

On Thermal Stability of Hairy Black Holes[†]

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Panagiotis Dorlis

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[†]Collaboration with N. Chatzifotis, N.E. Mavromatos and E. Papantonopoulos

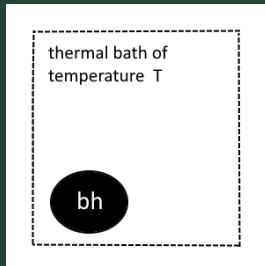


Plan

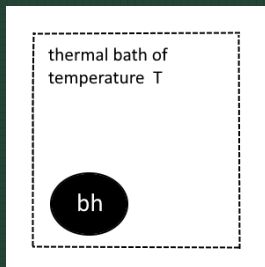
1. Black Holes in a Thermal Bath
2. Black Holes as Thermodynamical Topological Defects
3. Stabilizing Black Holes through Secondary Hair: Black Hole Relics?



Black Holes in a Thermal Bath

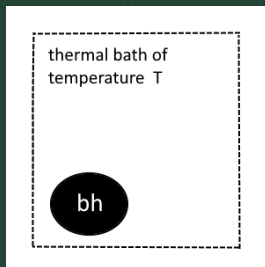


Black Holes in a Thermal Bath



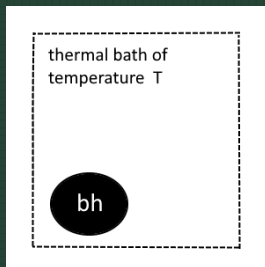
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- Black Hole Physics: $T_{BH} = \frac{\kappa_g}{2\pi}$

$$\kappa_g = 2\pi T \quad (1)$$

- A geometric quantity (surface gravity) is connected to the thermal bath's temperature

Black Holes in a Thermal Bath

- The surface gravity is a function of the event horizon radius, r_h

$$\kappa_g = \kappa_g(r_h) \quad (2)$$

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Black hole size \longleftrightarrow Heat bath's temperature

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Black hole size \longleftrightarrow Heat bath's temperature

- The dependence of κ_g on r_h can lead to different configurations of black holes that can be in thermal equilibrium with the heat bath having the same temperature, T .

r_h and T are not necessary in a "1 – 1" correspondence

Black Holes in a Thermal Bath

Defining the inverse temperature $\tau = T^{-1}$, for the known black holes:

- $\tau/4\pi = r_h$: Schwarzschild

- $\tau/4\pi = \frac{r_h^3}{r_h^2 - Q^2}$: RN

- $\tau/4\pi = \frac{r_h^3 + \alpha^2 r_h}{r_h^2 - \alpha^2}$: Kerr



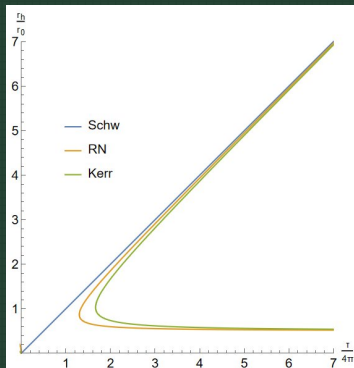
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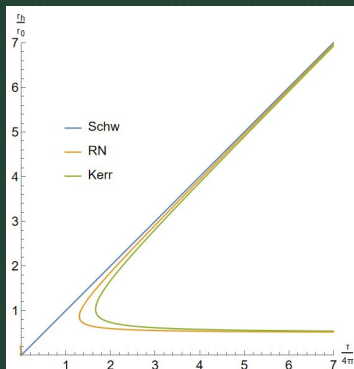
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- Extra charges
→ Two possible configurations are possible to exist in thermal equilibrium with the heat bath.



Small and Large black hole configuration



Black Holes in a Thermal Bath

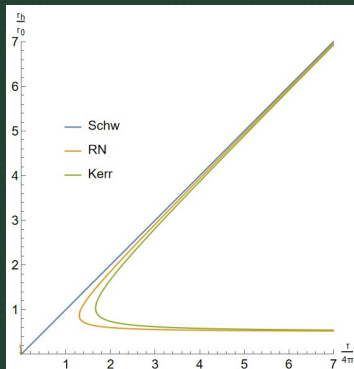
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Small and Large black hole configuration

What is the difference?



