

- From **Geometrizing** the **Cosmos** to **Micro-Cosmos**

- **Geometrizing the Cosmos**: . . . , Pythagoras (5c BC)

- **Geocentric versus Heliocentric System** [e.g., Van der Waerden '87]
  - Geocentric*: Anaximander (6c BC), . . . , Plato (4c BC), Aristotle (3c BC), Ptolemy (2c AD), . . . **Tycho** (16c AD)
  - Heliocentric*: Aristarchus (3c BC), Seleucus (2c BC), Copernicus (15c AD), Kepler (16c AD), Galileo (16c AD), . . .

- **Absolute versus Relative/Local Inertial Frame in Gravitation**

*Absolute*: Newton (17c AD), . . .

*Relative*: Einstein (20c AD), . . .

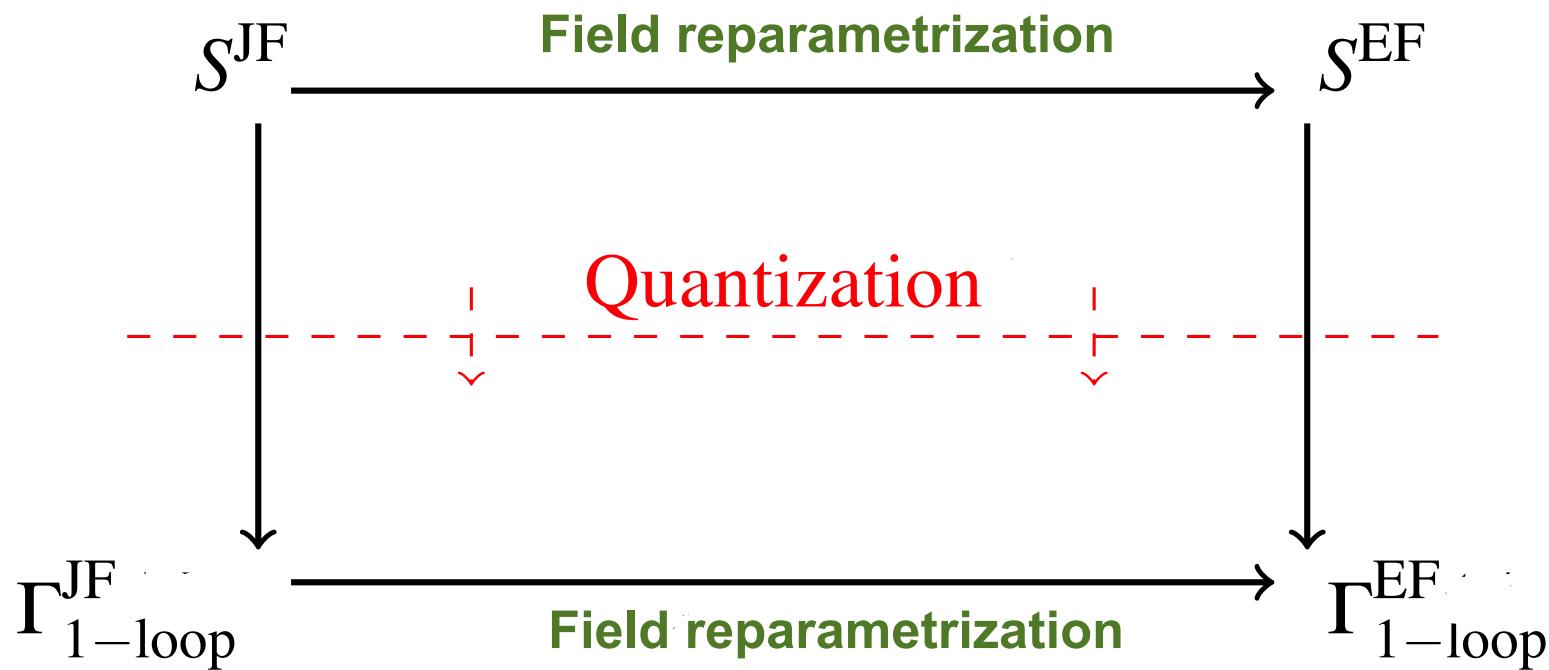
- **Geometrizing the Micro-Cosmos** as a solution to frame **problems** in Quantum Field Theory and Quantum Gravity

## – Einstein versus Jordan Frame

Action in Einstein Frame:  $S^{\text{EF}}[g_{\mu\nu}, \varphi] = \int_x \left[ -\frac{1}{2}M_P^2 R + \frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi) \right]$

Action in Jordan Frame:  $S^{\text{JF}}[\tilde{g}_{\mu\nu}, \tilde{\varphi}] = \int_x \left[ -\frac{1}{2}f(\tilde{\varphi})\tilde{R} + \frac{1}{2}(\partial_\mu \tilde{\varphi})^2 - \tilde{V}(\tilde{\varphi}) \right]$

Frame equivalence  $\implies S^{\text{JF}}[\tilde{g}_{\mu\nu}, \tilde{\varphi}] = S^{\text{EF}}[g_{\mu\nu}, \varphi]$  [R. H. Dicke '62]



$\Gamma^{\text{JF}}_{1-\text{loop}}[\tilde{g}_{\mu\nu}, \tilde{\varphi}] \neq \Gamma^{\text{EF}}_{1-\text{loop}}[g_{\mu\nu}, \varphi]$ : Effective action is frame dependent, except at extrema of the action.

## – Partial list of references

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- A. Karam, T. Pappas, K. Tamvakis, *Frame-dependence of higher-order inflationary observables in scalar-tensor theories*, Phys. Rev. D96 (2017) 064036.
- S. Karamitsos and A. Pilaftsis, *Frame Covariant Nonminimal Multifield Inflation*, Nucl. Phys. B927 (2018) 219.

