Entanglement and Expansion

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Introduction

- Consider a quantum mechanical system with many degrees of freedom, such as a spin chain or a quantum field.
- Assume it is in the ground state $|\Psi\rangle$, which is a pure state (zero temperature).
- The density matrix of the total system is $\rho_{\rm tot} = |\Psi\rangle\langle\Psi|$.
- Its von Neumann entropy $S_{\rm tot} = -{\rm tr}\rho_{\rm tot} \log \rho_{\rm tot}$ vanishes.
- Now divide the total system into subsystems A and B and assume that B is inaccessible to A.
- Trace out the part B of the Hilbert space in order to obtain the reduced density matrix of A: $\rho_A = {\rm tr}_B \rho_{\rm tot}$.
- The entropy $S_A = -{\rm tr}_A \rho_A \log \rho_A$ is a measure of the entanglement between A and B.
- It is nonvanishing and $S_A = S_B$.

- In a static background, the leading contribution to the entanglement entropy is proportional to the area of the entangling surface separating subsystems A and B:

$$S_A \sim \frac{\partial A}{\epsilon^{d-1}} + \text{subleading terms.}$$

- Massless scalar field in 3+1 dimensions and a spherical entangling surface:

$$S_A = s\,(R/\epsilon)^2 + c\,\log(R/\epsilon) + d$$

$s \simeq 0.3$ (scheme-dependent) (Srednicki 1993)
$c = -1/90$ (universal) (Lohmayer, Neuberger, Schwimmer, Theisen 2009).

- Conformal field theory in 1+1 dimensions, with central charge c: For a finite system of physical length L with boundaries, divided into two pieces of lengths ℓ and $L-\ell$:

$$S_A = \frac{c}{6}\ln\left(\frac{2L}{\pi\epsilon}\sin\frac{\pi\ell}{L}\right) + \bar{c}_1',$$

with $\bar{c}_1'$ scheme-dependent. (Cardy, Calabrese 2004)

- How does the entanglement entropy evolve in a time-dependent background?
- Volume term?
- de Sitter space (Maldacena, Pimentel 2013).
- Relevance for the expanding Universe.
- Explicit calculations are hard even in a static background. Analytical calculations mostly use the replica trick. They exist for low-dimensional or highly symmetric quantum field theories (CFTs).
- The Ryu-Takayanagi proposal provides a simpler framework in the context of the AdS/CFT correspondence. However, it applies only to theories that have a gravitational dual.
- We generalize Srednicki's approach to expanding backgrounds.

References

- K. Boutivas, G. Pastras and N. Tetradis,
  "Entanglement and expansion," [arXiv:2302.14666 [hep-th]]
- D. Katsinis, G. Pastras and N. Tetradis,
  in preparation
- K. Boutivas, G. Pastras and N. Tetradis,
  in preparation

Plan

- The quantum field as a collection of quantum oscillators.
- Cosmology primer.
- The oscillator wave function in an expanding background.
- Entanglement entropy of two quantum oscillators.
- Entanglement entropy of a quantum field in $1+1$ dimensions.
- Entanglement entropy of a quantum field in $3+1$ dimensions.
- Conclusions.

Expanding the field in momentum modes

- Consider a free scalar field $\phi(\tau, \mathrm{x})$ in a FRW background

$$\mathrm{ds}^2 = \mathrm{a}^2(\tau)\left(\mathrm{d}\tau^2 - \mathrm{dr}^2 - \mathrm{r}^2\mathrm{d}\Omega^2\right).$$

- With the definition $\phi(\tau, \mathrm{x}) = \mathrm{f}(\tau, \mathrm{x})/\mathrm{a}(\tau)$, the action becomes

$$\mathrm{S} = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3\mathrm{x}\left(\mathrm{f}'^2 - (\nabla\mathrm{f})^2 + \left(\frac{\mathrm{a}''}{\mathrm{a}} - \mathrm{a}^2\mathrm{m}^2\right)\mathrm{f}^2\right).$$

  The field $\mathrm{f}(\tau, \mathrm{x})$ has a canonically normalized kinetic term.

- For de Sitter: $\mathrm{a}(\tau) = -1/(\mathrm{H}\tau)$ with $-\infty < \tau < 0$, and

$$\mathrm{S} = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^3\mathrm{x}\left(\mathrm{f}'^2 - (\nabla\mathrm{f})^2 + \frac{2\kappa}{\tau^2}\mathrm{f}^2\right),$$

  where $\kappa = 1 - \mathrm{m}^2/2\mathrm{H}^2$.

