Classical and quantum aspects of a constrained FT

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Based mainly on (recent) work:

- with G. Georgiou, Nucl.Phys. **B987** (2023) 116096, arXiv:2208.01072 [hep-th]
- with G. Georgiou, A. Loukopoulos and K. Siampos, in progress

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General motivation I: Constrained systems

- In a classical FT we may have to impose constraints in the phase space.
- Dealing with constraints at the quantum level may be problematic when quantizing.
- Even if the original theory is a "good one", e.g. CFT, unitary, renormalizable etc, not clear these properties will be retained after constraints are imposed.

General motivation II: σ-models and integrable systems

- σ-models originated in particle physics [Gell-Mann & Levy 1960] and have a variety of applications: From string theory to condensed matter and statistical physics.

- A natural extension of CFT's but perhaps more interesting as QFT's since they have: non-vanishing beta-functions, operators aquiring anomalous dimensions. etc.. Also, their degrees of freedom reduce in the RG flow from the UV to the IR.

- A subclass of σ-models are integrable, i.e. in a sense, they have infinitely many independent conserved charges.

- Notable properties:
  - Factorizable S-matrix (all from two-particle scattering).
  - No-particle production, i.e. $a \not\to a + b$, neither particle transmutation, i.e. $a + b \not\to c + d$.

General framework: Integrability under constraints

- Integrable models seem the best arena for studying the fate of a QFT after a constraint is imposed.
- Issues to be addressed:
  - Is the system still integrable, classically and quantum mechanically?
  - Is it renormalizable? If yes, do the beta-functions resemble in any way the original ones? Do we see a reduction of d.o.f.?
- General remarks on the S-matrix:
  - Traditionally constructed for solitonic states; not derived from a Lagrangian, e.g. [Hasenfratz-Maggiore-Niedermayer 90].
  - Expanding the action of a generic integrable theory around the trivial vacuum the spectrum contains massless excitations.
  - Very effective for calculating β-functions and anomalous dimensions of operators [Georgiou-KS 19],
  - Connection to integrability is lost: The S-matrix obtained allows for particle production already at tree-level [Nappi 79] and is not factorizable [Hoare-Levine-Tseytlin 18].

The idea and outline

- Choose an integrable σ-model.
- Try to find a non-trivial vacuum so that elementary excitations become massive.
- Impose a classical constraint and compute the resulting action.
- Compute the scattering matrix at tree level and demonstrate no-particle production/transmutation.
- Compute the beta-function and see if one may infer a reduction of the d.o.f. from it.
- Concluding remarks.

The model, the pp-wave and post pp-wave correction

The general σ-model action is

$$S = \frac{1}{2\pi}\int d^2\sigma (G_{\mu\nu} + B_{\mu\nu})\partial_+ x^\mu \partial_- x^\nu\,, \quad \sigma^\pm = \tau \pm \sigma\ .$$

The specific model and its properties

The background fields are [KS 13]

- The metric

$$ds^2 = 2k\left(\frac{1+\lambda}{1-\lambda}d\alpha^2 + \frac{1-\lambda^2}{\Delta(\alpha)}\sin^2\alpha\left(d\beta^2 + \sin^2\beta d\gamma^2\right)\right)\ .$$

- The antisymmetric tensor

$$B = 2k\left(-\alpha + \frac{(1-\lambda)^2}{\Delta(\alpha)}\sin\alpha\cos\alpha\right)\sin\beta\, d\beta\wedge d\gamma\,,$$

  where

$$\Delta(\alpha) = (1-\lambda)^2\cos^2\alpha + (1+\lambda)^2\sin^2\alpha\,.$$

- The physical range of the parameter λ is $0 \leqslant \lambda < 1$.
- Geometrically, it describes a deformation of S^3 (for $\lambda = 0$).
- For $\lambda = 0$ it describes a CFT WZW model.
- Integrable [KS 13]: The eqs of motion can put in the Lax form. They exist infinitely many independent conserved charges.
- The trivial vacuum of the theory is at $\alpha = 0$. Expanding around it results in massless excitations.
- For our purposes need to do extras...

The pp-wave and post pp-wave corrections [Georgiou-KS 22]

We add time t and follow Penrose's procedure for a plane wave.

- A null geodesic is

$$\alpha = \beta = \frac{\pi}{2} , \qquad t = \gamma .$$

- Consider the change of variables $(t, \alpha, \beta, \gamma) \to (x^+, x^-, x_1, x_2)$

$$t = \frac{1}{2}\sqrt{\frac{1+\lambda}{1-\lambda}}\left(x^+ - \frac{x^-}{k}\right) , \qquad \gamma = \frac{1}{2}\sqrt{\frac{1+\lambda}{1-\lambda}}\left(x^+ + \frac{x^-}{k}\right) ,$$
$$\alpha = \frac{\pi}{2} + \sqrt{\frac{1-\lambda}{2k(1+\lambda)}}\, x_1 , \qquad \beta = \frac{\pi}{2} + \sqrt{\frac{1+\lambda}{2k(1-\lambda)}}\, x_2 .$$

