

Remarks on thermal CFTs and massless Feynman graphs

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- The context and the questions
- The technical work
- Results, some answers and many remarks



Context and questions

- Conformal field theory (CFT) is only . . .

defined on flat space $\eta_{\mu\nu}$, whose conformal isometries can be translated to quantum symmetry generators.

- Basic features of CFTs:

$$Q : \Delta_Q, s. // \langle QQ \rangle \sim \frac{1}{x^{2\Delta_Q}}$$

↳ scaling dimension

$$\langle QQQQ \rangle \sim f(\Delta_Q, g_Q) \rightarrow \text{coupling}$$

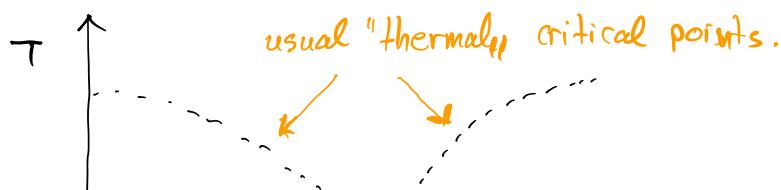
$$Q \times Q' \sim 1 + g_Q [Q + Q \cdot Q + \dots] + \dots \quad [\text{O.P.E.}]$$

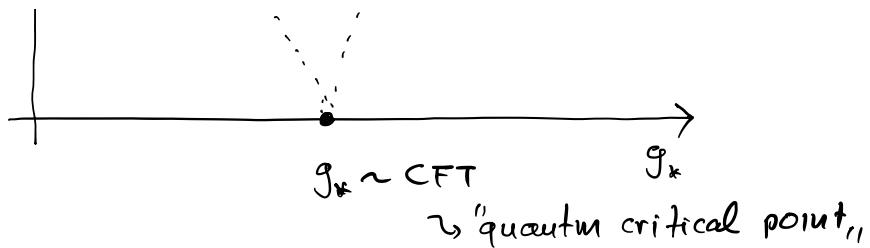
$$\langle Q'Q'Q'Q' \rangle \simeq 1 + g_Q^2 \langle \mathcal{E}_{\Delta_Q}(u, v) \rangle + \dots$$

- conformal block
- $u, v = \text{conformal ratios}$

\Rightarrow knowledge of Δ_Q, g_Q determines "in principle" all n-pt function \rightarrow the CFT.
(e.g. numerical bootstrap...)

- In nature we are interested in critical systems at finite temperature, in non-trivial geometries etc.





QUESTION:

To what extend CFTs determine the usual critical systems at finite temperature / size?

... an example ...

$$\mathbb{R}^2 \text{ is conformally equivalent to } S^1 \times \mathbb{R}.$$

\downarrow

$$ds^2 = dr^2 + r^2 d\theta^2 : r = L e^{s/L} \mapsto ds^2 = e^{2s/L} (dr^2 + L^2 d\theta^2)$$

\downarrow

Weyl factor

→ Side question: What is the relationship between theories on Weyl-related metrics?

}

or

}

If one knows everything for a CFT in $\eta_{\mu\nu}$, how much does she know for the same theory on $\underline{\Omega}^2 G \times \eta_{\mu\nu} ??$

Trivial only in $d=2$

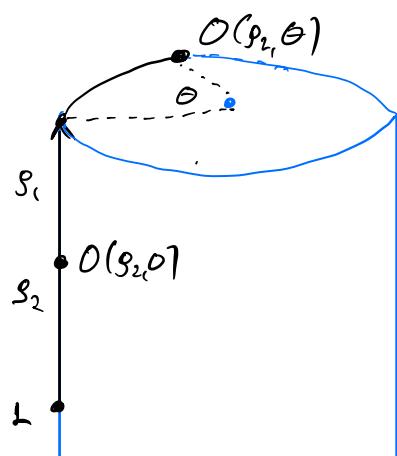
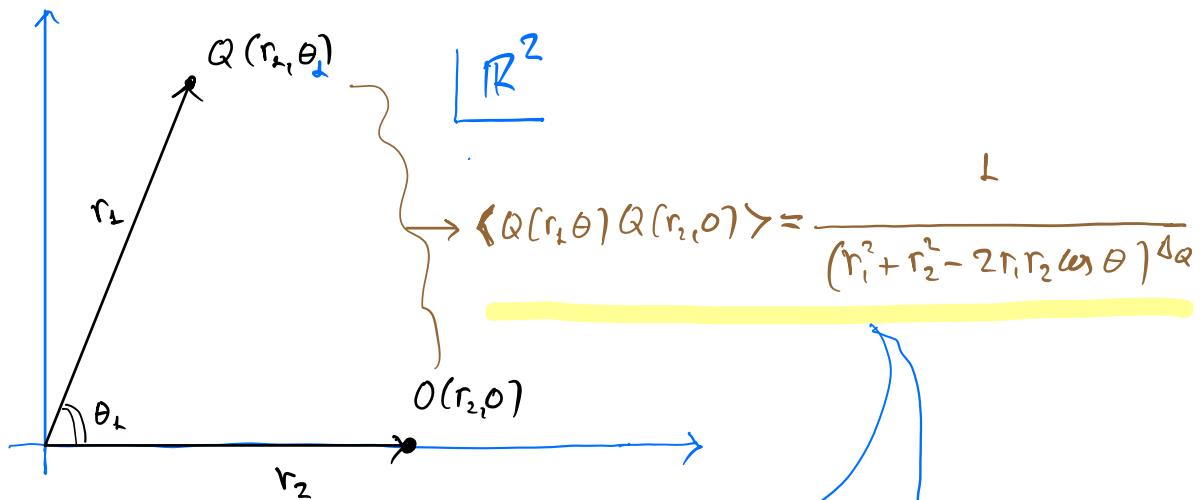
Non-trivial in $d > 2$...

