Inflation in Metric-Affine Quadratic Gravity

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The present framework of modern cosmology consists of classical *General Relativity (GR)* as a theory of gravitation and *Quantum Field Theory (QFT)* as the theory of matter. A common working assumption is that the quantum aspects of gravitation can be ignored for energies below the Planck energy of $10^{19} GeV$ and, therefore, gravity can be treated classically. In contrast, the full quantum character of particle interactions is considered within QFT. The quantum interactions of the matter fields coupled to the classical gravitational field introduce modifications to the standard GR action with cosmological implications. Such are non-minimal couplings of the inflaton field or higher power terms of the Ricci curvature in models of cosmological inflation. The *Metric-Affine* formulation of gravity, where the metric and the connection are independent variables, although equivalent to the standard (*metric*) GR in the case of the Einstein-Hilbert action, leads to different predictions when the above corrections are included.

Metric Versus Metric-Affine (Palatini) Formulation of Gravity

The **General Relativity Principle** states that all laws of physics should be invariant under general coordinate transformations. To implement such a principle we need to introduce a metric $g_{\mu\nu}$, which has to transform as

$$g'_{\alpha\beta}(x') = \left(\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\right)\left(\frac{\partial x^{\nu}}{\partial x'^{\beta}}\right) g_{\mu\nu}(x)\,, \tag{1}$$

as well as a **Connection** $\Gamma^{\rho}_{\mu\nu}$ in order to define covariant derivatives of tensors. In the standard **metric formulation** of gravity the connection is not an independent quantity but it is given by the **Levi-Civita relation** as

$$\Gamma^{\rho}_{\mu\nu}\Big|_{LC} = \frac{1}{2} g^{\rho\sigma}\left(\partial_{\mu} g_{\rho\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}\right)\,. \tag{2}$$

In contrast, in the so-called **Metric-Affine** theories of gravity the connection is an independent variable not related to the metric through (2). Note that $D_{\mu} g_{\nu\rho}\big|_{LC} = 0$ (**metricity**), while $D_{\mu} g_{\mu\nu} \neq 0$ in general for a metric-affine theory.

The Distortion Tensor

The difference between the independent connection of a metric-affine theory and the corresponding Levi-Civita one is a tensor called ***the Distrortion tensor***

$$C_{\mu\ \nu}^{\ \rho} = \Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\mu\ \nu}^{\ \rho}\Big|_{LC} \,. \tag{3}$$

The distortion tensor vanishes for metric theories. The Torsion is given by $T_{\mu}{}^{\rho}{}_{\nu} = C_{\mu}{}^{\rho}{}_{\nu} - C_{\nu}{}^{\rho}{}_{\mu} = 2C_{[\mu}{}^{\rho}{}_{\nu]} = \Gamma_{\mu}{}^{\rho}{}_{\nu} - \Gamma_{\nu}{}^{\rho}{}_{\mu}$.

The curvature tensor of a metric-affine theory is defined as

$$\mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \,. \tag{4}$$

The following two scalars, linear in the Riemann tensor, can be defined as

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} g^{\mu}_{\rho} g^{\nu\sigma} = \mathcal{R}_{\mu\nu}{}^{\mu\nu} \\ \tilde{\mathcal{R}} &= (-g)^{-1/2} \epsilon^{\mu\nu}{}_{\rho}{}^{\sigma} \mathcal{R}_{\mu\nu}{}^{\rho}{}_{\sigma} = (-g)^{-1/2} \epsilon^{\mu\nu\rho\sigma} \mathcal{R}_{\mu\nu\rho\sigma} \end{aligned} \tag{5}$$

The first corresponds to the usual Ricci scalar, while $\tilde{\mathcal{R}}$ is the so-called ***Holst invariant***, which vanishes identically in a metric theory due to the symmetry in the lower indices of the Levi-Civita connection.

The following expressions of the curvature scalars can be written in terms of the Distortion

$$
\begin{aligned}
\mathcal{R} &= R + D_\mu C_\nu{}^{\mu\nu} - D_\nu C_\mu{}^{\mu\nu} + C_\mu{}^\mu{}_\lambda C_\nu{}^{\lambda\nu} - C_\nu{}^\mu{}_\lambda C_\mu{}^{\lambda\nu} \\
\tilde{\mathcal{R}} &= 2(-g)^{-1/2}\epsilon^{\mu\nu\rho\sigma}\left(D_\mu C_{\nu\rho\sigma} + C_{\mu\rho\lambda}C_\nu{}^\lambda{}_\sigma\right)
\end{aligned}
\tag{6}
$$

where $R = R[g]$ is the standard metric Ricci scalar and the covariant derivatives are taken with respect to $\Gamma_\mu{}^\lambda{}_\nu\big|_{LC}$.