Black Holes in a Thermal Bath

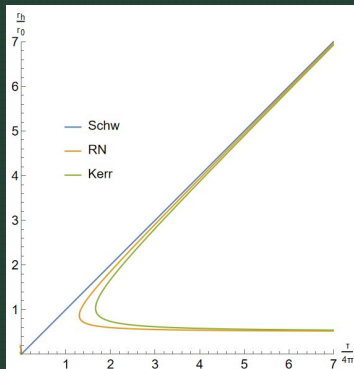
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Small and Large black hole configuration

What is the difference? **Stability**



Thermodynamical Stability Arguments

- Stability \rightarrow Potentials \Downarrow

Canonical Ensemble \rightarrow Free energy, \mathcal{F}

- $\mathcal{F} = E - TS$
- E : energy of the system (ADM mass)
- S : entropy (black hole's)
- T : heat bath's temperature

¹Fermi, E., Thermodynamics, Dover books in physics and mathematical physics, Dover Publications, 1956

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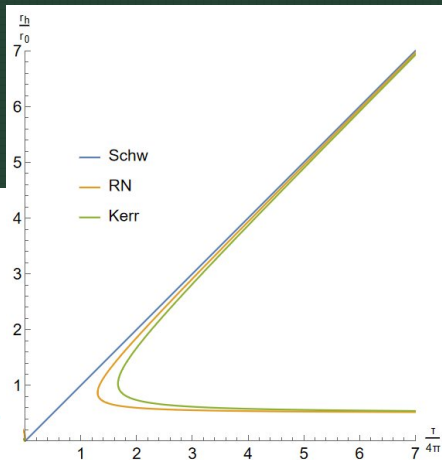
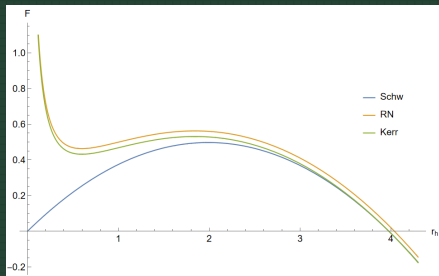
\Downarrow

- Equilibrium points $\rightarrow \frac{\partial\mathcal{F}}{\partial r_h} = 0$
- Stable $\rightarrow \frac{\partial^2\mathcal{F}}{\partial r_h^2} > 0$ (minimum) / Unstable $\rightarrow \frac{\partial^2\mathcal{F}}{\partial r_h^2} < 0$ (maximum)

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Black Holes as Thermodynamical Topological Defects

In a recent paper², a topological approach was introduced, in order to investigate the different branches of black holes in thermal equilibrium with the heat bath and classify solutions into *topologically equivalent classes*. \Downarrow

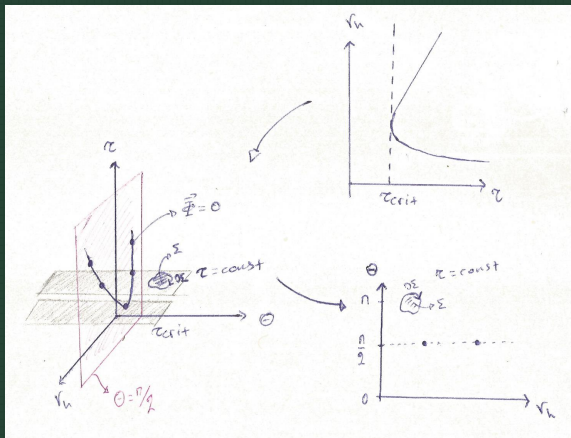
black hole solutions as zero points of some vector field, $\Phi^a(\vec{x})$

- Define the 3-D parameter space: $x^\mu = (\tau, r_h, \Theta)$
- $(\tau, r_h) \rightarrow$ Thermodynamic Parameters
- $0 \leq \Theta \leq \pi \rightarrow$ Auxiliary Parameter

²S. W. Wei, Y. X. Liu and R. B. Mann, “Black Hole Solutions as Topological Thermodynamic Defects,” Phys. Rev. Lett. **129**, no.19, 191101 (2022)
doi:10.1103/PhysRevLett.129.191101

Black Holes as Thermodynamical Topological Defects

- 3-D parameter space: $x^\mu = (\tau, r_h, \Theta)$, $0 \leq \Theta \leq \pi$, $r_h, \tau \geq 0$,
- Define on (r_h, Θ) the vector field: $\vec{\Phi} = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot \Theta \csc \Theta \right)$
- $\vec{\Phi} = 0 \rightarrow \left(\frac{\partial \mathcal{F}}{\partial r_h} = 0, \Theta = \pi/2 \right)$.