Requires the notion

of Weyl theories | e.g. in flat space, Weyl \mapsto conformal

which go beyond CFTs

but conformal \rightarrow Weyl



$S^1 \times \mathbb{R}$

$$\langle Q(s_1, \theta) Q(s_2, 0) \rangle = \frac{1}{L^{2\Delta_Q}} \frac{1}{(2\cosh \frac{s_1 s_2}{L} - 2\cos \theta)^{\Delta_Q}}$$

... but $S^1 \times \mathbb{R}$ is the thermal geometry \Rightarrow the above is the thermal correlator!

*** The $T=0$ CFT₂ suffices to determine the $T \neq 0$ CFT₂

\Rightarrow What about higher dimensions? \Leftarrow

• n^d . . . -1 n^{d-k} / \sim •

\mathbb{R}^d and $S^d \times \mathbb{R}$ ($d > 1$)

are not related by a Weyl rescaling
(as \mathbb{R}^d and $\mathbb{R} \times S^{d-1}$ are)

Hence: thermal correlators in $d > 2$ depend on extra parameters.

QUESTION: What are they? How are they determined?

QUESTION: Can we learn something about thermalisation from zero-temperature CFT?

Technicalities

- Given the OPE, the thermal 2-pt function of scalars can be written as:

$$\langle \varphi(r, \theta) \varphi(0, 0) \rangle \equiv g(r, \cos\theta) = \sum_{Q_s} \frac{1}{r^{d-2}} r^{\Delta_{Q_s}} a_{Q_s} C_s(\cos\theta)$$

Gegenbauers

* $\Delta_\varphi = v = d/2 - L$ \rightarrow elementary scalar of massless free CFT_d.

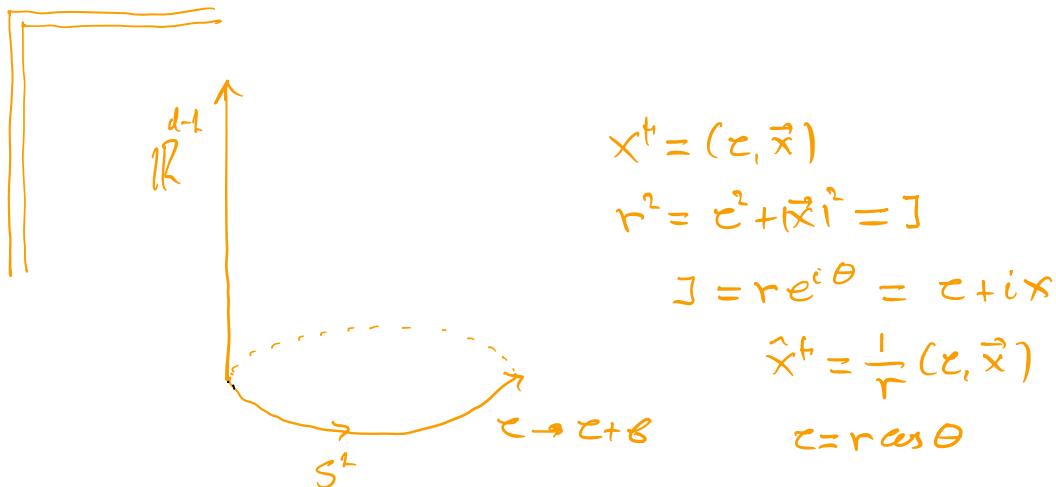
$$* a_{Q_s} = \frac{s!}{2^s (v)_s} \frac{g_{QQ_s} b_{Q_s}}{c_{Q_s}}, \quad L = \beta = 1/T = L$$

$$* \langle Q_s(0) \rangle = b_{Q_s} (\hat{e}_{\mu_1} \cdots \hat{e}_{\mu_s} - \text{traces}) \quad \hat{e} \cdot \hat{e} = L$$

These are the new parameters that are needed to describe the thermal correlators.

They are the non-zero Lpt functions

$\langle Q \rangle \sim \frac{b\alpha}{L^{\Delta_Q}}$, which are allowed in $S^1 \times \mathbb{R}^{d-1}$
(translation inv. is broken).



e.g. $\hat{x}^t \hat{x}^v T_{\mu\nu} = \hat{x}^t \hat{x}^v T_{tt} + \hat{x}^v \hat{x}^t T_{vv}$

$T_{tt} + (d-1)T_{vv} = 0, \quad \hat{x}^t \hat{x}^v = L$

$\Rightarrow \frac{1}{d-1} (\hat{x}^t \hat{x}^v - \frac{1}{d}) T_{tt} \propto (e_2^{d-1} \cos \theta) T_{tt}$

One can do a spectral analysis of the parameters a_{qs} :

$$g(r, \cos \theta) = \sum_s \left\{ \frac{d\Delta}{2\pi i} a(\Delta, s) \right\}_{-\epsilon-i\omega}^{-\epsilon+i\omega} \frac{C_s(\cos \theta)}{r^{2s\alpha-\Delta}}$$

$$\downarrow -\frac{\alpha_{QS}}{\Delta - \Delta_{QS}}$$

Then, using the orthogonality of the Gegenbauers

$$\alpha(\Delta, s) = \frac{1}{N} \int_0^L \frac{dr}{r^{\Delta-2s+2}} \int_{-1}^1 dx (1-x^2)^{-1/2} C_s(x) g(r, x)$$

$x = \cos \theta$

In principle this can be used to do a spectral analysis of the thermal 2pt function and possibly a (kind of) bootstrap...