The metric-affine version of the **Einstein-Hilbert action** is

$$
\mathcal{S}_{EH} = \frac{1}{2}\int d^4x\sqrt{-g}\mathcal{R} = \frac{1}{2}\int d^4x\sqrt{-g}\left\{R + D_\mu C_\nu{}^{\mu\nu} - D_\nu C_\mu{}^{\mu\nu} + C_\mu{}^\mu{}_\lambda C_\nu{}^{\lambda\nu} - C_\nu{}^\mu{}_\lambda C_\mu{}^{\lambda\nu}\right\}
$$

Variation with respect to the distortion gives

$$
\frac{\delta\mathcal{S}}{\delta C} = 0 \implies \delta^\alpha_\beta C_{\nu\gamma}{}^\nu + \delta^\alpha_\gamma C_\nu{}^\nu{}_\beta - C_{\beta\gamma}{}^\alpha - C_\gamma{}^\alpha{}_\beta = 0\,,
$$

which has the general solution $C_{\mu\nu\rho} = U_\mu\, g_{\nu\rho}$ in terms of the arbitrary vector U_μ. Substituting this solution into Eq.(2.8) the $C-$dependent terms vanish. Therefore, **$\mathcal{S}_{EH}$ is entirely equivalent to the standard metric GR**. Nevertheless, this is not the case for quadratic actions.

Non-Minimal Coupling to Scalars

One can derive the **metric-equivalent** of any metric-affine theory based on an action, where gravity couples to a scalar field,

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}\Omega^2(\phi)\mathcal{R} + \mathcal{L}(\phi, g_{\mu\nu}, \partial_\mu\phi) \right\} . \tag{7}$$

Note that any $F(\mathcal{R})$ theory can also be set in this form. Indeed the action $\mathcal{S} = \frac{1}{2}\int d^4x \sqrt{-g}\, F(\mathcal{R})$, corresponding to the metric-affine formulation of $f(R)$ theories studied in the standard metric formulation. The action can be set in the form

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} F'(\chi)\, \mathcal{R} - V(\chi) \right\} \text{ where } V(\chi) = \frac{1}{2}\left(\chi F'(\chi) - F(\chi)\right) , \tag{8}$$

in terms of the auxiliary scalar χ.

Substituting the expression of $\mathcal{R}$ in terms of the distortion, we obtain

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}\Omega^2(\phi)R(g) + \frac{1}{2}\Omega^2(\phi)\left(D_\mu C_\nu{}^{\mu\nu} - D_\nu C_\mu{}^{\mu\nu} \right.\right.$$

$$\left.\left. + C_\mu{}^\mu{}_\lambda C_\nu{}^{\lambda\nu} - C_\nu{}^\mu{}_\lambda C_\mu{}^{\lambda\nu}\right) + \mathcal{L}(\phi, g_{\mu\nu}, \partial_\mu\phi) \right\} , \tag{9}$$

Variation with respect to the distortion gives us the equation

$$\Omega^2 \left(\delta^\alpha_\beta C_{\nu\gamma}{}^{\nu} + \delta^\alpha_\gamma C_\nu{}^\nu{}_\beta - C_{\beta\gamma}{}^\alpha - C_\gamma{}^\alpha{}_\beta\right) = \delta^\alpha_\gamma \partial_\beta \Omega^2 - \delta^\alpha_\beta \partial_\gamma \Omega^2 \qquad (10)$$

with a solution (up to a term $U_\mu \, g_{\nu\rho}$ of an arbitrary vector U_μ)

$$C_\mu{}^\rho{}_\nu = \frac{1}{2}\left(g^\rho_\mu \partial_\nu \ln \Omega^2 + g^\rho_\nu \partial_\mu \ln \Omega^2 - g_{\mu\nu}\partial^\rho \ln \Omega^2\right) \qquad (11)$$

Substituting C back into the action we obtain

$$\mathcal{S} = \int d^4x \sqrt{-g}\left\{\frac{1}{2}\Omega^2(\phi)R(g) + \frac{3}{4}\frac{(\nabla\Omega^2)^2}{\Omega^2} + \mathcal{L}(\phi, g_{\mu\nu}, \partial_\mu\phi)\right\}. \qquad (12)$$

This is a metric theory and the appearing connection is the Levi-Civita one. Note that the extra term has the form of the extra kinetic term that appears when we Weyl-rescale the metric theory to the Einstein frame, albeit *with an opposite sign*. The inequivalence of the two formulations rests on this term. Only in the case of the Einstein-Hilbert action this term vanishes and the two formulations are equivalent.