Black Holes as Thermodynamical Topological Defects

- $\vec{\Phi} = 0 \rightarrow$ Equilibrium states of black hole in the thermal bath
- Considering τ as "time" in the parameter space \rightarrow The equilibrium states ($\vec{\Phi} = 0$) are moving on the parameter space "like point particles"



Duan's Φ — mapping³

Define the topological current on the 3 — D parameter space:

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \epsilon_{ab} \partial_\nu n^a \partial_\rho n^b, \mu\nu\rho = 0, 1, 2, a, b = 1, 2 \quad (3)$$

where $n^a = \frac{\Phi^a}{\|\Phi\|}$ and ϵ the Levi-Civita symbol (flat).

³Y. S. Duan, *The structure of the topological current*, SLAC-PUB-3301, (1984)

Black Holes as Thermodynamical Topological Defects

- $j^\mu = \delta \left(\vec{\Phi} \right) J^\mu \left(\frac{\Phi}{x} \right)$
 \rightarrow point-like delta structure
 where $J^\mu \left(\frac{\Phi}{x} \right) = \frac{1}{2} \epsilon_{ab} \epsilon^{\mu\nu\rho} \partial_\nu \Phi^a \partial_\rho \Phi^b$
 (Jacobian tensor)

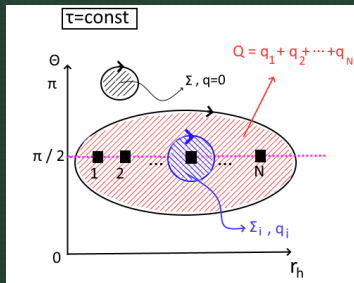
- Identically conserved: $\partial_\mu j^\mu = 0$
- Conserved
 charge: $q = \int_\Sigma j^0 d^2x$, where Σ
 some surface on the $\Theta - r_h$ plane.

- Local*

topological charge $q_i = \int_{\Sigma_i} j^0 d^2x$

- Global topological charge*

$$Q = \sum_i q_i$$



Black Holes as Thermodynamical Topological Defects

- local topological charge $q_i = \int_{\Sigma_i} j^0 d^2x$

After some algebra⁴ \Downarrow

$$q_i = \text{sgn} \left(\frac{\partial^2 \mathcal{F}}{\partial r_h^2} \right) \quad (4)$$

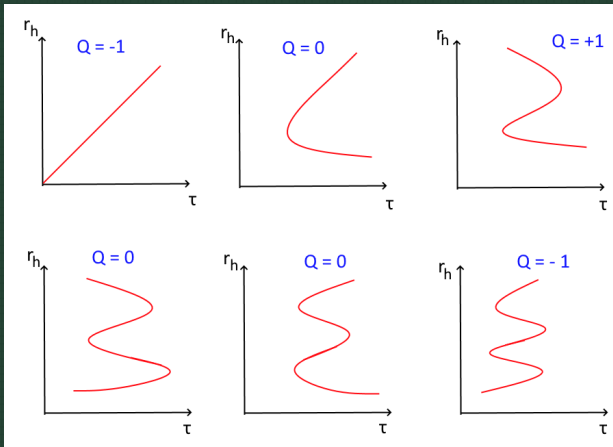
→ The local topological charge of each black hole branch concerns its thermodynamical stability:

$$\begin{aligned} \text{Unstable branch} &\rightarrow q_i = -1 \\ \text{Stable branch} &\rightarrow q_i = +1 \end{aligned} \quad (5)$$

⁴N. Chatzifotis, P. Dorlis, N. E. Mavromatos and E. Papantonopoulos, “On Thermal Stability of Hairy Black Holes,” [arXiv:2302.03980 [gr-qc]].