... here I will content myself with the simplest questions raised by thermal CFTs.

→ How are α_{QS} determined if free CFTs?

$$g(r, \cos \theta) = \frac{1}{r^{d-2}} + [\text{shadows}] + \frac{g_{\phi\phi} e^2}{C_\phi^2} \langle \phi^2 \rangle + \dots$$

At this point we need to know/find the spectrum of operators that appear in

$$\underline{\phi \times \phi = \dots}$$

This consists of higher-spin operators

with $\Delta_c = d-2+s : s=0, 2, 4, \dots$

Not there in free theory [along with their higher-twist generalizations.]

There are no shadow operators for free CFTs.

- The massless free thermal 2-pt function:

$$g(r, \cos\theta) = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{d\vec{p}}{(2\pi)^{d-1}} e^{-i\omega_n c - i\vec{p} \cdot \vec{x}} \frac{1}{\omega_n^2 + \vec{p}^2}$$

$$\omega_n = 2\pi n, \quad n \in \mathbb{Z}$$

One can also show that:

$$g(r, \cos\theta) = \sum_{m=-\infty}^{\infty} \frac{1}{[m + c r^2 + \vec{x}^2]^{d_\alpha}}$$

and recall the Legendre polynomials:

$$\frac{1}{(1 - 2xt + t^2)^{\alpha}} = \sum_{n=0}^{\infty} C_n^\alpha(x) t^n$$

to get:

$$g(r, \cos\theta) = \frac{1}{r^{2d_\alpha}} + \sum_{j=0,2,\dots} (-1)^j \left(\sum_{m \neq 0} \left(\frac{m}{|m|} \right)^j \frac{1}{|m|^{2d_\alpha+1}} \right) C_j^\alpha(n) r^j$$

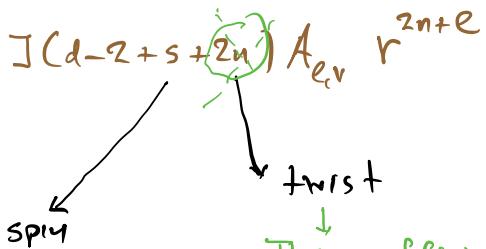
zeta functions

$$\Delta_\alpha = \nu = \frac{d}{2} - 1, \quad n = \cos\theta = \frac{c}{r}$$



the result reads:

$$g(r, \cos\theta) = \frac{1}{r^{d-2}} + \sum_{n=0} \sum_{s=0,2,\dots} J(d-2+s+2n) A_{e,v} r^{2n+s}$$



The coefficients
are zero

- * Gives the contributions of operators with $\Lambda_\phi = d - 2 + s + 2n$

- * The leading contribution comes from $: \Phi^2 :$

- * For $d = \text{odd}$ $\rightarrow J(2k+1)$, $k=0, 1, 2, \dots$
 \rightarrow odd-zeta functions are thought to be transcendental...

- * For $d=3$ the contribution from $: \Phi^2 :$ diverges!
 \Rightarrow the free boson cannot thermalise in $d=3$!

This is closely related to the absence of SSB in $d=2$, or equivalently the breaking of global symmetries, or equivalently the IR divergence of the 2-d boson propagator.

- * For $\ell=2, n=0 \rightarrow$ the $J(d)$ term is the expectation value $\langle T_{cc} \rangle \sim$ free energy density thermal.

- Consider the simplest deformation of the thermal 2-pt function

$$g(r, \cos\theta; m) = \sum_u \int \frac{d^4 \vec{p}}{(2\pi)^{d-1}} e^{-i\omega_u c - i\vec{p} \cdot \vec{x}} \frac{L}{\omega_u^2 + \vec{p}^2 + m_{th}^2}$$

thermal mass

Can such a 2-pt function be conformal?

→ only if $m_{th} \cdot L$ is fixed. How? →

→ compare with the expected spectrum of
the zero temperature non-trivial CFT.

We find:

$$g(r, \cos\theta; m) = \frac{1}{(2\pi)^{d/2}} \sum_{n=-\infty}^{+\infty} \left(\frac{m_{th}}{\sqrt{(r-n)^2 + |\vec{x}|^2}} \right)^{d/2-1} K_{d/2-1} [m_{th} \sqrt{(r-n)^2 + |\vec{x}|^2}]$$

Bessel K

$$= \frac{1}{r^{d-2}} + [\text{shadows}] + \frac{g_{\varphi\varphi\varphi^2}}{C_\varphi^2} \langle \varphi^2 \rangle + \dots$$

... after some algebra we find:

$$\frac{g_{\varphi\varphi\varphi^2}}{C_\varphi^2} \langle \varphi^2 \rangle = \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-2)} \left[\frac{\Gamma(d-\frac{1}{2})}{2\sqrt{\pi}} m_{th}^{d-2} + I_d(z, \bar{z}) \right]$$

$$I_d(z, \bar{z}) = \sum_{n=0}^{\frac{d-3}{2}} \frac{(d-1)^n (d-3-n)!}{(\frac{d-3}{2}-n)!} \frac{z^n \ln^{\frac{n}{2}} |z|}{n!} \left[L_{d-2-n}(z) + L_{d-2-n}(\bar{z}) \right]$$

$$z = e^{-m_{th}}$$

$$L_n(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

polylogarithm.