Quadratic Metric-Affine Theories

Consider the following metric-affine generalization of the Starobinsky model

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}\alpha\mathcal{R} + \frac{1}{2}\beta\tilde{\mathcal{R}} + \frac{1}{4}\gamma\mathcal{R}^2 + \frac{1}{4}\delta\tilde{\mathcal{R}}^2 \right\} , \qquad (13)$$

where $\mathcal{R}$ is the Ricci scalar curvature and $\tilde{\mathcal{R}}$ is the Holst invariant. This is a general quadratic action of these scalars. In what follows we shall use Planck-mass units taking $\alpha = 1$. An equivalent way to express the action is in terms of the auxiliary scalars χ and ζ as

$$\mathcal{S} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2}(1+\gamma\chi)\mathcal{R} + \frac{1}{2}(\beta+\delta\zeta)\tilde{\mathcal{R}} - \frac{1}{4}\left(\gamma\chi^2 + \delta\zeta^2\right) \right\} . \qquad (14)$$

Next, we may use the expressions of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ in terms of the Distortion C, given in (6), and obtain

$$\begin{aligned}\mathcal{S} = \int d^4x\sqrt{-g}\,\{&\tfrac{1}{2}(1+\gamma\chi)R \\ &+\tfrac{1}{2}(1+\gamma\chi)\left(D_\mu C_\nu{}^{\mu\nu} - D_\nu C_\mu{}^{\mu\nu} + C_\mu{}^\mu{}_\lambda C_\nu{}^{\lambda\nu} - C_\nu{}^\mu{}_\lambda C_\mu{}^{\lambda\nu}\right) \\ &+(\beta+\delta\zeta)(-g)^{-1/2}\epsilon^{\mu\nu\rho\sigma}\left(D_\mu C_{\nu\rho\sigma} + C_{\mu\rho\lambda}C_\nu{}^\lambda{}_\sigma\right) - \tfrac{1}{4}\left(\gamma\chi^2+\zeta^2\right)\}\end{aligned} \tag{15}$$

where $R = R(g)$ and the covariant derivatives are with respect to $\Gamma^\lambda_{\mu\nu}\big|_{LC}$. Variation with respect to $C_\alpha^{\ \beta\gamma}$ gives

$$\frac{1}{2}\Omega^2\left(\delta^\alpha_\beta C_{\nu\gamma}{}^\nu + \delta^\alpha_\gamma C_\nu{}^\nu{}_\beta - C_{\beta\gamma}{}^\alpha - C_\gamma{}^\alpha{}_\beta\right) - \frac{\overline{\Omega}^2}{\sqrt{-g}}\left(\epsilon^{\mu\alpha\sigma}{}_\beta C_{\mu\gamma\sigma} + \epsilon^{\mu\alpha\sigma}{}_\gamma C_{\mu\sigma\beta}\right) = J_\beta{}^\alpha{}_\gamma \tag{16}$$

where

$$\begin{aligned} J_\beta{}^\alpha{}_\gamma &= \tfrac{1}{2}\delta^\alpha_\gamma\partial_\beta\Omega^2 - \tfrac{1}{2}\delta^\alpha_\beta\partial_\gamma\Omega^2 + \frac{\epsilon^{\mu\alpha}{}_{\beta\gamma}}{\sqrt{-g}}\partial_\mu\overline{\Omega}^2 \\ \Omega^2 &\equiv 1+\gamma\chi, \quad \overline{\Omega}^2 = \beta+\delta\zeta \end{aligned} \tag{17}$$

A Solution for the Distortion and the Corresponding Metric Action

Note that in the previous equation the RHS is antisymmetric in the lower indices, i.e. $J_{\beta\ \gamma}^{\ \alpha} = -J_{\gamma\ \beta}^{\ \alpha}$. Then, we obtain the following solution

$$C_{\mu\nu\rho} = \frac{g_{\mu\nu}}{2\Delta}\left(\Omega^2\partial_\rho\Omega^2 + 4\overline{\Omega}^2\partial_\rho\overline{\Omega}^2\right) - \frac{g_{\mu\rho}}{2\Delta}\left(\Omega^2\partial_\nu\Omega^2 + 4\overline{\Omega}^2\partial_\nu\overline{\Omega}^2\right) + \frac{\epsilon_{\mu\nu\rho\sigma}}{\Delta\sqrt{-g}}\left(\Omega^2\partial^\sigma\overline{\Omega}^2 - \overline{\Omega}^2\partial^\sigma\Omega^2\right) , \tag{18}$$

where $\Delta \equiv \Omega^4 + 4\overline{\Omega}^4$. Note that

$$C_{\mu\nu\rho} = -C_{\mu\rho\nu} . \tag{19}$$

Substituting C back into the action, we obtain the corresponding metric action

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\Omega^2 R(g) + \frac{3}{4}\frac{(\nabla\Omega^2)^2}{\Omega^2} - \frac{3}{\Omega^2\Delta}\left(\Omega^2\nabla\overline{\Omega}^2 - \overline{\Omega}^2\nabla\Omega^2\right)^2 - \frac{1}{4\gamma}(\Omega^2-1)^2 - \frac{1}{4\delta}(\overline{\Omega}^2-\beta)^2\right\} \tag{20}$$