- The eom (Mukhanov-Sasaki equation) in Fourier space is

$$f_k'' + k^2 f_k - \frac{2\kappa}{\tau^2} f_k = 0.$$

- Its general solution is

$$f_k(\tau) = A_1\sqrt{-\tau}\, J_\nu(-k\tau) + A_2\sqrt{-\tau}\, Y_\nu(-k\tau) \qquad \nu = \frac{1}{2}\sqrt{1+8\kappa}.$$

- Bunch-Davies vacuum: $A_1 = -\frac{\sqrt{\pi}}{2}$, $A_2 = -\frac{\sqrt{\pi}}{2}i$. For $\tau \to -\infty$ we get a positive-frequency mode function similarly to flat space:

$$f_k(\tau) \simeq \frac{1}{\sqrt{2k}} e^{-ik\tau}.$$

- For $\kappa = 1$ (massless scalar), the full solution reads

$$f_k(\tau) = \frac{1}{\sqrt{2k}} e^{-ik\tau}\left(1 - \frac{i}{k\tau}\right).$$

For $k\tau \to 0^-$ the mode becomes superhorizon and the oscillations stop. The mode freezes.

- The quantum field can be expressed as

$$\hat{f}(\tau, x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[f_k(\tau)\hat{a}_k + f_k^*(\tau)\hat{a}_k^\dagger \right] e^{ik\cdot x}$$

where $\hat{a}_k^\dagger$, $\hat{a}_k$ are standard creation and annihilation operators.

- The variance of the field is

$$\langle \hat{f}^2 \rangle = \int d\ln k \, \frac{k^3}{2\pi^2} |f_k(\tau)|^2, \qquad |f_k(\tau)|^2 = -\frac{\pi}{4}\tau \left[J_\nu^2(-k\tau) + Y_\nu^2(-k\tau) \right].$$

For a massless field ($\nu = 3/2$), it results in the known scale-invariant inflationary power spectrum.

- For superhorizon modes with $k\tau \to 0^-$ the second term in the mode function of a massless scalar dominates. If only this term is retained one obtains

$$f_k(\tau) = -\frac{i}{\sqrt{2}k^{3/2}}\frac{1}{\tau} = -\tau f_k'(\tau)$$

and

$$\hat{\pi}(\tau, x) = -\frac{1}{\tau}\hat{f}(\tau, x).$$

- The fact that the dominant term of the field and the dominant term of its conjugate momentum commute indicates that for most of its properties it can be viewed as a classical stochastic field instead of a quantum one.
- However, the full quantum field and its conjugate always obey the canonical commutation relation. This is guaranteed by the presence of the subleading first term in the mode function.
- The entanglement entropy is of purely quantum origin, for which a classical description is inadequate. It does not vanish for superhorizon modes.

- One may consider the momentum-space entanglement between high and low-momentum modes, such as between modes with physical momenta below and above the Hubble scale H.
- For a free field described by a quadratic action, where the momentum modes do not interact, the entanglement entropy would vanish, as long as the initial state can be written as a tensor product of one state for each momentum mode, as in the Minkowski vacuum.
- The reduced density matrix, when some modes are traced over, would be one of a pure state, namely the state of the modes which have not been traced out.
- We are interested in the entanglement between degrees of freedom localized within two spatial regions separated by an entangling surface. For a dS background one may consider the entanglement between the interior of a horizon-size region of radius $1/H$ and the exterior.

Expanding the field in coordinate space

- For spherical entangling surfaces, define the spherical moments

$$\mathrm{f}_{\mathrm{lm}}(\mathrm{r}) = \mathrm{r}\int d\Omega\, \mathrm{Y}_{\mathrm{lm}}(\theta,\varphi)\,\mathrm{f}(\mathrm{x}), \qquad \pi_{\mathrm{lm}}(\mathrm{r}) = \mathrm{r}\int d\Omega\, \mathrm{Y}_{\mathrm{lm}}(\theta,\varphi)\,\pi(\mathrm{x}),$$