## • Brief History on Covariant Methods in QFT

- B DeWitt '67: . . . *The Manifestly Covariant Quantum Field Theory*, introducing the well-known *Background Field Method*.
- S Weinberg '68, . . . J Honerkamp '72 . . . :  
*Geometry of Field Space and Amplitudes in Non-linear  $\sigma$ -Models*  
⇒ Little Higgs Models, HEFT  $\supset$  SMEFT, Multifield Inflation . . .
- L Alvarez-Gaumé, DZ Friedman, S Mukhi '81:  
*Geometry and UV Finiteness of SUSY Non-linear  $\sigma$ -Models*
- G Vilkovisky '84, B DeWitt '85, AO Barvinsky, G Vilkovisky '85,  
CP Burgess, G Kunstatter '87, A Rebhan '87 . . . :  
*Covariant VDW Effective Action* [this talk]
- MK Gaillard '86: *Gauge-Covariant Derivative Approach to EFT*  
⇒ Universal effective action, threshold effects in SMEFT . . .

- JM Cornwall '82, JM Cornwall, J Papavassiliou '89, AP '97:  
*Pinch Technique: a Gauge-Covariant Diagrammatic Approach to Off-Shell Amplitudes*  
⇒ Effective gluon mass, Non-Abelian effective charges,  
Unstable particle dynamics and High-energy unitarity . . .
  - G Sigl, G Raffelt '93, PSB Dev, P Millington, AP, D Teresi '14:  
*Flavour Covariant Transport Equations*  
⇒ Neutrino flavour dynamics in astrophysics and cosmology,  
Leptogenesis . . .
  - D Teresi, AP '13:  
*Symmetry Improved Cornwall-Jackiw-Tomboulis Effective Action*  
⇒ Massless Goldstones, 2nd order phase transitions for  $\mathbb{O}(N)$  theories,  
RG exactness and IR safe effective potentials . . .
- ⋮

- **Grand Covariance in Quantum Gravity**

[G Vilkovisky '84, B DeWitt, '85;  
K Finn, S Karamitsos, AP, '19]

$$S = \int d^Dx \sqrt{-g} \left[ -\frac{f(\varphi)}{2}R + \frac{1}{2}k_{AB}(\varphi) g^{\mu\nu}(\nabla_\mu\varphi^A)(\nabla_\nu\varphi^B) - V(\varphi) \right],$$

$S = S[g_{\mu\nu}, \varphi; f(\varphi), k(\varphi), V(\varphi)]$ : *classical action*

$f(\varphi)$ ,  $k_{AB}(\varphi)$ ,  $V(\varphi)$ : *model functions*

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### Grand or Frame Covariance:

#### (i) *Spacetime diffeomorphisms*

$$x^\mu \rightarrow \tilde{x}^\mu = \tilde{x}^\mu(x^\nu), \quad \text{with} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu$$

#### (ii) *Field reparametrizations*

$$\begin{aligned} g_{\mu\nu} &\rightarrow \tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(g_{\kappa\lambda}, \varphi) = \Omega^2(\varphi) g_{\mu\nu} \\ \varphi^A &\rightarrow \tilde{\varphi}^A = \tilde{\varphi}^A(g_{\mu\nu}, \varphi) = \tilde{\varphi}^A(\varphi) \end{aligned}$$

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- Introduce new model function  $\ell(\varphi)$  to restore (i)

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu , \quad \text{with} \quad \bar{g}_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\ell^2(\varphi)}$$

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- Transformation of model functions

$$\tilde{\ell}(\varphi) = \Omega \ell(\varphi) ,$$

$$\tilde{f}(\varphi) = \Omega^{-2} f(\varphi) ,$$

$$\tilde{k}_{\widetilde{A}\widetilde{B}}(\varphi) = \left[ k_{AB} - 6f(\ln \Omega)_{,A}(\ln \Omega)_{,B} + 3f_{,A}(\ln \Omega)_{,B} + 3(\ln \Omega)_{,A}f_{,B} \right] \partial^A \varphi_{\widetilde{A}} \partial^B \varphi_{\widetilde{B}} ,$$

$$\tilde{V}(\varphi) = \Omega^{-4} V(\varphi) .$$

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- Frame invariance of the classical action  $S$ :

$$S[\tilde{g}_{\mu\nu}, \tilde{\varphi}; \tilde{\ell}(\varphi), \tilde{f}(\varphi), \tilde{k}(\varphi), \tilde{V}(\varphi)] = S[g_{\mu\nu}, \varphi; \ell(\varphi), f(\varphi), k(\varphi), V(\varphi)] \quad *$$

Models related by a frame transformation define an *equivalence class*

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[K Falls, M Herrero-Valea '19]

\*  $S = \int d^D x \sqrt{-g} \mathcal{L} = \int d^D x \sqrt{-\bar{g}} \bar{\mathcal{L}}$  is independent of  $\ell(\varphi)$  (only at tree level).

## – Coordinates of the **Grand Configuration Space**

$$\Phi^i \equiv \Phi^I(x_I) = \begin{pmatrix} g^{\mu\nu}(x) \\ \phi^A(x_A) \end{pmatrix}, \text{ with } i = \{I, x_I\}, I = \{\mu\nu, A\}, x_I = \{x, x_A\}.$$

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## – The **Grand Configuration Space Metric**

$$g_{ij} \equiv \frac{\bar{g}_{\mu\nu}}{D} \frac{\bar{\delta}^2 S}{\bar{\delta}(\partial_\mu \Phi^i) \bar{\delta}(\partial_\nu \Phi^j)} = \ell^2 \begin{pmatrix} f P_{\mu\nu\rho\sigma} & -\frac{3}{4} f_{,B} g_{\mu\nu} \\ -\frac{3}{4} f_{,A} g_{\rho\sigma} & k_{AB} \end{pmatrix} \bar{\delta}^{(D)}(x_I - x_J),$$

where  $\bar{\delta}^{(D)}(x_I - x_J) \equiv \delta^{(D)}(x_I - x_J)/\sqrt{-\bar{g}}$  is *frame invariant*, and

$$P_{\mu\nu\rho\sigma} \equiv G_{(\mu\nu)(\rho\sigma)} = \frac{1}{2} \left( g_{\mu\rho} g_{\sigma\nu} + g_{\mu\sigma} g_{\rho\nu} - \alpha g_{\mu\nu} g_{\rho\sigma} \right)$$

is the *gravitational field-space metric*.