Black Holes as Thermodynamical Topological Defects

- *Global topological charge* $Q = \sum_i q_i$
- Characterizes the $r_h = r_h(\tau)$ graph of the solution.
- Some examples:



Black Holes as Thermodynamical Topological Defects

Global topological charge $Q = \sum_i q_i$

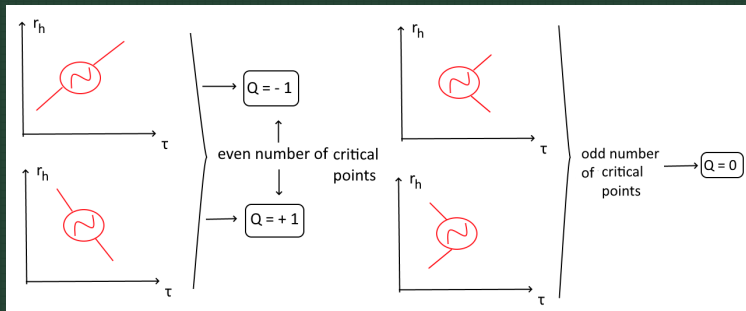
- Q depends on how many critical points appear in the graph.
- If the number of critical points is odd $\rightarrow Q = 0$
- If the number of critical points is even $\rightarrow Q = \pm 1$



Black Holes as Thermodynamical Topological Defects

Global topological charge $Q = \sum_i q_i$

- Q depends on how many critical points appear in the graph.
- If the number of critical points is odd $\rightarrow Q = 0$
- If the number of critical points is even $\rightarrow Q = \pm 1$
- Continuity of the function $r_h = r_h(\tau) \implies$ the number of critical points depends on the asymptotic behavior (i.e. the slope) \downarrow



Black Holes as Thermodynamical Topological Defects

- Thus, for a well behaved asymptotic behavior the black hole solution can have $Q = 0, \pm 1$.
- In this sense, black hole solutions with the same topological charge construct are *topologically equivalent*⁵

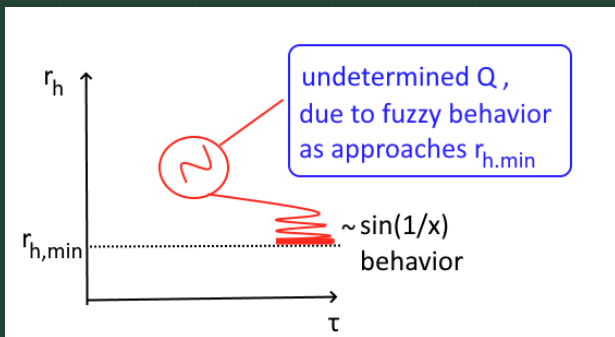
Examples:

$Q = 0$	RN, Schw-AdS, Kerr
$Q = -1$	Schw
$Q = +1$	RN-AdS, Kerr-AdS

⁵S. W. Wei, Y. X. Liu and R. B. Mann, “Black Hole Solutions as Topological Thermodynamic Defects,” Phys. Rev. Lett. **129**, no.19, 191101 (2022)
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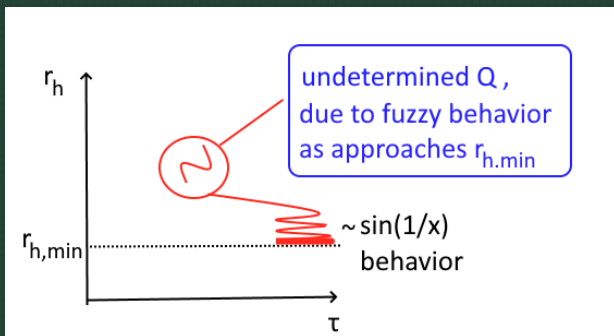
Black Holes as Thermodynamical Topological Defects

- Beyond these classes?



Black Holes as Thermodynamical Topological Defects

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A toy model can be constructed

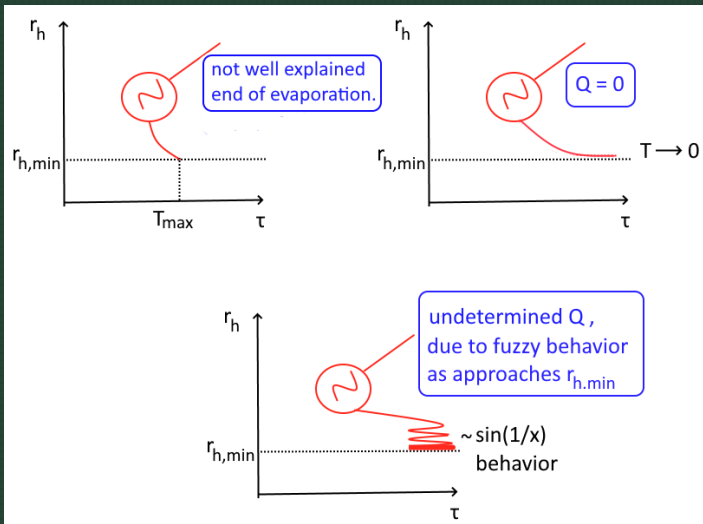
Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- Primordial black holes can be formed at the early stages of the universe⁶
- Assuming Schwarzschild black holes → **Total evaporation**
 $t_{\text{evap}} < t_{\text{universe}}$ (*basic constraint for the mass window*)
- **Overcoming this constraint** → The temperature - mass relation is modified at the late stages of evaporation, due to higher order interactions, leading to Black hole relics⁷
- **In the perspective of the $r_h = r_h(\tau)$ graph** → Black hole relics if a minimum horizon radius is induced