  where Y_{lm} are real spherical harmonics.
- Discretize the radial coordinate as $\mathrm{r_j} = \mathrm{j}\epsilon$, where $1 \leq \mathrm{j} \leq \mathrm{N}$.
- UV cutoff: $1/\epsilon$. IR cutoff: $1/\mathrm{L}$ with $\mathrm{L} = \mathrm{N}\epsilon$. We set $\epsilon = 1$.
- Define the canonically commuting degrees of freedom

$$\mathrm{f}_{\mathrm{lm}}(\mathrm{j}\epsilon) \to \mathrm{f}_{\mathrm{lm,j}}, \qquad \pi_{\mathrm{lm}}(\mathrm{j}\epsilon) \to \frac{\pi_{\mathrm{lm,j}}}{\epsilon}.$$

- Hamiltonian:

$$\mathrm{H} = \frac{1}{2\epsilon}\sum_{\mathrm{l,m}}\sum_{\mathrm{j=1}}^{\mathrm{N}}\Bigg[\pi^2_{\mathrm{lm,j}} + \left(\mathrm{j}+\frac{1}{2}\right)^2\left(\frac{\mathrm{f}_{\mathrm{lm,j+1}}}{\mathrm{j}+1} - \frac{\mathrm{f}_{\mathrm{lm,j}}}{\mathrm{j}}\right)^2 + \left(\frac{\mathrm{l}(\mathrm{l}+1)}{\mathrm{j}^2} - \frac{2\kappa}{(\tau/\epsilon)^2}\right)\mathrm{f}^2_{\mathrm{lm,j}}\Bigg].$$

- We would like to trace out the oscillators with $j\epsilon < R$.
- The 'ground state' of the system is the product of the 'ground states' of the modes that diagonalize the Hamiltonian.
- In the Bunch-Davies vacuum as a 'ground state' of a mode we must define the solution of the time-dependent Schrödinger equation which reduces to the usual simple harmonic oscillator ground state as $\tau \to -\infty$.
- One must determine first the eigenmodes of this system of coupled oscillators. The wave function of each mode depends on a linear combination of the various $f_{lm,j}$. Various (l, m) do not mix.
- In summary, the discretized Hamiltonian for the free field in an inflationary background has the form

$$H = \frac{1}{2\epsilon} \sum_{l,m} \sum_{j=1}^{N} \left[\tilde{\pi}_{lm,j}^2 + \left(\omega_{lm,j}^2 - \frac{2\kappa}{(\tau/\epsilon)^2} \right) \tilde{f}_{lm,j}^2 \right], \qquad (1)$$

with $\tilde{f}_{lm,j}$ the canonical coordinates.

- We need to solve for the harmonic oscillator with a time-dependent eigenfrequency of the form $\omega_0 - 2\kappa/\tau^2$.

de Sitter era

- Oscillator with time-dependent frequency

$$\omega^2(\tau) = \omega_0^2 - \frac{2\kappa}{\tau^2}.$$

- Find the general solution of the Ermakov equation

$$\mathrm{b}''(\tau) + \omega^2(\tau)\mathrm{b}(\tau) = \frac{\omega_0^2}{\mathrm{b}^3(\tau)}$$

  in terms of two linearly independent solutions of

$$\mathrm{y}''(\tau) + \omega^2(\tau)\mathrm{y}(\tau) = 0,$$

  as

$$\mathrm{b}^2(\tau) = \mathrm{c}_1\,\mathrm{y}_1^2(\tau) + \mathrm{c}_2\,\mathrm{y}_2^2(\tau) + 2\mathrm{c}_3\,\mathrm{y}_1(\tau)\mathrm{y}_2(\tau).$$

  c_1, c_2 and c_3 must obey $\mathrm{c}_1\mathrm{c}_2 - \mathrm{c}_3^2 = \mathrm{A}$, with A a constant that depends on the form of $\omega(\tau)$.
- For the problem at hand $\mathrm{y}_1 = \sqrt{-\tau}\,\mathrm{J}_\nu\,(-\omega_0\tau)$ and $\mathrm{y}_2 = \sqrt{-\tau}\,\mathrm{Y}_\nu\,(-\omega_0\tau)$, where $\mathrm{A} = \pi^2\omega_0^2/4$.