Einstein Frame

The Weyl rescaling $g_{\mu\nu} = \Omega^{-2}\bar{g}_{\mu\nu}$ takes us to the Einstein frame. Note that $R(g) = \Omega^2\bar{R} - 6\Omega^3\Box\Omega^{-1}$. The action is

$$\mathcal{S} = \int d^4x\sqrt{-\bar{g}}\left\{\frac{1}{2}\bar{R}(\bar{g}) - \frac{3}{\Omega^4\Delta}\left(\Omega^2\bar{\nabla}\overline{\Omega}^2 - \overline{\Omega}^2\bar{\nabla}\Omega^2\right)^2 \right.$$

$$\left. -\frac{1}{\Omega^4}\left(\frac{1}{\gamma}(\Omega^2 - 1)^2 + \frac{1}{\delta}(\overline{\Omega}^2 - \beta)^2\right)\right\} \qquad (21)$$

Introducing the field

$$\sigma \equiv \frac{\overline{\Omega}^2}{2\Omega^2}\,, \qquad (22)$$

thescalar part of the Lagrangian becomes

$$\mathcal{L} = -\frac{12(\bar{\nabla}\sigma)^2}{(1+16\sigma^2)} - \frac{1}{4}\left(\frac{1}{\gamma}(\Omega^{-2} - 1)^2 + \frac{1}{\delta}(2\sigma - \beta\Omega^{-2})^2\right). \qquad (23)$$

Variation with respect to the non-dynamical Ω^2 gives

$$\frac{\delta\mathcal{L}}{\delta\Omega^2} = 0 \implies \Omega^{-2} = \frac{\delta + 2\beta\gamma\sigma}{\delta + \beta^2\gamma} \implies \mathcal{L} = -\frac{12(\bar{\nabla}\sigma)^2}{(1+16\sigma^2)} - \frac{1}{4}\frac{(2\sigma - \beta)^2}{(\delta + \beta^2\gamma)}. \qquad (24)$$

The theory can be expressed in terms of a canonical scalar s defined by

$$\sigma = \frac{1}{4}\sinh(\sqrt{2/3}\, s) \tag{25}$$

as

$$\mathcal{L} = -\frac{1}{2}(\nabla s)^2 - \frac{1}{16}\frac{\left(\sinh(\sqrt{2/3}\, s) - 2\beta\right)^2}{(\delta + \beta^2\gamma)} . \tag{26}$$

At least one of γ and δ has to be included in order to generate the additional pseudoscalar degree of freedom represented by σ. The inflationary behaviour of this model has been studied by G.Pardisi and A.Salvio (2022). Note that the parameters γ and δ, associated with $\mathcal{R}^2$ and $\tilde{\mathcal{R}}^2$, can only have a secondary role in a possible inflationary behaviour, which would be controlled by β.

$\mathcal{R}$	GR
$\mathcal{R} + \tilde{\mathcal{R}}$	GR
$\mathcal{R} + \mathcal{R}^2$	GR
$\mathcal{R} + \tilde{\mathcal{R}}^2$	σ, No Inflation
$\mathcal{R} + \tilde{\mathcal{R}} + \mathcal{R}^2$	σ, Inflation possible
$\mathcal{R} + \tilde{\mathcal{R}} + \tilde{\mathcal{R}}^2$	σ, Inflation possible
$\mathcal{R} + \tilde{\mathcal{R}}^2 + \mathcal{R}^2$	σ, No Inflation
$\mathcal{R} + \tilde{\mathcal{R}} + \mathcal{R}^2 + \tilde{\mathcal{R}}^2$	σ, Inflation possible

I.Antoniadis, A.Karam, A.Lykkas, KT (2018)

G.Pradisi, A.Salvio (2022)

Coupling to a Fundamental Scalar

We consider a scalar ϕ coupled to quadratic metric-affine gravity non-minimally. The action is

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}f(\phi)\mathcal{R} + \frac{1}{2}h(\phi)\tilde{\mathcal{R}} + \frac{\gamma}{4}\mathcal{R}^2 + \frac{\delta}{4}\tilde{\mathcal{R}}^2 + \mathcal{L}_\phi\right\}, \tag{27}$$

with

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi). \tag{28}$$

Introducing the auxiliaries χ and ζ, we arrive at

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}(\gamma\chi + f(\phi))\mathcal{R} + \frac{1}{2}(\delta\zeta + h(\phi))\tilde{\mathcal{R}} - \frac{1}{4}\left(\gamma\chi^2 + \delta\zeta^2\right) + \mathcal{L}_\phi\right\} \tag{29}$$

or, introducing

$$\Omega^2 = \gamma\chi + f(\phi), \quad \overline{\Omega}^2 = \delta\zeta + h(\phi), \tag{30}$$

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\Omega^2\mathcal{R} + \frac{1}{2}\overline{\Omega}^2\tilde{\mathcal{R}} - \frac{1}{4}\left(\frac{1}{\gamma}(\Omega^2 - f(\phi))^2 + \frac{1}{\delta}(\overline{\Omega}^2 - h(\phi))^2\right) + \mathcal{L}_\phi\right\} \tag{31}$$