⁶B. J. Carr and S. W. Hawking, Mon. Not. Roy. Astron. Soc. **168**, 399-415 (1974) doi:10.1093/mnras/168.2.399

⁷J. D. Barrow, E. J. Copeland and A. R. Liddle, Phys. Rev. D **46**, 645-657 (1992) doi:10.1103/PhysRevD.46.645

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?



Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- We shall consider in a *quite* general form interactions (or self-interactions) of the gravitational field with matter fields of the form:

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{matter} + A\mathcal{L}_{int} \quad (6)$$

- \mathcal{L}_{EH} : Einstein-Hilbert
- \mathcal{L}_{matter} : Matter Lagrangian
- \mathcal{L}_{int} : possible interactions
- A : dimensionfull coupling constant
- A introduces a length scale that characterizes the interactions

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- A which denotes the strength of the interaction
- r_h : characteristic length scale of the black hole
- The impact of an interaction to a black hole becomes stronger, when these length scales become comparable.
- $\mathcal{L}_{EH} \rightarrow$ Planck length is introduced, $\kappa^2 = 8\pi G$.



Claim: the impact of an interaction to a black hole is determined by the dimensionless parameter, γ :

$$\gamma = \frac{A\kappa}{r_h^2} \quad (7)$$

Examples of backreactions...

- Linear scalar-Gauss-Bonnet:

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial\phi)^2 + A\phi \mathcal{R}_{GB} \right] \quad (8)$$

where $\mathcal{R}_{GB} \equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$ denotes the GB topological term. The solution for the g_{tt} component, up to second order in A ⁸ takes the following form:

$$g_{tt} = -1 + \frac{r_h}{r} + \gamma^2 \left[\frac{20}{3} \left(\frac{r_h}{r} \right)^7 - \frac{16}{5} \left(\frac{r_h}{r} \right)^6 - \frac{22}{5} \left(\frac{r_h}{r} \right)^5 - \frac{52}{3} \left(\frac{r_h}{r} \right)^4 - \frac{4}{3} \left(\frac{r_h}{r} \right)^3 + \frac{49}{5} \frac{r_h}{r} \right] + \mathcal{O}(\gamma^4) \quad (9)$$

⁸T. P. Sotiriou and S. Y. Zhou, Phys. Rev. Lett. **112**, 251102 (2014)
doi:10.1103/PhysRevLett.112.251102 [arXiv:1312.3622 [gr-qc]].

Examples of backreactions...

- Linear scalar-Chern-Simons (CS)

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} - \frac{1}{2}(\partial\phi)^2 - A\phi\mathcal{R}_{CS} \right] \quad (10)$$

where $\mathcal{R}_{CS} = R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma}$ (C-S topological term)⁹. The backreaction (up to $\mathcal{O}(\alpha)$)

$$g_{t\phi}(r, \theta) = - \left[\frac{r_h}{r} + \sum_{n=4} \frac{d_n}{2^{n-2}} \left(\frac{r_h}{r} \right)^{n-2} \right] \alpha \sin^2 \theta \quad (11)$$

► α : rotation parameter coefficients

► $d_n \sim \gamma^{2n}$: fully determined by a recurrence relation¹⁰

⁹explained previously in Chatzifotis talk

¹⁰N. Chatzifotis, P. Dorlis, N. E. Mavromatos and E. Papantonopoulos, Phys. Rev. D **105**, no.8, 084051 (2022)

Examples of backreactions...

- Extended GB gravity¹¹:

$$S = \int d^4x \frac{\sqrt{-g}}{2\kappa^2} \left[R - \beta e^{2\phi} (R + 6(\nabla\phi)^2) - 2\lambda e^{4\phi} \right. \\ \left. - \tilde{a} \left(\phi \mathcal{R}_{GB} - 4G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - 4\Box\phi (\nabla\phi)^2 - 2(\nabla\phi)^4 \right) \right] \quad (12)$$

- Redefine $\tilde{a} \rightarrow A\kappa$; then, A is the coupling constant of the theory with dimensions of length, as previously.