- * There is also an infinite series of contributions of the form:

$$\frac{1}{r^{d-2}} \cdot r^{2k} m_{th}^{2k} b_{6^k} : k=1, 2, \dots$$

These come from the shadow operators G^k , $k=1, 2, \dots$

$$|\Delta_G + \Delta_{\phi^2} = d \Rightarrow \Delta_G = 2|$$

- * For a given d , a finite # of G^k operators are relevant:

\Rightarrow So, what is the expected spectrum and how should we determine m_{th} ?

\hookrightarrow All (?) nontrivial CFT_d : $d=3, 5, \dots$ do not have the operator ϕ^2 in their spectrum.

For $d=3$:

$$\frac{g_{\phi\phi\phi^2}}{C_{\phi^2}} \langle \phi^2 \rangle = 0 \Leftrightarrow |\frac{1}{2} m_{th} b + \ln(1 - e^{-m_{th} b}) = 0|$$

— * — $(1, 1 + \sqrt{5})$

$$\Rightarrow m_{th} = \frac{1}{\beta} \ln \left(\frac{e^{\beta h}}{2} \right)$$

$$\frac{a+b}{b} = \frac{b}{a} \Rightarrow \frac{b}{a} = \frac{1+\sqrt{5}}{2}$$

golden mean.

For $d=5$:

$$\frac{1}{6}(m_{th} \cdot 6)^3 + Li_3(e^{-m_{th} \cdot 6}) + (m_{th} \cdot 6) Li_2(e^{-m_{th} \cdot 6}) = 0$$

$$\Rightarrow m_{th}^* \approx 1.17 \pm i 1.19 \dots$$

* General behaviour:

For $d=3, 7, 11 \dots$ there is a unique real solution $\underline{m_{th}^*}$.

For $d=5, 9, 13, \dots$ there are pairs of complex conjugate solutions.

* Physical interpretation.

Critical points in $d=2r+1$ dimensions, $r=1, 2, \dots$

(i.e. the Ising model in $d=3 \dots$)

are accessible via the ε -expansion from

$D=d+1$ - dimensions

i.e. $D=4 : \lambda \varphi^4 \xrightarrow{4-\varepsilon} \text{IR Wilson-Fisher}$

fixed point
 $\epsilon \rightarrow 0$
 * Ising critical point.
 $d=3$

The study of $\lambda\phi^{2n}$ deformations in D-dimensions is done using a Hubbard-Stratonovich transformation:

$$I = -\frac{1}{2} \int \phi (-\partial^2) \phi + G \phi^2 + g_* G^{\frac{D}{2}}$$

gives the marginal deformation $\phi^{\frac{2D}{D-2}}$

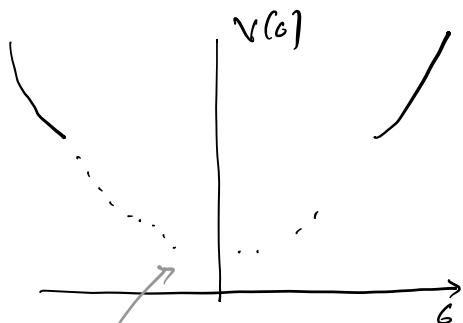
Integrate ϕ^2

$$\boxed{\text{Tr}_0 \ln (-\partial^2 + G) + g_* G^{\frac{D}{2}}}$$

For $D=4, 8, 12$

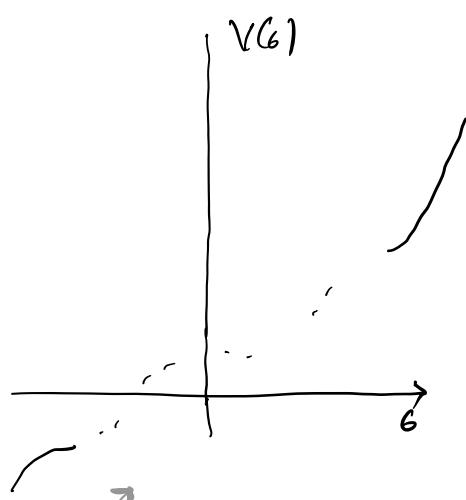
this gives: $G^{\frac{D}{2}} \ln G^2$

which is positive and dominates for $G \rightarrow \infty$



There exists a global minimum.

For $D=6, 10, 14 \dots$
the term $G^{\frac{D}{2}}$ gives an unbounded potential



No global minimum
~ metastable states.

[The above was described with A. Stergiou in 2018]
including fermions



- * There exist more traditional methods to study thermal averages in QFTs. After all:

$$\langle \hat{Q} \rangle_e = \frac{1}{Z} \text{Tr} (\hat{Q} e^{-\beta \hat{H}})$$

\Rightarrow the fancy polylog results found above e.g. for $: \bar{\phi}^2 :$
should be also accessible from thermal partition function calculations.