The Corresponding Metric Theory

Rewriting the action in terms of the Distortion and solving for it we arrive at the action of the corresponding metric theory in the Jordan frame

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}\Omega^2 R(g) + \frac{3}{4}\frac{(\nabla\Omega^2)^2}{\Omega^2} - \frac{3}{\Omega^2\Delta}\left(\Omega^2\nabla\overline{\Omega}^2 - \overline{\Omega}^2\nabla\Omega^2\right)^2 \right.$$
$$\left. -\frac{1}{4}\left(\frac{1}{\gamma}(\Omega^2 - f(\phi))^2 + \frac{1}{\delta}(\overline{\Omega}^2 - h(\phi))^2\right) + \mathcal{L}_\phi\right\} \tag{32}$$

The Weyl rescaling $g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu}$ takes the action into the Einstein frame

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R(g) - \frac{3}{\Omega^4\Delta}\left(\Omega^2\nabla\overline{\Omega}^2 - \overline{\Omega}^2\nabla\Omega^2\right)^2 \right.$$
$$\left. -\frac{1}{4\Omega^4}\left(\frac{1}{\gamma}(\Omega^2 - f(\phi))^2 + \frac{1}{\delta}(\overline{\Omega}^2 - h(\phi))^2\right) - \frac{1}{2}\frac{(\nabla\phi)^2}{\Omega^2} - \frac{V(\phi)}{\Omega^4}\right\} \tag{33}$$

Introducing the field $\sigma = \frac{\overline{\Omega}^2}{2\Omega^2}$, we get the action in the form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R - \frac{12(\nabla\sigma)^2}{(1+16\sigma^2)} - \frac{1}{2}\frac{(\nabla\phi)^2}{\Omega^2} - \frac{\sigma^2}{\delta}\right.$$

$$\left. - \frac{1}{4\gamma\Omega^4}\left(f(\phi) - \Omega^2\right)^2 - \frac{h(\phi)}{4\delta\Omega^4}\left(h(\phi) - 4\sigma\Omega^2\right) - \frac{V(\phi)}{\Omega^4}\right\} \tag{34}$$

Note that no kinetic term for Ω^2 appears. Solving for it we obtain

$$\frac{\delta\mathcal{S}}{\delta\Omega^2} = 0 \Longrightarrow \Omega^2 = \frac{f(\phi)^2 + 4\gamma V(\phi) + \gamma h^2(\phi)/\delta}{f(\phi) + 2\gamma\sigma h(\phi)/\delta - \gamma(\nabla\phi)^2} \tag{35}$$

Substituting Ω^2 into the action we get it in the form

$$\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}K_\phi(\phi,\sigma)(\nabla\phi)^2 + \frac{1}{4}L_\phi(\phi)(\nabla\phi)^4 - \frac{1}{2}K_\sigma(\sigma)(\nabla\sigma)^2 - U(\phi,\sigma)\right\} \tag{36}$$

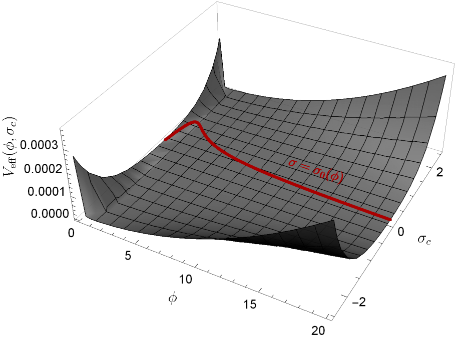
where

$$\begin{cases} K_\phi(\phi,\sigma) = \frac{f(\phi)+2\gamma\sigma h(\phi)/\delta}{\gamma h^2(\phi)/\delta+f^2(\phi)+4\gamma V(\phi)} \\ L_\phi = \frac{\gamma}{\gamma h^2(\phi)/\delta+f^2(\phi)+4\gamma V(\phi)} \\ K_\sigma(\sigma) = \frac{24}{1+16\sigma^2} \\ U(\phi,\sigma) = \frac{V(\phi)}{f^2(\phi)+4\gamma V(\phi)} + \frac{1}{\delta}\left(\frac{f^2(\phi)+4\gamma V(\phi)}{\gamma h^2(\phi)/\delta+f^2(\phi)+4\gamma V(\phi)}\right)(\sigma-\sigma_0(\phi))^2 \end{cases} \tag{37}$$

where

$$\sigma_0(\phi) = \frac{h(\phi)f(\phi)/2}{f^2(\phi)+4\gamma V(\phi)}\,. \tag{38}$$

Note that the potential is positive-definite with a minimum line along $\sigma = \sigma_0(\phi)$.



3D plot of $U(\phi,\sigma)$ for $f(\phi)=1+\xi\phi^2$, $h(\phi)=\bar{\xi}\phi+\bar{\xi}'\phi^3$ and $V(\phi)=\frac{\lambda}{4}\phi^4$

The one-field Lagrangian.