- In the limit $k \gg 1$, the Lagrangian has the expansion

$$\mathcal{L} = \mathcal{L}^{(0)} + \frac{1}{k}\mathcal{L}^{(1)} + \mathcal{O}(1/k^2) ,$$

- $\mathcal{L}^{(0)}$ has a metric and an antisymmetric tensor (omitted below)

$$ds^{(0)2} = 2dx^+dx^- + dx_1^2 + dx_2^2 - \frac{1}{4}\left(\left(\frac{1-\lambda}{1+\lambda}\right)^3 x_1^2 + \frac{1+\lambda}{1-\lambda}x_2^2\right)(dx^+)^2 \ .$$

  - Represents a plane wave in its Brinkmann form.
  - A deformation of the [Nappi-Witten 92] CFT (for $\lambda = 0$)

- Interactions are encoded in the $\mathcal{O}(1/k)$ terms with

$$ds^{(1)2} = -\frac{1}{2}\left(\left(\frac{1-\lambda}{1+\lambda}\right)^3 x_1^2 + \frac{1+\lambda}{1-\lambda}x_2^2\right)dx^+dx^- - \frac{1}{2}\left(\frac{1-\lambda}{1+\lambda}\right)^3 x_1^2\, dx_2^2$$
$$+\frac{1}{24}\left[\frac{(1-\lambda)^4(1-10\lambda+\lambda^2)}{(1+\lambda)^6}x_1^4 + \left(\frac{1+\lambda}{1-\lambda}\right)^2 x_2^4 + 3\left(\frac{1-\lambda}{1+\lambda}\right)^2 x_1^2 x_2^2\right](dx^+)^2$$

  and an antisymmetric tensor (omitted).
  Interactions quartic in the fields.

They are still massless excitations.... associated with $x^{\pm}$.

Imposing the constraint [Georgiou-KS 22]

- We impose the constraints (Virasoro) following from varying the σ-model w.r.t world-sheet metric

- Since the background is curved the world-sheet metric has to be chosen to be consistent with choosing the light cone gauge

$$x^+ = \tau \ .$$

as the eq. of motion with respect to x^-.

- Using the above the light-cone action is

$$\mathcal{L}_{\text{l.c.}} = \mathcal{L}_{\text{l.c.}}^{(0)} + \frac{1}{k}\mathcal{L}_{\text{l.c.}}^{(1)} + \mathcal{O}(1/k^2) \ .$$

- The free part is

$$\mathcal{L}^{(0)}_{\text{l.c.}} = \frac{1}{2}\Big(\dot{x}_1^2 + \dot{x}_2^2 - x_1'^2 - x_2'^2\Big) + \frac{1}{2}(m_1^2 x_1^2 + m_2^2 x_2^2) + g x_1 x_2' \ ,$$

where

$$m_1^2 = \frac{1}{4}\Big(\frac{1-\lambda}{1+\lambda}\Big)^3 , \quad m_2^2 = \frac{1}{4}\frac{1+\lambda}{1-\lambda} , \quad g = -\frac{1+\lambda^2}{\sqrt{(1-\lambda)(1+\lambda)^3}} \ .$$

- We have obtained the desired non-diagonal mass matrix.

- The interacting part $\mathcal{L}^{(1)}_{\text{l.c.}}$ contains quartic in the x's terms with 0 to 3 derivatives w.r.t. τ and σ.

- The light cone action is not manifestly Lorentz invariant since $x^+ = \tau$ singles out the time coordinate.

Computing the S-matrix

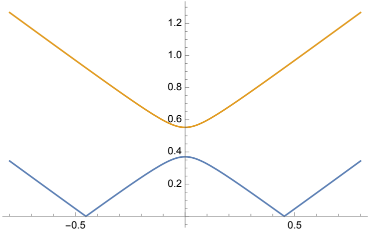
The spectrum

- We pass to momentum space and diagonalize the mass matrix.
- The dispersion relations are ($p^\alpha = (E, p)$)

$$Y_1 : \quad E = \sqrt{p^2 + \frac{m_1^2 + m_2^2}{2} - \sqrt{(m_2^2 - m_1^2)^2/4 + g^2 p^2}} \ ,$$

$$Y_2 : \quad E = \sqrt{p^2 + \frac{m_1^2 + m_2^2}{2} + \sqrt{(m_2^2 - m_1^2)^2/4 + g^2 p^2}} \ ,$$

- Dispersion curves for particle 1 (lower) and particle 2 (upper)



No particle transmutation or production - tree level

The interaction in momentum space is

$$S^{(1)}_{\text{l.c.}} = \int \frac{d^2 p_1 \cdots d^2 p_4}{(2\pi)^2}\, \delta^{(2)}(p_1 + \cdots + p_4) \Big(\mathcal{J}_1\, X_1(p_1) X_1(p_2) X_1(p_3) X_1(p_4) + \dots \Big) \ ,$$

where the five couplings $\mathcal{J}_a(\lambda, p_i)$, are certain functions of λ and of the spatial momenta p_i, $i = 1, 2, 3, 4$ of the particles.