- * Moreover, there are scaling arguments that determine the form of the thermal free energy in CFTs.

$$Z(\beta) = e^{-\beta F(\epsilon)} = e^{-\beta \cdot \frac{Vd-1}{\epsilon^d} C_d(\epsilon)}$$

↑
spatial volume ↓
dimensionless parameter.

- * The quantity $C_d(\epsilon)$ has been considered many times as a thermal C-function. It does not

quite work for $d > 2$.

- * For massless-free theories it is proportional to

$$C_d(t) \propto J(d)$$

The question is how it changes along the RG flow.

- * The massive deformation is the simplest one and we expect:

$$Z(\ell) \rightarrow Z(\ell; m) = e^{-\frac{V_{d-1}}{\ell^{d-1}} C_d(m, \ell)}$$

$$S[m] = S_0 + \frac{1}{2} m^2 \int \varphi^2$$

$$\cancel{V_{d-1}} \langle \varphi^2 \rangle = - \frac{1}{\ell} \frac{\partial}{\partial m} \ln Z(\ell; m)$$

\downarrow
uniform
thermal average

$$\Rightarrow \langle \varphi^2 \rangle = \frac{1}{\ell^d} \frac{\partial}{\partial m^2} C_d(m, \ell) = \frac{b \ell^2}{\ell^{d-2}}$$

This calculation should give the same results as above.



Before we proceed, an observation.

Recall that:

$$\langle \varphi^2 \rangle_{d=3} = \frac{1}{2} m \ell + \ln(1 - e^{-m \ell}) \quad \xrightarrow{\text{familiar...}}$$

Consider 1d harmonic oscillator.

$$\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \omega^2 \hat{x}^2 \mapsto \hat{H} = \omega (\mathbb{I}_2 + \hat{N}), \hat{N} = \hat{a}^\dagger \hat{a}$$

$$\hbar = m = 1$$

$$Z = \text{Tr } e^{-\beta \hat{H}} = \frac{e^{-\frac{\beta \omega}{2}}}{1 - e^{-\beta \omega}} = e^{-\beta \cdot \frac{1}{\hbar} C_1(\hbar \omega)}$$

$$\Rightarrow F = -\frac{1}{\hbar} \ln Z = \frac{\omega}{2} + \frac{1}{\hbar} \ln (1 - e^{-\beta \omega})$$

$$\Rightarrow \frac{\partial}{\partial \omega} C_3 \sim \langle \dot{\phi}^2 \rangle_3 \sim C_1$$

We could interpret the dropping out of the thermal spectrum of $\langle \dot{\phi}^2 \rangle$ as the signal that the 1-d h.o. is fully thermalised.

$$F = E - TS = 0 \Rightarrow \underline{E = TS}.$$

- * The above observation encourages us to continue in the direction of thermal partition functions..

Thermal partition function of free massive scalars.

$$Z_d(\beta; m) = \frac{1}{Z_d(0,0)} \int D\phi e^{-\int_0^\beta d\tau \int d^d x \frac{1}{2} \phi (-\partial_\tau^2 - \vec{\nabla}^2 + m^2) \phi}$$

$$= e^{-\frac{\sqrt{d-1}}{C^{d-1}} C_d(\beta m)}$$

$$C_d(0) = \frac{\Gamma(d/2)}{\pi^{d/2}} J(d)$$

$$C_d(\ell m) = -\frac{1}{2} k_d (-1)^d (\ell m)^d - \frac{S_{d-1}}{(2\pi)^{d-1}} \int_0^{\ell m} \frac{dw}{w} \ell m w [\ell m^2 - w^2]^{\frac{d-3}{2}} \ell e(l-w)$$

$$k_d = \frac{\pi S_d}{d(2\pi)^d} \frac{1}{\sin(\frac{\pi d}{2})} \quad , \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

If satisfies the recursive relations:

$$\frac{d}{dm} C_d(\ell m) = -\frac{\ell^2}{2\pi} m C_{d-2}(\ell m) \quad , \quad d=3, 5, 7, \dots$$

The boundary condition is:

$$\begin{aligned} \frac{d}{dm} C_1(\ell m) \equiv C_{-1}(\ell m) &= m \langle \hat{x}^2 \rangle = m \frac{1}{2m} [l + 2\langle \hat{n} \rangle] \\ &= m \cdot \frac{1}{2m} [l + 2 \frac{e^{-\ell m}}{1-e^{-\ell m}}] \end{aligned}$$

Recall virial theorem:

$$\frac{1}{2} \omega^2 \langle \hat{x}^2 \rangle = \frac{1}{2} \langle \hat{H} \rangle$$

→ For each d , the question is whether $C_d(\ell m)$ has a global minimum such that it defines a thermal CFT $_d$.