Condition on the inverse metric  $G^{(\mu\nu)(\rho\sigma)}$ :

$$G^{(\mu\nu)(\rho\sigma)} = g^{\alpha\mu} g^{\beta\nu} g^{\kappa\rho} g^{\lambda\sigma} G_{(\alpha\beta)(\kappa\lambda)} \implies \alpha = 0 \text{ or } 1.$$

## – Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int [\mathcal{D}\phi] \exp\left\{\frac{i}{\hbar}\left[S[\phi] + \frac{\delta\Gamma[\varphi]}{\delta\varphi^a}(\varphi^a - \phi^a)\right]\right\}.$$

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**Not** invariant under frame transformations.

## – VDW Quantum Effective Action

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int [\overline{D}\Phi] \mathcal{M}[\Phi] \exp\left\{\frac{i}{\hbar}\left[S[\Phi] + \frac{\bar{\delta}\Gamma[\varphi]}{\bar{\delta}\varphi^i} \Sigma^i[\varphi, \Phi]\right]\right\},$$

with  $\varphi = (g^{\mu\nu}, \phi)$ ,

$$[\overline{D}\Phi] = \exp\left[ \sum_I \int d^Dx \sqrt{-\bar{g}(x)} \ln \mathcal{D}\Phi^I(x) \right], \quad \mathcal{M}[\Phi] = V_{\text{FP}} \sqrt{\det(\mathcal{G}_{ij})},$$

and  $V_{\text{FP}}$  is the *Faddeev–Popov determinant* [for  $SU(N)$ , see Rebhan '87].

## – One- and Two-Loop VDW Effective Actions

$$\Gamma^{(1)}[\varphi] = \frac{i}{2} \ln \overline{\det}(\nabla^a \nabla_b S), \quad [\text{consistent with P Ellicott, T Toms '89}]$$

$$\begin{aligned} \Gamma^{(2)}[\varphi] &= \text{Diagram of two circles connected by a horizontal line} + \text{Diagram of a circle with a horizontal chord} \\ &= -\frac{1}{8} \Delta^{ab} \Delta^{cd} \nabla_{(a} \nabla_b \nabla_c \nabla_d) S \\ &\quad + \frac{1}{12} \Delta^{ab} \Delta^{cd} \Delta^{ef} (\nabla_{(a} \nabla_c \nabla_e) S) (\nabla_{(b} \nabla_d \nabla_f) S), \end{aligned}$$

with  $\Delta^{ab} \equiv (\nabla_a \nabla_b S)^{-1} = \Delta^{ba}$ .

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## – Grand Covariance of the VDW Effective Action

[K Finn, S Karamitsos, AP, '19]

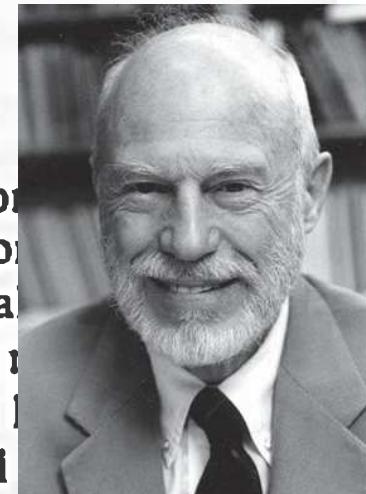
$$\Gamma[\varphi; \ell(\phi), f(\phi), k_{AB}(\phi), V(\phi)] = \Gamma[\tilde{\varphi}(\varphi); \tilde{\ell}(\phi), \tilde{f}(\phi), \tilde{k}_{AB}(\phi), \tilde{V}(\phi)]$$

with  $\varphi = (g^{\mu\nu}, \phi)$ .

- The Fermion problem: Living on a Supermanifold?

# The Effective Action

Bryce De Witt '85



## 14 DISCUSSION

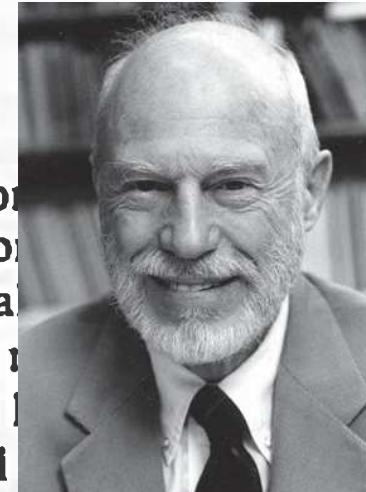
This completes the basic outline of how Vilkovisky's idea for invariant scalar effective action works, or can be made to work. In discussing the advantages of such an effective action, I shall take some of its possible defects. First of all, how unique is it? I do not believe it is *in principle* as unique as Vilkovisky has claimed. Choices have to be made for three quantities: the starting metric  $\gamma_{ij}$ , the functional measure  $\mu_K[I, K]$ ; all else follows from these. The last two have no effect on the final form of  $\Gamma$ . The measure  $\mu_I$ , and hence  $\mu$ , is determined by unitarity requirements. Expression (13.11) for  $\mu$  appears to depend on a fourth arbitrary quantity,  $g^+$ , but in fact is independent of  $g^+$ . To show this just vary  $a_\alpha f_\beta$  and use (13.15).