Along the minimum line (red) the potential is just $\frac{V(\phi)}{f^2(\phi)+4\gamma V(\phi)}$. However the kinetic term of ϕ is modified by $-\frac{1}{2}K_\sigma(\sigma_0(\phi))(\nabla\sigma_0(\phi))^2$. The effective Lagrangian is

$$\mathcal{L}_{eff} = -\frac{1}{2}\overline{K}(\phi)(\nabla\phi)^2 + \frac{1}{4}L(\phi)(\nabla\phi)^4 - U_0(\phi) \tag{39}$$

where

$$\begin{cases} \overline{K}(\phi) = \left(\frac{12}{1+\frac{4h(\phi)^2 f^2(\phi)}{[f^2(\phi)+4\gamma V(\phi)]^2}}\right)\left(\frac{h'(\phi)f(\phi)+h(\phi)f'(\phi)}{f^2(\phi)+4\gamma V(\phi)}\right. \\ \left. -\frac{h(\phi)f(\phi)}{[f^2(\phi)+4\gamma V(\phi)]^2}\left(2f'(\phi)f(\phi)+4\gamma V'(\phi)\right)\right)^2 \\ L(\phi) = \frac{\gamma}{\gamma h^2(\phi)/\delta+f^2(\phi)+4\gamma V(\phi)} \\ U_0(\phi) = \frac{V(\phi)}{f^2(\phi)+4\gamma V(\phi)} \end{cases} \tag{40}$$

Inflation

In what follows we shall addopt the following leading terms of $f(\phi)$ and $h(\phi)$, namely

$$f(\phi) = 1 + \xi\phi^2, \quad h(\phi) = \bar{\xi}\phi + \bar{\xi}'\phi^3 . \tag{41}$$

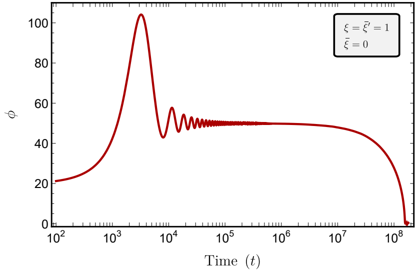
Note that $h(\phi)$ is chosen this way to counteract the parity-odd coupling $h(\phi)\tilde{\mathcal{R}}$. We also replace σ with the canonical field

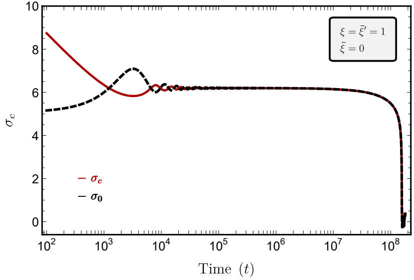
$$\sigma_c = 2\sqrt{6}\int \frac{d\sigma}{\sqrt{1+16\sigma^2}} \Longrightarrow \sigma = \frac{1}{4}\sinh\left(\sqrt{\frac{2}{3}}\,\sigma_c\right) . \tag{42}$$

In an FRW background the equations of motion read

$$\begin{aligned}
&(K_\phi + 3L_\phi\dot{\phi}^2)\ddot{\phi} + 3H(K_\phi + L_\phi\dot{\phi}^2)\dot{\phi} + \dot{\phi}\dot{\sigma}_c\frac{\partial K_\phi}{\partial\sigma_c} + \left(\frac{1}{2}\frac{\partial K_\phi}{\partial\phi} + \frac{3}{4}\frac{\partial L_\phi}{\partial\phi}\dot{\phi}^2\right)\dot{\phi}^2 + \frac{\partial U}{\partial\phi} = 0\\
&\ddot{\sigma}_c + 3H\dot{\sigma}_c - \frac{1}{2}\frac{\partial K_\phi}{\partial\sigma_c}\dot{\phi}^2 + \frac{\partial U}{\partial\sigma_c} = 0\\
&H^2 = \frac{1}{3}\rho\,, \qquad \rho = \frac{1}{2}K_\phi\dot{\phi}^2 + \frac{3}{4}L_\phi\dot{\phi}^4 + \frac{1}{2}\dot{\sigma}_c^2 + U\\
&\dot{H} = -\frac{1}{2}(\rho + p)\,, \qquad p = \frac{1}{2}K_\phi\dot{\phi}^2 + \frac{1}{4}L_\phi\dot{\phi}^4 + \frac{1}{2}\dot{\sigma}_c^2 - U
\end{aligned} \tag{43}$$