→ The recursive relations point towards a resummation

$$Z = Z_1 Z_3 Z_5 \dots$$

$$= e^{-C_1 - \frac{V_2}{\epsilon^2} C_3 - \left(\frac{V_2}{\epsilon^2}\right)^2 C_5 - \dots}$$

$$\lambda = \frac{V_2}{\epsilon^2}$$

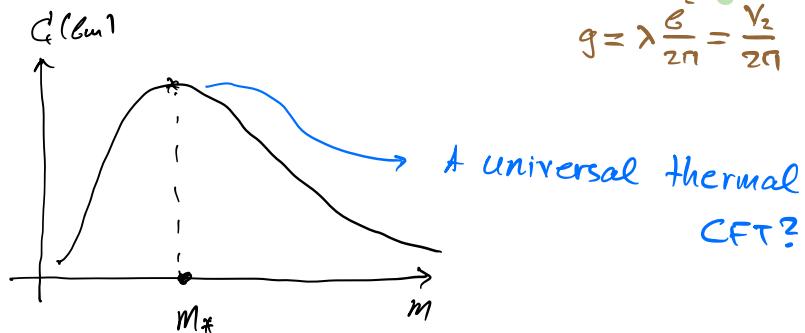
$$= e^{-C(Gm)}$$

$$C(Gm) = C_1 + \lambda C_3 + \lambda^2 C_5 + \dots$$

$$\Rightarrow \frac{d}{dm} G(m) + \lambda \frac{\epsilon^2}{2n} m G(m) = C_1(m)$$

$$\Rightarrow G(m) = e^{-\frac{V_2}{2} g m^2} \int_0^m e^{\frac{V_2}{2} g m'^2} m' \langle \hat{x}^2 \rangle_{m'} dm'$$

$$g = \lambda \frac{\epsilon^2}{2n} = \frac{V_2}{2n}$$



$$\rightarrow G(m) = \sum_{n=0}^{\infty} \lambda^n G_n(Gm)$$

is a generating function for d-dimensional thermal free energies.

$$G_n \sim \int_{m_{n-1}}^{m_n} \int_{m_{n-2}}^{m_{n-1}} \dots \int_{m_0}^{m_1} \langle \hat{x}^2 \rangle dm_0 \dots dm_{n-1}$$

\swarrow
Dyson series

Recap:

~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~ ~

- We want to study thermal 1-pt functions
in CFTs: $\langle \phi \phi \rangle \sim \frac{1}{r^{2\Delta_\phi}} + \dots + \langle \phi^2 \rangle + \dots$
- We find that the 1-pt functions are given by intriguing polylog formulae:
- We associate the thermal 1-pt functions with derivatives of the thermal partition function
 $\langle \phi^2 \rangle \sim \frac{\partial}{\partial \mu^2} \ln Z$
- We find some intriguing recursive relations for
 $\langle \phi^2 \rangle_d \leftrightarrow \langle \phi^2 \rangle_{d-2} \sim C_d \rightarrow C_{d-2}, \dots$
- We suggest a resummation/Dyson series towards a universal CFT.

● → Time to dwell deeper...

Imaginary chemical potential:

$$Z_c(\beta, Q) = \text{Tr} [\delta(Q - \hat{Q}) e^{-\beta \hat{H}}]$$

$$= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta Q} \underbrace{\text{Tr} [e^{-\beta (\hat{H} - i\mu \hat{Q})}]}_{\downarrow} \boxed{\mu = \frac{\theta}{\beta}}$$

$$Z_{g.c.}(\beta, \mu)$$

• $\hat{Q} \sim \text{global } U(1)$

- if $Q = \text{integer}$ (i.e. QCD)

$\Rightarrow \theta \rightarrow \theta + \frac{2\pi}{N}$ periodic $\Rightarrow \mathbb{Z}_N$ vacua.

- if $\theta \not\rightarrow \theta + \frac{2\pi}{N}$ \rightarrow deconfining phase.

→ Main example: massive free complex scalar with imaginary chemical potential $\simeq \simeq$ in the presence of A₀-component of a real Euclidean gauge field (i.e. Chern-Simons in $d=3$).

$$S_E(\ell; m, \mu) = \int_0^\ell d\zeta \int d^d \bar{x} \left(|(\partial_\zeta - i\mu)\varphi|^2 + |\bar{\partial}\varphi|^2 + m^2 |\varphi|^2 \right)$$

$$Z_{g.c.} = \frac{1}{Z(0; 0, 0)} \int D\varphi D\bar{\varphi} e^{-S_E} = e^{-\frac{\sqrt{d-1}}{\ell^{d-1}} C_d(\ell m, \ell \mu)}$$

Use the variables:

$$z = e^{-\ell m - i\ell \mu}, \quad \bar{z} = e^{\ell m + i\ell \mu} : C_d \rightarrow \boxed{C_d(z, \bar{z})}$$

$$C_d(z, \bar{z}) = -k_d \ln|z| - \frac{S_{d-1}}{(2\pi)^{d-1}} [i_d(z, \bar{z}) + \bar{i}_d(\bar{z}, z)]$$

$$i_d(z, \bar{z}) = \int_0^z \frac{dw}{w} \left(\ln w - \frac{1}{2} \ln \frac{z}{\bar{z}} \right) \left[\left(\ln w - \frac{1}{2} \ln \frac{z}{\bar{z}} \right)^2 - \ln^2 |z| \right]^{\frac{d-3}{2}} \ln(1-w)$$

$$i_d(z) = -\ln(1-z), \quad \bar{i}_d(\bar{z}) = -\ln(1-\bar{z})$$

Result for $d = 2k+L$, $k=1, 2, \dots$

$$i_d + \bar{i}_d = - \frac{\Gamma\left(\frac{d+1}{2}\right)}{d-1} I_{d+2}(z, \bar{z})$$

$$I_d(z, \bar{z}) = \sum_{n=0}^{\frac{d-3}{2}} \frac{(-1)^n (d-3-n)! 2^n}{\left(\frac{d-3}{2}-n\right)!} \frac{e^{i\mu|z|}}{n!} [L_{d-2-n}(z) + L_{d-2-n}(\bar{z})]$$