That leaves  $\gamma_{ij}$ . Vilkovisky (1984) has suggested that  $\gamma_{ij}$  should be determined by the coefficient of the highest derivatives in the superclassical field equations. This cannot be correct in the fermion sector of supergravity theory, where the highest-order derivative is first order, because the coefficient of a first-order derivative cannot yield a tensor of even rank having

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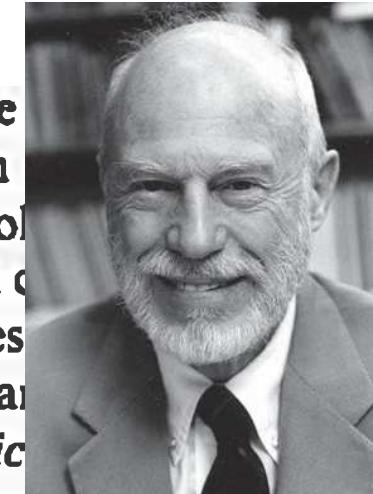
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⇒ **No known metric for theories with fermions:**  $g_{XY}(\phi) \bar{\psi}^X i\gamma^\mu \partial_\mu \psi^Y$

# The Effective Action

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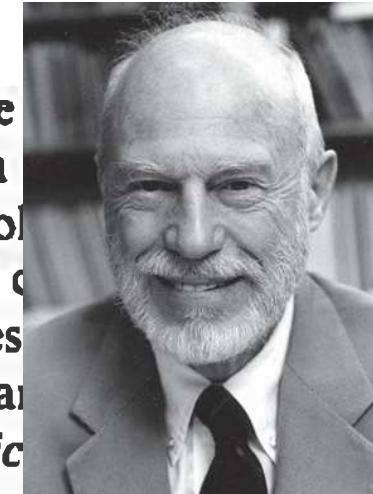


the right mass dimension. Such coefficients are much more what they say about the singularity structures of the Green and  $\hat{G}^+$ , which are of relevance for unitarity.  $\gamma_{ij}$  plays no role in these questions. Its only function is to provide (equation (11.18)) a choice of gauge for the fields  $\Phi$  which respects the orbit decomposition and which enables one to obtain results to be obtained that are independent of how the various gauge choices are chosen. Having said this, I must in fairness add that *in practice* there is little freedom in the choice of  $\gamma_{ij}$ . In all gauge theories certain choices stand out as superior to all others for making the whole scheme work smoothly. Although I cannot give a general algorithm for these metrics I believe that Vilkovisky's effective action is *effectively unique*.

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**But, are we really living on a Supermanifold?**

- Living on a **Supermanifold**

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572]

- Fermions as Coordinates in the Field-Space **Supermanifold**

$$\Phi \equiv \{\Phi^\alpha\} = (\phi^A, \psi_a^1, \bar{\psi}_{\dot{a}}^1, \psi_a^2, \bar{\psi}_{\dot{a}}^2, \dots)^T,$$

where  $\phi^A$  are scalars and  $\psi_a^X$  are Dirac (or Weyl) fermions.

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where  $\phi^A$  are scalars and  $\psi_a^X$  are Dirac (or Weyl) fermions.

- Frame-covariant Lagrangian of a scalar theory with **fermions**

$$\mathcal{L} = \underbrace{\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^\alpha{}_\alpha k_\beta(\Phi) \partial_\nu \Phi^\beta}_{:\text{scalars}} + \underbrace{\frac{i}{2} \zeta_\alpha^\mu(\Phi) \partial_\mu \Phi^\alpha}_{:\text{fermions}} - U(\Phi).$$

### Model functions:

${}_\alpha k_\beta(\Phi)$  : rank-2 tensor of the would-be bosonic metric (with  ${}_\alpha k_X = 0$ )

$\zeta_\alpha^\mu(\Phi)$  : mixed spacetime and field-space vector

$U(\Phi)$  : a scalar describing the potential and Yukawa sector

– Extracting the model functions  ${}_\alpha k_\beta$  and  $\zeta_\alpha^\mu$

$${}_\alpha k_\beta = \frac{g_{\mu\nu}}{D} \frac{\overrightarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)} \mathcal{L} \frac{\overleftarrow{\partial}}{\partial(\partial_\nu \Phi^\beta)}, \quad \zeta_\alpha^\mu = \frac{2}{i} \left( \mathcal{L} - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi^\alpha {}_\alpha k_\beta \partial_\nu \Phi^\beta \right) \frac{\overleftarrow{\partial}}{\partial(\partial_\mu \Phi^\alpha)}$$

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- Free theory as an example

$$\begin{aligned} \mathcal{L} &= \sum_{A \in \text{Nscalars}} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^A - \frac{1}{2} m_A^2 (\phi^A)^2 \\ &+ \sum_{X \in \text{Mfermions}} \frac{i}{2} \left( \bar{\psi}^X \gamma^\mu \partial_\mu \psi^X - \partial_\mu \bar{\psi}^X \gamma^\mu \psi^X \right) - m_X \bar{\psi}^X \psi^X. \end{aligned}$$

Model functions:

$$\begin{aligned} {}_\alpha k_\beta &= \begin{pmatrix} \delta_{AB} & \mathbf{0}_{N \times 8M} \\ \mathbf{0}_{8M \times N} & \mathbf{0}_{8M \times 8M} \end{pmatrix}, \\ \zeta_\alpha^\mu &= \left( \mathbf{0}_N, \bar{\psi}_a^1 \gamma_{aa}^\mu, \gamma_{aa}^\mu \psi_a^1, \bar{\psi}_b^2 \gamma_{bb}^\mu, \gamma_{bb}^\mu \psi_b^2, \dots \right) \end{aligned}$$

## – Deriving the Grand Metric

Define the rank-1 field-superspace tensor,

[to be improved by V Gattus, AP, to appear in May 2023]

$$\zeta_\alpha(\Phi) = \frac{1}{4} \frac{\delta \zeta_\alpha^\mu(\Phi)}{\delta \gamma^\mu} \quad \xrightarrow{\text{improved}} \quad \zeta_\alpha(\Phi) = \zeta_\beta^\mu(\Phi) \sum_{\mu,\alpha}^\beta,$$

to derive the rank-2 anti-supersymmetric tensor (in analogy to  $F_{\mu\nu}$  in QED)

$${}_\alpha \lambda_\beta(\Phi) = \frac{1}{2} \left( \overrightarrow{\frac{\partial}{\partial \Phi^\alpha}} \zeta_\beta(\Phi) - (-1)^{\alpha+\beta+\alpha\beta} \overrightarrow{\frac{\partial}{\partial \Phi^\beta}} \zeta_\alpha(\Phi) \right), \quad \text{with } \lambda^{\text{sT}} = -\lambda.$$