- Particle transmutation (as well as with reverse arrows)

$$1 + 1 \to 1 + 2\,, \quad 1 + 1 \to 2 + 2\,, \quad 2 + 2 \to 2 + 1\,,$$

- Particle creation (or destruction by reversing the arrows)

$$1 \to 1 + 1 + 1\ , \quad 2 \to 2 + 1 + 1\,, \quad 2 \to 1 + 1 + 1\,,$$

and

$$1 \to 1 + 1 + 2\,, \quad 1 \to 1 + 2 + 2\,, \quad 1 \to 2 + 2 + 2\,,$$
$$2 \to 2 + 2 + 1\,, \quad 2 \to 2 + 2 + 2\ .$$

- All of the above processes have vanishing amplitudes even when allowed by kinematics.
- This is due to the specific expressions for the couplings $\mathcal{J}_i(\lambda, p_i)$ following from integrability.
  The slightest modification leads to non-vanishing amplitudes.
- In addition, due to the absence of terms of $\mathcal{O}(1/k^{1/2})$ and $\mathcal{O}(1/k^{3/2})$ the 3- and 5-point *contact* amplitudes that would have been originated from such terms are zero.
- These are strong hints that the integrability of the parent theory in preserved even after imposing the constraints.
- The non-vanishing amplitudes correspond to

$$1+1 \to 1+1\,, \quad 1+2 \to 1+2\,, \quad 2+2 \to 2+2\,.$$

  The S-matrix is unitary, i.e. $S = \mathbb{1} + \frac{i}{k}(\cdots) + \ldots$.

Renormalization and the beta-function

[Georgiou-Loukopoulos-KS-Siampos, to appear]

From classical to quantum the simplest not trivial issue to address.

- Can we compute the beta-function for the coupling λ?
- The (Virasoro) constraint is imposed classically not on a CFT. The theory has broken manifest Lorentz invariance. Hence not obvious that this will be possible.
- The beta-function of the original theory is [Itsios-Siampos-KS 14]

$$\beta_\lambda = \frac{d\lambda}{d\ln \mu^2} = -\frac{2}{k}\,\frac{\lambda^2}{(1+\lambda)^2} + \mathcal{O}(1/k^2)\ ,$$

  where μ is the RG energy scale.
- For the case at hand, does it resemble this, if renormalizable ?

The computation is done using the heat kernel method.

- One finds the fluctuations under $x_a \to x_a + \delta x_a$, $a = 1, 2$ of the classical eqs. of motion and casts them in the form

$$D_{ab} \delta x_b = 0 \ ,$$

  where D_{ab} is a certain differential 2×2 matrix operator.

- Going in momentum space and integrating out the fluctuations gives the effective Lagrangian

$$-\mathcal{L}_{\rm eff} = \mathcal{L}^{(0)}_{\rm l.c.} + \frac{1}{k}\mathcal{L}^{(1)}_{\rm l.c.} + \int_0^{\mu} \frac{d^2 p}{(2\pi)^2} \ln(\det D)^{-1/2} \ ,$$

  where μ is the UV cut-off.

- Expand at high momenta and pick up $\ln \mu^2$-terms.

- After a specific coordinate transformation of (τ, σ) and wave function renormalization for the fields we find

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 - x_1'^2 - x_2'^2\right) + \frac{1}{2}m_1^2\left(1 + \frac{\ln \mu^2}{k}N_1\right)x_1^2$$
$$+ \frac{1}{2}m_2^2\left(1 + \frac{\ln \mu^2}{k}N_2\right)x_2^2) + \ g\left(1 + \frac{\ln \mu^2}{k}N_g\right)x_1 x_2' + \dots \ ,$$

  for certain functions $N_{1,2,g}(\lambda)$.

- Recall also that $m_{1,2}(\lambda)$ and $g(\lambda)$ are functions of λ.

- Physics is scale independent, i.e. $\partial_\mu \mathcal{L}_{\text{eff}} = 0$. Hence

$$\beta_\lambda = -\frac{1}{k}\,\frac{\lambda^2}{(1+\lambda)^2} + \mathcal{O}(1/k^2)\ .$$

- Non-trivial that this suffices for all three terms in $\mathcal{L}_{\text{eff}}$!

- This is half of that of the original model. Indicates a reduction of the d.o.f due to imposing the constraint. (there is no C-theorem [A. Zamolodchikov 86]).

Concluding remarks

and a couple of future directions...

- We have studied classical and quantum mechanical properties of a constrained σ-model.
- Classical integrability persists after imposing the constrained. Is this picture retained when loops are included?
- The beta-function indicates a reduction of d.o.f. Is there an analogue of Zamolodchikov's C-theorem?
- What about other models?
  Can we draw more general conclusions for constrained systems?

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