Thermal 1-pt functions of the operators

$$\hat{O} = |\varphi^2 \alpha|, \quad \hat{Q} = i \bar{\varphi} \overleftrightarrow{\partial}_t \varphi$$

are moments of the partition function.

$$\frac{\partial}{\partial m^2} = \frac{e^2}{2\mu|z|} (z\partial_z + \bar{z}\partial_{\bar{z}}) = e^2 \hat{D}$$

$$\frac{\partial}{\partial \mu} = -ie(z\partial_z - \bar{z}\partial_{\bar{z}}) = -ie \hat{L}$$

We find:

$$\langle O \rangle_d = \frac{1}{e^{d-2}} \hat{D} \cdot C_d(z, \bar{z})$$

$$\langle Q \rangle_d = \frac{1}{e^{d-1}} \hat{L} \cdot C_d(z, \bar{z})$$

For $d=L$ we have a twisted pair of harmonic oscillators.

$$\hat{H} = \sum_{i=1}^2 \frac{1}{2} \hat{p}_i^2 + \frac{1}{2} m^2 \hat{x}_i^2$$

$$\hat{Q} = \hat{Q}_1 - \hat{Q}_2 = \hat{a}_1^+ \hat{a}_1 - \hat{a}_2^+ \hat{a}_2$$

$$\hat{O} = \frac{1}{2} (\hat{x}_1^2 + \hat{x}_2^2) = \frac{1}{2} (1 + \hat{a}_1^+ \hat{a}_1 + \hat{a}_2^+ \hat{a}_2)$$

We take the Hilbert space as $\{|n_1\rangle \otimes |n_2\rangle\}$

and the partition function is

$$\begin{aligned} Z_{g.c.} &= \text{Tr } e^{-\beta(\hat{H}_0 + m^2 \hat{O}) + i\beta \mu \hat{Q}} \\ &= \sum_{n_1, n_2} e^{-\beta(E_{n_1} + E_{n_2}) + i\beta \mu (E_{n_1} - E_{n_2})} \\ &\equiv Z_1(\beta - i\beta \mu) \overline{Z_2(\beta - i\beta \mu)} \\ &= e^{-C_L(z, \bar{z})} \end{aligned}$$

$$C_L(z, \bar{z}) = -\ln|z| + \ln(1-z) + \ln(1-\bar{z})$$

$$\langle \hat{O} \rangle = -\frac{\beta}{2\ln|z|} \left[1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} \right]$$

$$\langle \hat{Q} \rangle = \frac{\bar{z}}{1-\bar{z}} - \frac{z}{1-z} = \langle \hat{N}_1 \rangle - \langle \hat{N}_2 \rangle$$

Recursive relations:

$$\hat{D} \cdot C_L(z, \bar{z}) = -\frac{1}{4\pi} C_{d-2}, \quad C_d = -\frac{4\pi}{\beta} \langle \hat{x}^2 \rangle$$

$$\langle \hat{Q}^2 \rangle_{d+2} = -\frac{1}{4\pi} \frac{1}{6^d} C_d(z, \bar{z})$$

$$\langle \hat{Q} \rangle_d = -4\pi 6^2 D \cdot \langle \hat{Q} \rangle_{d+2}$$

==== The most intriguing result =====

Having a complex scalar gives more structure
in the OPE and hence in the thermal 2pt function.

$$\langle \bar{\phi}(r, \omega, \theta), \phi(0, 0) \rangle = \frac{1}{r^{d-2}} + [\text{shadows}] + \frac{g_{\phi\phi\phi}}{c_\phi} |\phi|^2$$

$$z \leftarrow +r \cos \theta \quad \frac{g_{\phi\phi\phi}}{c_\phi} \hat{Q} + \dots$$

spin-l current

We find:

$$\frac{g_{\phi\phi\phi}}{c_\phi} \langle \phi \rangle = -\frac{1}{2} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-2)} H_d(z, \bar{z})$$

$$H_d(z, \bar{z}) = \sum_{n=0}^{\frac{d-1}{2}} \frac{(-1)^n (d-1-n)! 2^n}{(\frac{d-1}{2}-n)!} \frac{L_{d-1-2n}^n(z)}{n!} [L_{d-1-2n}(z) - L_{d-1-2n}(\bar{z})]$$

Where did we see this ??

We can verify that:

$$\partial_z \partial_{\bar{z}} C_d(z, \bar{z}) = -\frac{d-1}{8\pi} \frac{1}{|z|^2} C_{d-2}(z, \bar{z})$$

Setting: $2L = d-1$: $L=1, 2, 3, \dots$

We can show that:

$$\partial_z \partial_{\bar{z}} \langle Q \rangle_L = \frac{1}{4\pi} \left[\frac{z}{L-2} - \frac{\bar{z}}{L-2} \right] \frac{1}{|z|^2}$$

$$\partial_z \partial_{\bar{z}} \langle Q \rangle_{n+1} = -\frac{n}{4\pi} \frac{1}{|z|^2} \langle Q \rangle_n.$$

$n=2, 3, \dots$

These are the famous equations satisfied by L-loop ladder integrals and studied by F. Brown, that are solved by single-valued polylogs.