Introduce the *non-singular* rank-2 tensor:

$${}_\alpha \Lambda_\beta \equiv {}_\alpha k_\beta + {}_\alpha \lambda_\beta \xrightarrow{\text{free theory}} {}_\alpha N_\beta \equiv \begin{pmatrix} 1_N & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1_4 & 0 & 0 & \cdots \\ 0 & 1_4 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1_4 & \cdots \\ 0 & 0 & 0 & 1_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

## – Properties of the Grand Field-Space Metric $\alpha G_\beta(\Phi)$

The Grand Metric  $\alpha G_\beta(\Phi)$  should:

1. Be *uniquely* determined from the *action*.
2. Transform as a proper *rank-2 field-space tensor*.
3. Be *supersymmetric* and *non-singular* to produce a non-zero line element.
4. Be *ultralocal*, i.e. it should not depend on  $\partial_\mu \Phi$ .
5. Have the *local form* on each point of the field-space Supermanifold

$${}^a H_b \equiv \begin{pmatrix} 1_N & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1_4 & 0 & 0 & \cdots \\ 0 & -1_4 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1_4 & \cdots \\ 0 & 0 & 0 & -1_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

## – The Grand Field-Space Metric

Determine first the *field-space vielbeins*  ${}_{\alpha}e^a(\Phi)$  from

$${}_{\alpha}\Lambda_{\beta}(\Phi) = {}_{\alpha}e^a(\Phi) {}_aN_b{}^b e_{\beta}^{sT}(\Phi),$$

and use these to obtain the **Grand Field-Space Metric**:

$${}_{\alpha}G_{\beta}(\Phi) = {}_{\alpha}e^a(\Phi) {}_aH_b{}^b e_{\beta}^{sT}(\Phi)$$

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## – The Christoffell Symbols

$${}^\alpha \Gamma_{\beta\gamma} = \frac{1}{2} G^{\alpha\delta} \left[ {}_\delta G_\beta \overleftarrow{\partial}_\gamma + (-1)^{\beta\gamma} {}_\delta G_\gamma \overleftarrow{\partial}_\beta - (-1)^\beta \overrightarrow{\partial}_\delta {}_\beta G_\gamma \right]$$

## – The Riemann Tensor

$$\begin{aligned} R^\alpha_{\beta\gamma\delta} = & - {}^\alpha \Gamma_{\beta\gamma} \overleftarrow{\partial}_\delta + (-1)^{\gamma\delta} {}^\alpha \Gamma_{\beta\delta} \overleftarrow{\partial}_\gamma + (-1)^{\gamma(\beta+\epsilon)} {}^\alpha \Gamma_{\epsilon\gamma} {}^\epsilon \Gamma_{\beta\delta} \\ & - (-1)^{\delta(\epsilon+\beta+\gamma)} {}^\alpha \Gamma_{\epsilon\delta} {}^\epsilon \Gamma_{\beta\gamma} \end{aligned}$$

- **Grand Covariant Effective Action with Fermions**

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572]

$$\exp(i\Gamma[\Phi]) = \int [D\Phi_q] \sqrt{|\text{sdet}G|} \exp \left( iS[\Phi_q] + i \int d^4x \sqrt{-g} \Gamma[\Phi] \frac{\overleftarrow{\partial}}{\partial \Phi^\alpha} \Sigma^\alpha[\Phi, \Phi_q] \right)$$

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- One- and Two-Loop Grand Covariant Effective Actions

$$\Gamma^{(1)}[\Phi] = \frac{i}{2} \ln \text{sdet} \left( \vec{\nabla}^{\hat{\alpha}} S \overleftarrow{\nabla}_{\hat{\beta}} \right) = \frac{i}{2} \text{str} \ln \left( \vec{\nabla}^{\hat{\alpha}} S \overleftarrow{\nabla}_{\hat{\beta}} \right)$$

$$\Gamma^{(2)}[\Phi] = \text{Diagram of two circles} + \text{Diagram of one circle with a horizontal line}$$

$$\begin{aligned} &= -\frac{1}{8} S \overleftarrow{\nabla}_{\{\hat{\alpha}} \overleftarrow{\nabla}_{\hat{\beta}} \overleftarrow{\nabla}_{\hat{\gamma}} \overleftarrow{\nabla}_{\hat{\delta}\}} \Delta^{\hat{\delta}\hat{\gamma}} \Delta^{\hat{\beta}\hat{\alpha}} \\ &\quad + (-1)^{\hat{\gamma}\hat{\beta} + \hat{\epsilon}(\hat{\delta} + \hat{\beta})} \frac{1}{12} \left( S \overleftarrow{\nabla}_{\{\hat{\epsilon}} \overleftarrow{\nabla}_{\hat{\gamma}} \overleftarrow{\nabla}_{\hat{\alpha}\}} \right) \Delta^{\hat{\alpha}\hat{\beta}} \Delta^{\hat{\gamma}\hat{\delta}} \Delta^{\hat{\epsilon}\hat{\zeta}} \left( \vec{\nabla}_{\{\hat{\zeta}} \vec{\nabla}_{\hat{\delta}} \vec{\nabla}_{\hat{\beta}\}} S \right) \end{aligned}$$

$$\hat{\alpha} \Delta_{\hat{\beta}}^{-1} \equiv \vec{\nabla}_{\hat{\alpha}} S \overleftarrow{\nabla}_{\hat{\beta}}, \text{ with } {}^{\hat{\alpha}} \Delta^{\hat{\beta}} = \Delta^{\hat{\alpha}\hat{\beta}} \text{ defined through: } {}^{\hat{\alpha}} \Delta^{\hat{\gamma}} {}_{\hat{\gamma}} \Delta_{\hat{\beta}}^{-1} = {}^{\hat{\alpha}} \delta_{\hat{\beta}}.$$