Expanding the analytic solution, we get:

$$-g_{tt}(r) = 1 - \frac{r_h}{r} + \left[\frac{r_h}{r} - 2 \left(\frac{r_h}{r} \right)^2 + \left(\frac{r_h}{r} \right)^4 \right] \gamma \\ - 2 \frac{r_h}{r} \left[\left(\frac{r_h}{r} \right)^3 - 2 \left(\frac{r_h}{r} \right)^4 + \left(\frac{r_h}{r} \right)^6 \right] \gamma^2 + \mathcal{O}(\gamma^3) \quad (13)$$

¹¹P. G. S. Fernandes, Phys. Rev. D **103**, no.10, 104065 (2021)

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

Our goal is to compress the following properties to a general metric.

1. The only dimensionfull parameters of the solution to be l_P , A and mass, M .
2. The corresponding local solutions have a smooth limit for a vanishing coupling constant, which we assume to be the Schwarzschild black hole.
3. The dependence of the backreacting terms vanish asymptotically, which means that they depend only to inverse powers of r
4. The coupling constant of the interaction appears only into the dimensionless factor γ
5. The r —dependence of the back-reaction terms appear only into $x = r_h/r$.



Stabilizing Black Holes through Secondary Hair: Black Hole Relics?



$$g_{tt}(r) = -(1-x) \left(1 + \sum_n \gamma^n f_n(x) \right) \quad (14)$$

where $x = r_h/r$ and $f_n(x)$ some polynomial functions.

- Black hole's exterior: $r > r_h \rightarrow 0 < x < 1$
- Black hole's interior: $r < r_h \rightarrow x > 1$
- Event horizon: $r = r_h \rightarrow x = 1$
- Asymptotic flatness implies that $f_n(0) = 0$
- $1 + \sum_n \gamma^n f_n(x) > 0$, for $0 < x < 1$.
- If an inner horizon exists: $1 + \sum_n \gamma^n f_n(x_0) = 0$, for some $x_0 > 1$.

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- The surface gravity reads:

$$\kappa_g(r_h) = \frac{|g'_{tt}|}{2\sqrt{-g_{tt}g_{rr}}}\Big|_{r \rightarrow r_h} . \quad (15)$$

- In order to acquire a zero temperature for a finite r_h , $|g'_{tt}| \rightarrow 0$

Calculating for the above metric:

$$g'_{tt}(r_h) = -\frac{1}{r_h} \left(1 + \sum_n \gamma^n f_n(1) \right) . \quad (16)$$

\Rightarrow There have to be a critical value, γ_c , for which

$$1 + \sum_n \gamma^n f_n(1) \rightarrow 0, \text{ as } \gamma \rightarrow \gamma_c \quad (17)$$

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- The γ_c defines the minimum horizon radius:

$$r_{h,min} = \frac{A\kappa}{\gamma_c} \quad (18)$$

- The condition (17) implies that the g_{tt} metric component tends to acquire a double root at $x = 1$, which is a characteristic of extremal black holes.



If the interactions, \mathcal{L}_{int} , produce an inner horizon inside the black hole, the runaway evaporation stops and a near extremal black hole relic is left over.

Stabilizing Black Holes through Secondary Hair: Black Hole Relics?

- $r_{h,min} = \frac{A\kappa}{\gamma_c} \implies \boxed{r_{h,min} \sim |\tilde{A}| l_P}$, where $A = \tilde{A}^2 l_P$

Thus, whether the stable black hole minimal size exceeds or not the Planck length depends on the order of magnitude of the coupling of the pertinent interaction.

- **Electric charge and rotation**, also induce an inner horizon. However, electric charge and rotation probably do not survive at the final stages of evaporation¹², so they **cannot stabilize the black hole**.
- The difference here is that there is no extra charge beyond the mass, $M \rightarrow$ **The inner horizon can survive until the final stages and stabilize (thermodynamically) the black hole.**

¹²Calmet, X., Carr, B., Winstanley, E. (2014). Hawking Radiation and Black Hole Evaporation. In: Quantum Black Holes. SpringerBriefs in Physics. Springer, Berlin, Heidelberg.

THANK YOU!