$$I^{(4)} = \int \frac{dx_1 dx_2 dx_3 dx_4}{x_1 x_2} = \frac{1}{x_{12}^2}$$

→ "Magic" identities $\Pi_2 \frac{1}{x_{12}^2} = -4\pi \delta(x_{12})$

→ Conformal invariance:

$$x_{3u}^2 I^{(L)}(x_1, \dots, x_L) = \frac{1}{x_{12}^2} g^{(L)}(v, u)$$

\downarrow
 $x_3 \rightarrow \infty$

$$v = \frac{x_{12}^2 x_{3u}^2}{x_{1u}^2 x_{23}^2} \rightarrow \frac{x_{12}^2}{x_{1u}^2}$$

$$u = \frac{x_{12}^2 x_{3u}^2}{x_{13}^2 x_{2u}^2} \rightarrow \frac{x_{12}^2}{x_{2u}^2}$$

$$I^{(L)}(x_1, x_2, x_u) = \frac{1}{x_{12}^2} g^{(L)}(v, u)$$

$$\Pi_2 I^{(L)}(x_1, x_2, x_u) = -\frac{4}{x_{12}^2} I^{(L-1)}(x_1, x_2, x_u)$$

Comparing this with the action of the Laplacian
acting on u, v .

$$uv \hat{\Delta}_2 g^{(L)}(u, v) = -g^{(L-1)}(u, v)$$

$$\hat{\Delta}_2 = 2(\partial_u + \partial_v) + u \partial_u^2 + v \partial_v^2 - (1-u-v) \partial_u \partial_v$$

change variables:

$$u = \frac{1-z^2}{(1-\bar{z})(1-\bar{\bar{z}})}, \quad v = \frac{1}{(1-z)(1-\bar{z})}$$

$$\Rightarrow \partial_z \partial_{\bar{z}} \left[\frac{z-\bar{z}}{(1-z)(1-\bar{z})} g^{(L)}(z, \bar{z}) \right] = -\frac{1}{|z|^2} \frac{z-\bar{z}}{(1-z)(1-\bar{z})} g^{(L-1)}(z, \bar{z})$$



Results, answers and remarks

- We have argued that thermal CFT 2-pt functions are determined by the $T=0$ OPE, i.e. by matching the corresponding operator spectra.



the matching is based on some non-trivial algebraic equations (gap eqs) involving polylogs.

The solutions carry the expected structure of conformal manifolds in d -even dimensions.

- The thermal 1-pt functions of global conserved charges (when they exist), are given by expressions that are in $L \leftrightarrow L$ correspondence with massless Feynman graphs in four-dimensions as:

$$\langle Q \rangle_d \longleftrightarrow \frac{z - \bar{z}}{(1-z)(1-\bar{z})} g^{(L)}(z, \bar{z})$$

$$z = e^{-\theta u - i \theta p} \quad \longleftrightarrow \quad u = \frac{|z|^2}{(1-z)(1-\bar{z})}$$

$\dots \dots$

$$\bar{z} = e^{\text{outward}} \quad \longleftrightarrow \quad v = \frac{i}{(1-z)(1-\bar{z})}$$

$$d \quad \longleftrightarrow \quad L$$

$$2L = d + L$$

↓

This correspondence suggests that it may make sense to resum over d -dimensions \leftrightarrow resum over loops.
 \Rightarrow a primitive result is encouraging.

It appears that the system of two charged harmonic oscillators (\sim related to thermofield double) is the generator of the polylogarithmic expressions for the thermal 1-pt functions \rightarrow through iterative integrations \longleftrightarrow Dyson series evolution.

The picture in $d=3$

$$\langle \bar{\varphi} \varphi \rangle_3 = \frac{1}{r} + \dots + \langle \bar{\varphi}^2 \rangle_3 + r \cos\theta \langle Q \rangle_3 + \dots$$

$$Z_{qc}^{(3)} = e^{-\frac{V_2}{6} C_3(z, \bar{z})}$$

$$C_3(z, \bar{z}) \approx \frac{1}{6\pi} \ln|z| + \frac{1}{2\pi} [2 \operatorname{Re} \operatorname{Li}_3(z) - 2 \operatorname{Re} (\ln|z| \operatorname{Li}_2(z))]$$

↷ $\langle \bar{\varphi}^2 \rangle_3 \approx C_2(z, \bar{z}) \approx -\ln|z| + \ln(1-z) + \ln(1-\bar{z})$

$$\langle Q \rangle_3 = \operatorname{Im} \text{Li}_2(z) + \ell \operatorname{Re}|z| \operatorname{Arg}(1-z) \leftarrow \text{Bloch-Wigner function!}$$

This diverges for real $|z|=1$, but

it can vanish for $|z|=1 \rightarrow z^* = e^{i\pi/3}, e^{i2\pi/3}$

For these values, the Bloch-Wigner function reaches its maximum (max volume of ideal tetrahedra) on the unit circle!

{ The same story for all d ! ? }

Feynman graphs

There is a lot of work trying to expand multi-loop Feynman graphs into suitable basis of functions i.e. multiple polylogs.

Above we have observed that:

$$\langle \hat{Q} \rangle_d \approx \sum_i (\text{Li}_{i_1} - \text{Li}_{i_1}) \xrightarrow{\quad} \begin{array}{c} \text{Feynman graph} \\ 2L=d+1 \end{array}$$

so, what is the graph
of

$$\langle :Q^2: \rangle \approx \sum_i \text{Li}_{i_1} + \text{Li}_{i_2} \xrightarrow{\quad} \{??\}$$

In other words, can one ask: given
a polylog expression \rightarrow what are the graphs?