## – Single Fermion Model

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} k(\phi) \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} g(\phi) \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi \right) \\ & - \frac{1}{2} h(\phi) \bar{\psi} \gamma^\mu \psi \partial_\mu \phi - Y(\phi) \bar{\psi} \psi - V(\phi)\end{aligned}$$

**Supermanifold:**  $\Phi^\alpha = (\phi, \psi, \bar{\psi})$ , with **grand** field-space metric

$${}_\alpha G_\beta = \begin{pmatrix} k - \frac{g'^2 + h^2}{2g} \bar{\psi} \psi & -\frac{1}{2}(g' - ih) \bar{\psi} & \frac{1}{2}(g' + ih) \psi \\ \frac{1}{2}(g' - ih) \bar{\psi} & 0_4 & g1_4 \\ -\frac{1}{2}(g' + ih) \psi & -g1_4 & 0_4 \end{pmatrix}$$

**But,**  $R^\alpha_{\beta\gamma\delta} = 0 \implies$  field-space is flat

- Frame-reparametrization to a **Cartesian Frame**,  $\tilde{\Phi}^\alpha = (\tilde{\phi}, \tilde{\psi}, \tilde{\bar{\psi}})^\top$ :

$$\phi \rightarrow \tilde{\phi} = \int_0^\phi \sqrt{k(\phi)} d\phi, \quad \psi \rightarrow \tilde{\psi} = \sqrt{g(\phi)} \exp\left(\frac{i}{2} \int_0^\phi \frac{h(\phi)}{g(\phi)} d\phi\right) \psi$$

Lagrangian in the **Cartesian Frame**:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} + \frac{i}{2} \left( \tilde{\bar{\psi}} \gamma^\mu \partial_\mu \tilde{\psi} - \partial_\mu \tilde{\bar{\psi}} \gamma^\mu \tilde{\psi} \right) - \tilde{Y}(\tilde{\phi}) \tilde{\bar{\psi}} \tilde{\psi} - \tilde{V}(\tilde{\phi}),$$

with  $\tilde{Y}(\tilde{\phi}) = Y(\phi)/g(\phi)$  and  $\tilde{V}(\tilde{\phi}) = V(\phi)$ .

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with  $\tilde{Y}(\tilde{\phi}) = Y(\phi)/g(\phi)$  and  $\tilde{V}(\tilde{\phi}) = V(\phi)$ .

- **Grand Effective Action up to one-loop level**

$$\begin{aligned} \Gamma[\Phi] = S[\Phi] &+ \frac{i}{2} \text{Tr} \ln \left\{ \square + \tilde{V}'' - \tilde{\bar{\psi}} \left[ 2 \tilde{Y}'^2 \left( -i \not{\partial} + \tilde{Y} \right)^{-1} - \tilde{Y}'' \right] \tilde{\psi} \right\} \\ &- i \text{Tr} \ln \left( -i \not{\partial} + \tilde{Y} \right) \end{aligned}$$

## – Model with Multiple Fermions

General *frame-invariant* Lagrangian (up to quadratic kinetic terms)

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} g^{\mu\nu} k_{AB}(\Phi) \partial_\mu \phi^A \partial_\nu \phi^B + \frac{i}{2} g_{XY}(\Phi) \left( \bar{\psi}^X \gamma^\mu \partial_\mu \psi^Y - \partial_\mu \bar{\psi}^X \gamma^\mu \psi^Y \right) \\ & - \frac{1}{2} h_{AXY}(\Phi) \bar{\psi}^X \gamma^\mu \psi^Y \partial_\mu \phi^A + \frac{i}{2} j_{WXYZ}(\Phi) \bar{\psi}^W \gamma^\mu \psi^X \left( \bar{\psi}^Y \partial_\mu \psi^Z - \partial_\mu \bar{\psi}^Y \psi^Z \right) \\ & - Y_{XY}(\Phi) \bar{\psi}^X \psi^Y - V(\phi)\end{aligned}$$

New kinetic model functions:  $h_{AXY}(\Phi)$  and  $j_{WXYZ}(\Phi)$ .

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New kinetic model functions:  $h_{AXY}(\Phi)$  and  $j_{WXYZ}(\Phi)$ .

[K Finn, S Karamitsos, AP, '21; V Gattus, AP, to appear]

Grand field-space metric for single scalar  $\phi$  and  $j_{WXYZ} = 0$ :

$${}_\alpha G_\beta = \begin{pmatrix} k - \frac{1}{2} \bar{\psi}(\mathbf{g}' - i\mathbf{h}) \mathbf{g}^{-1} (\mathbf{g}' + i\mathbf{h}) \psi & -\frac{1}{2} \bar{\psi} (\mathbf{g}' - i\mathbf{h}) & \frac{1}{2} \psi^\top (\mathbf{g}'^\top + i\mathbf{h}^\top) \\ \frac{1}{2} (\mathbf{g}'^\top - i\mathbf{h}^\top) \bar{\psi}^\top & 0 & \mathbf{g}^\top \mathbf{1}_4 \\ -\frac{1}{2} (\mathbf{g}' + i\mathbf{h}) \psi & -\mathbf{g} \mathbf{1}_4 & 0 \end{pmatrix},$$

with  $\psi = \{\psi^X\}$ ,  $\mathbf{g}(\phi) = \{g_{XY}\} = \mathbf{g}^\dagger(\phi)$  and  $\mathbf{h}(\phi) = \{h_{XY}\} = \mathbf{h}^\dagger(\phi)$ .

But,  $R^\alpha{}_{\beta\gamma\delta} = 0 \implies$  field super-space has zero curvature.