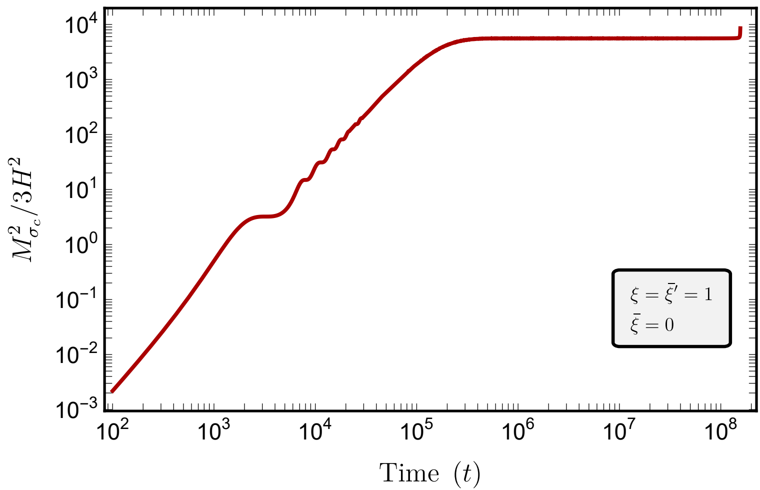
Solving numerically the equations of motion (with $V = \lambda\phi^4/4$) we obtain the plots





showing that very soon the system falls along the minimum line $\sigma_0(\phi)$.

$M^2_{\sigma_c}/3H^2$



Single-Field Inflation

Starting with large initial values for the fields, the system, in a relatively short time, drops into the valey defined by the minimum line $\sigma = \sigma_0(\phi)$ and its further evolution is described by the single field effective Lagrangian $-\frac{1}{2}\overline{K}(\phi)(\nabla\phi)^2 + \frac{1}{4}L(\phi)(\nabla\phi)^4 - U_0(\phi)$, where $\overline{K},\ L$ and U_0 are given in (40). The equations of motion in a FRW background are

$$(\overline{K} + 3L\dot{\phi}^2)\ddot{\phi} + 3H(\overline{K} + L\dot{\phi}^2)\dot{\phi} + \frac{1}{2}\overline{K}'\dot{\phi}^2 + \frac{3}{4}L'\dot{\phi}^4 + U_0' = 0$$

$$H^2 = \frac{1}{3}\rho, \qquad \rho = \frac{1}{2}\overline{K}\dot{\phi}^2 + \frac{3}{4}L\dot{\phi}^4 + U_0 \tag{44}$$

$$\dot{H} = -\frac{1}{2}(\rho + p), \qquad p = \frac{1}{2}\overline{K}\dot{\phi}^2 + \frac{1}{4}L\dot{\phi}^4 - U_0$$

Next, we aim at calculating the inflationary observables, namely, the amplitude of the scalar power spectrum A_s, the spectral index n_s and the tensor-to-scalar ratio r. Note that, due to the presence of the quartic kinetic term, the speed of sound is not a constant but given by

$$c_s^2 = (1 + \dot{\phi}^2 L/\overline{K})/(1 + 3\dot{\phi}^2 L/\overline{K})\,. \tag{45}$$

Nevertheless, the deviations from unity will turn out to be insignificant for the inflationary observables(I.Gialamas, A.Lahanas (2020)).

In order to calculate the observables we consider the Hubble flow equations[1]

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\dot{\epsilon}_1}{\epsilon_1 H}, \quad s_1 = \frac{\dot{c}_s}{c_s H}. \tag{46}$$

Keeping only the first order terms in these equations we can arrive at the following expressions for the tensor and scalar amplitudes at a pivot scale $k_\star = a_\star H_\star / c_s^\star$

$$A_s^\star = \frac{H_\star^2}{8\pi^2 \epsilon_1^\star c_s^\star} \left(1 - 2(D+1)\epsilon_1^\star - D\epsilon_2^\star - (2+D)s_1^\star\right) \qquad (D = 7/19 - \ln 3)$$

$$A_t^\star = \frac{2H_\star^2}{\pi^2} \left(1 - 2(D + 1 - \ln c_s^\star)\epsilon_1^\star\right) \tag{47}$$

From the Planck 2018 data we have $A_s^\star = (2.10 \pm 0.03) \times 10^{-9}$ at $k_\star = 0.05 Mpc^{-1}$. The tensor-to-scalar ratio and the spectral index are

$$r = A_t^\star / A_s^\star = 16\epsilon_1^\star c_s^\star \left(1 + 2\epsilon_1^\star \ln c_s^\star + D\epsilon_2^\star + (2+D)s_1^\star\right)$$

$$n_s = 1 - 2\epsilon_1^\star - \epsilon_2^\star - s_1^\star \tag{48}$$

[1] J.Martin, C.Ringeval, V.Vennim (2013)

The recent release of BICEP/Keck data imposes the bound $r(0.05Mpc^{-1}) < 0.036$ at 95%. The combination of WMAP, Planck and BICEP/Keck data constrains the spectral index as $0.958 < n_s < 0.975$ at 95% for $r = 0.004$.