– Minimal 2D Model with Fermionic Curvature [V Gattus, AP, to appear soon]

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} \left( g(\phi) + \underline{\mathbf{g}_\psi(\phi)} \underline{\bar{\psi}\psi} \right) \left( \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi \right) \\ & - Y(\phi) \bar{\psi} \psi - V(\phi), \quad \text{with } \gamma^\mu = (\sigma^1, -i\sigma^2).\end{aligned}$$

Chart:  $\Phi = (\phi, \psi^\top, \bar{\psi})$ ,  $\psi^\top = (\psi_1, \psi_2)$  and  $\bar{\psi} \equiv \psi^\dagger \gamma^0$ .

Grand field-space metric:

$${}_\alpha G_\beta = \begin{pmatrix} 1 - \frac{1}{2} \bar{\psi} \mathbf{d}' \mathbf{d}^{-1} \mathbf{d}' \psi & -\frac{1}{2} \bar{\psi} \mathbf{d}' & \frac{1}{2} \psi^\top \mathbf{d}'^\top \\ \frac{1}{2} \mathbf{d}'^\top \bar{\psi}^\top & \mathbf{0}_2 & \mathbf{d}^\top \\ -\frac{1}{2} \mathbf{d}' \psi & -\mathbf{d} & \mathbf{0}_2 \end{pmatrix},$$

where  $\mathbf{d} = (g + g_\psi \bar{\psi} \psi) \mathbf{1}_2 + g_\psi \psi \bar{\psi} = \bar{\mathbf{d}}$ , with  $\bar{\mathbf{d}} \equiv \gamma^0 \mathbf{d}^\dagger \gamma^0$  and  $\mathbf{d}' \equiv \partial \mathbf{d} / \partial \phi$ .

Set  $g(\phi) = 1$  for simplicity:

Super-Ricci scalar:  $\mathfrak{R} = 4g_\psi - \frac{1}{2} g_\psi'^2 (\bar{\psi} \psi)^2 \neq 0 \implies$  field super-space is curved.

## • Conclusions

- Re-formulation of a Grand Covariant Effective Action for Scalar–Tensor Theories, with a complete set of model functions
- New model function  $\ell = \ell(\Phi)$  determines the uniqueness of the VDW path-integral measure, with  $ds^2 = g_{\mu\nu}/\ell^2 dx^\mu dx^\nu$
- Rigorous Algorithms for calculating the field-space metric from the Classical Action  $S$  for both bosons and fermions
- Extension of the VDW formalism on Supermanifolds to describe realistic theories that include fermions, such as the SM
- Derivation of One- and Two-Loop Grand Covariant Effective Action for Theories with Fermions

# Yes, we may live on a Supermanifold

$$D\Sigma^2 = d\Phi^{\hat{\alpha}} \, {}_{\hat{\alpha}}G_{\hat{\beta}}(\Phi) \, d\Phi^{\hat{\beta}}$$

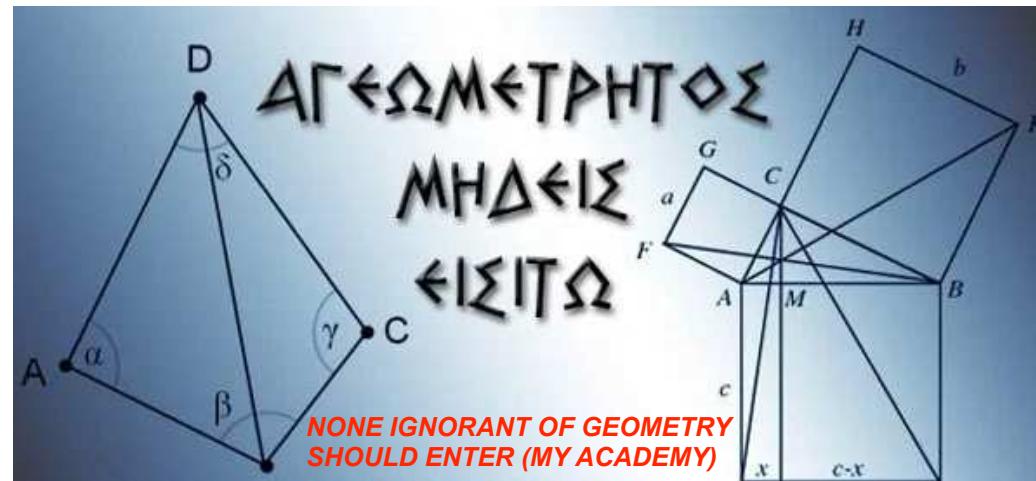
[K Finn, S Karamitsos, AP, EPJC81 (2021) 572;  
Viola Gattus, AP, to appear in May 2023.]

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$$D\Sigma^2 = d\Phi^{\hat{\alpha}} \hat{\alpha} G_{\hat{\beta}}(\Phi) d\Phi^{\hat{\beta}}$$

[K Finn, S Karamitsos, AP, EPJC81 (2021) 572;  
Viola Gattus, AP, to appear in May 2023.]

from Plato



# Back-Up Slides

- Effects of  $\ell(\varphi)$  (from  $\bar{g}_{\mu\nu} \equiv g_{\mu\nu}/\ell^2(\varphi)$ )

- Frame-invariant Dirac-delta function

$$\int d^D x \sqrt{-\bar{g}} \delta^{(D)}(x) = 1, \quad \text{with} \quad \delta^{(D)}(x_I - x_J) \equiv \frac{\delta^{(D)}(x_I - x_J)}{\sqrt{-\bar{g}}}$$

- Functional derivative

$$\frac{\bar{\delta}F[\Phi(x)]}{\bar{\delta}\Phi(y)} \equiv \lim_{\epsilon \rightarrow 0} \frac{F[\Phi(x) + \epsilon \bar{\delta}^{(D)}(x - y)] - F[\Phi(x)]}{\epsilon}$$

- Functional determinant

$$\overline{\det}(M_{xy}) \equiv \exp \left[ i \int d^D x \sqrt{-\bar{g}} \ln(M)_{xx} \right]$$

- **Field-Space Riemann Tensor  $\mathfrak{R}^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)}$  for General Relativity**

[K Finn, S Karamitsos, AP, '19]

$$\begin{aligned}
 \mathfrak{R}^{(\mu\nu)}_{(\alpha\beta)(\rho\sigma)(\gamma\delta)} = & -\frac{1}{32}\delta_\rho^\mu\delta_\beta^\nu g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{32}\delta_\sigma^\mu\delta_\beta^\nu g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta_\beta^\mu\delta_\sigma^\nu g_{\rho\gamma}g_{\alpha\delta} - \frac{1}{32}\delta_\beta^\mu\delta_\rho^\nu g_{\sigma\gamma}g_{\alpha\delta} \\
 & - \frac{1}{32}\delta_\rho^\mu\delta_\beta^\nu g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{32}\delta_\sigma^\mu\delta_\beta^\nu g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta_\beta^\mu\delta_\sigma^\nu g_{\rho\delta}g_{\alpha\gamma} - \frac{1}{32}\delta_\beta^\mu\delta_\rho^\nu g_{\sigma\delta}g_{\alpha\gamma} \\
 & - \frac{1}{32}\delta_\alpha^\mu\delta_\rho^\nu g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta_\alpha^\mu\delta_\sigma^\nu g_{\rho\gamma}g_{\beta\delta} - \frac{1}{32}\delta_\rho^\mu\delta_\alpha^\nu g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{32}\delta_\sigma^\mu\delta_\alpha^\nu g_{\rho\gamma}g_{\beta\delta} \\
 & - \frac{1}{32}\delta_\alpha^\mu\delta_\rho^\nu g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta_\alpha^\mu\delta_\sigma^\nu g_{\rho\delta}g_{\beta\gamma} - \frac{1}{32}\delta_\rho^\mu\delta_\alpha^\nu g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{32}\delta_\sigma^\mu\delta_\alpha^\nu g_{\rho\delta}g_{\beta\gamma} \\
 & + \frac{1}{32}\delta_\gamma^\mu\delta_\beta^\nu g_{\rho\delta}g_{\sigma\alpha} + \frac{1}{32}\delta_\delta^\mu\delta_\beta^\nu g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta_\beta^\mu\delta_\delta^\nu g_{\rho\gamma}g_{\sigma\alpha} + \frac{1}{32}\delta_\beta^\mu\delta_\gamma^\nu g_{\rho\delta}g_{\sigma\alpha} \\
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 & + \frac{1}{32}\delta_\alpha^\mu\delta_\gamma^\nu g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta_\alpha^\mu\delta_\delta^\nu g_{\rho\beta}g_{\sigma\gamma} + \frac{1}{32}\delta_\gamma^\mu\delta_\alpha^\nu g_{\rho\beta}g_{\sigma\delta} + \frac{1}{32}\delta_\delta^\mu\delta_\alpha^\nu g_{\rho\beta}g_{\sigma\gamma} \\
 & + \frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\beta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\gamma}g_{\beta\delta} + \frac{1}{4D}g_{\rho\alpha}g^{\mu\nu}g_{\sigma\delta}g_{\beta\gamma} \\
 & + \frac{1}{4D}g_{\rho\gamma}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\delta} + \frac{1}{4D}g_{\rho\delta}g^{\mu\nu}g_{\sigma\alpha}g_{\beta\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\delta}g_{\alpha\gamma} + \frac{1}{4D}g_{\rho\beta}g^{\mu\nu}g_{\sigma\gamma}g_{\alpha\delta} \\
 & - \frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\gamma}g_{\alpha\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\beta}g_{\sigma\delta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\alpha}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\alpha}g_{\beta\gamma} \\
 & - \frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\gamma}g_{\beta\delta} - \frac{1}{4D}g^{\mu\nu}g_{\rho\alpha}g_{\sigma\delta}g_{\beta\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\delta}g_{\sigma\beta}g_{\alpha\gamma} - \frac{1}{4D}g^{\mu\nu}g_{\rho\gamma}g_{\sigma\beta}g_{\alpha\delta}
 \end{aligned}$$

- Field-Space Ricci Tensor  $\mathfrak{R}_{(\mu\nu)(\rho\sigma)}$

$$\mathfrak{R}_{(\mu\nu)(\rho\sigma)} = \frac{1}{4}g_{\mu\nu}g_{\rho\sigma} - \frac{D}{8}g_{\mu\rho}g_{\nu\sigma} - \frac{D}{8}g_{\mu\sigma}g_{\nu\rho}$$

- Field-Space Ricci Scalar  $\mathfrak{R}$

$$\mathfrak{R} = \frac{D}{4} - \frac{D^2}{8} - \frac{D^3}{8} < 0,$$

for spacetime dimensions  $D > 1$ .

⇒ Gravity has a **curved** field space, with ***negative*** scalar curvature.

- Partial differentiation with respect to  $\gamma^\mu$

Any matrix  $M_{a\dot{a}}$  in spinor space can be uniquely expressed as

$$M_{a\dot{a}} = \sum_{i=S,P,V,A,T} a_{(i)} \Gamma_{a\dot{a}}^{(i)}, \quad \text{with } \Gamma^{(i)} \in \{I_4, \gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}\},$$

where  $\Gamma^{(i)}$  are the 16 orthogonal Lorentz-covariant bilinears,  $a_{(i)}$  is a coefficient, and  $\sigma^{\mu\nu} \equiv i/2 [\gamma^\mu, \gamma^\nu]$ .

- **Γ-matrix partial differentiation:**

$$\frac{\delta M}{\delta \Gamma^{(i)}} \equiv \lim_{\epsilon^{(i)} \rightarrow 0} \frac{M[\Gamma^{(i)} \rightarrow \Gamma^{(i)} + \epsilon^{(i)} I_4] - M[\Gamma^{(i)}]}{\epsilon^{(i)}}.$$

For  $\Gamma^{(i)} = \Gamma^{(V)} = \gamma^\mu$ ,

$$\frac{\delta M_{a\dot{a}}}{\delta \gamma^\mu} = a_\mu \delta_{a\dot{a}}.$$