After the end of inflation, the Universe enters a radiation dominated era through a reheating phase. We shall assume that reheating is instantaneous (i.e. $a_{end} = a_{reh}$). The number of e-folds during inflation can be computed from

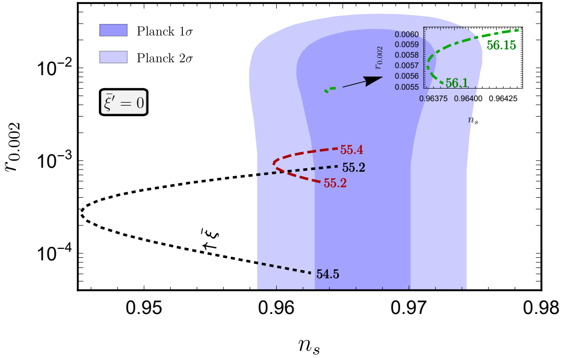
$$N_\star = \ln(a_{end}/a_\star) \quad where \quad k_\star = a_\star H_\star / c_s^\star \tag{49}$$

or

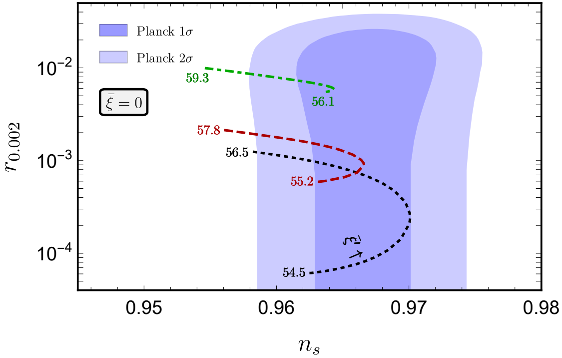
$$N_\star = 66.89 - \ln c_s^\star - ln\left(k_\star / a_0 H_0\right) + \frac{1}{2}\left(\frac{3H_\star^2}{\rho_{end}^{1/3}}\right) - \frac{1}{12}\ln g_{reh}^{(s)}\,, \tag{50}$$

where $a_0,\ H_0$ refer to the present epoch and $g_{reh}^{(s)}$ is the entropy density degrees of freedom (106.75 for the Standard Model) at 1 TeV or higher. This formula can be further reduced to

$$N_{0.05} \approx 55.8 + \frac{1}{4}\ln(r/0.036) - \frac{1}{4}\ln(U_0^{end}/U_0^{0.05})\,. \tag{51}$$



Predictions of the model using pivot scales $0.05\,\mathrm{Mpc}^{-1}$ for n_s and $0.002\,\mathrm{Mpc}^{-1}$ for r. Shaded regions are the allowed parameter regions at 68% and 95% confidence coming from the latest combination of Planck, BICEP/Keck and BAO data. The values of the parameters are $\bar{\xi}' = 0$ and $\gamma = 10^6$, while $\xi = 0.1$ (green dashed-dotted line), $\xi = 1$ (red dashed line) and $\xi = 10$ (black dotted line). The parameter $\bar{\xi}$ varies from 10^{-3} to 10^3 in each curve in a clockwise direction indicated by the arrow. The small numbers at the edges of the curves indicate the number of $e-$folds $N_{0.05}$.



Predictions of the model using pivot scales 0.05 Mpc^{-1} for n_s and 0.002 Mpc^{-1} for r. Shaded regions are the allowed parameter regions at 68% and 95% confidence coming from the latest combination of Planck, BICEP/Keck and BAO data. The values of the parameters are $\bar{\xi} = 0$ and $\gamma = 10^6$, while $\xi = 0.1$ (green dashed-dotted line), $\xi = 1$ (red dashed line) and $\xi = 10$ (black dotted line). The parameter $\bar{\xi}'$ varies from 10^{-3} to 10^3 in each curve in a counterclockwise direction indicated by the arrow. The small numbers at the edges of the curves indicate the number of $e-$folds $N_{0.05}$.

ξ	$\bar{\xi}$	$n_{s0.05}$	$r_{0.05}$	$N_{0.05}$	$r_{0.002}$	$N_{0.002}$
0.1	3	0.9644	6.72×10^{-3}	56.15	6.04×10^{-3}	59.37
1	30	0.9645	1.50×10^{-3}	55.39	1.35×10^{-3}	58.61
10	300	0.9645	9.66×10^{-4}	55.20	8.66×10^{-4}	58.42

Summary

Summarizing, we have considered a general quadratic *Metric-Affine* theory, featuring an extra dynamical degree of freedom, and coupled it non-minimally to a scalar field. We studied inflation in the resulting two-field model and found that it effectively reduces to a single-field model, with a potential of the Palatini-$\mathcal{R}^2$ form with its characteristic inflationary plateau and a modified kinetic term. We find that the inflationary predictions of this model fall within the latest observational bounds for a wide range of parameters. Furthermore, it allows for an increase in the tensor-to-scalar ratio.

I.Gialamas and K.Tamvakis, arXiv:2212.09896(gr-qc)