- c_1, c_2 are fixed through appropriate initial conditions.
- $b(\tau)$ must tend to 1 for $\tau \to -\infty$.
- $$b^2(\tau) = -\frac{\pi}{2}\omega_0\tau\left(J_\nu^2(-\omega_0\tau) + Y_\nu^2(-\omega_0\tau)\right).$$
- The solution of the Schrödinger equation can now be expressed as
$$F(\tau, f) = \frac{1}{\sqrt{b(\tau)}} \exp\left(\frac{i}{2}\frac{b'(\tau)}{b(\tau)}f^2\right) F^0\left(\int \frac{d\tau}{b^2(\tau)}, \frac{f}{b(\tau)}\right),$$
where $F^0(\tau, f)$ is a solution with constant frequency ω_0.
- Variance of the conjugate operators $\hat{f}$ and $\hat{\pi} = -i\partial/\partial f$:
$$\langle\hat{f}^2\rangle = \frac{b^2(\tau)}{2\omega_0} = -\frac{\pi}{4}\tau\left(J_\nu^2(-\omega_0\tau) + Y_\nu^2(-\omega_0\tau)\right), \quad \langle\hat{\pi}^2\rangle = \frac{\omega_0}{2b^2(\tau)} + \frac{b'^2(\tau)}{2\omega_0}$$
- For $\kappa > 0$ and $\tau \to -\infty$, we have $\Delta f\,\Delta\pi \to 1/2$.
- When $b(\tau)$ diverges for $\tau \to 0^-$, we have $\Delta f = \to \infty$, $\Delta\pi \to \infty$.
- For $\kappa > 0$ and $\tau \to 0^-$, we have $\Delta f/\Delta\pi \to 0$. The uncertainty is much larger in the determination of the momentum.
- Squeezed state.

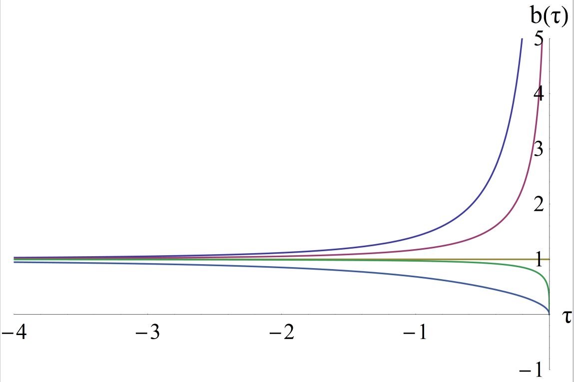

Figure: The form of the function $\mathrm{b}(\tau)$ for $\omega_0 = 1$ and κ =1, 0.5, 0, -0.1, -2 (from top to bottom).

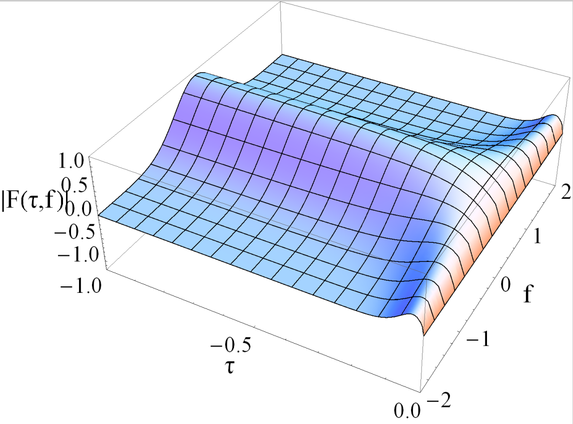

Figure: The amplitude of the 'ground-state' wave function for $\omega_0 = 5$.

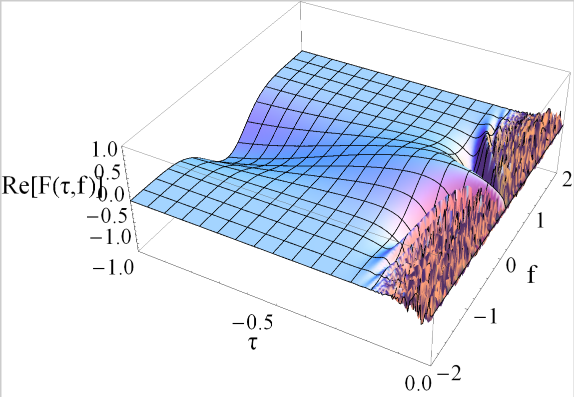

Figure: The real part of the 'ground-state' wave function for $\omega_0 = 5$.

Radiation and matter domination

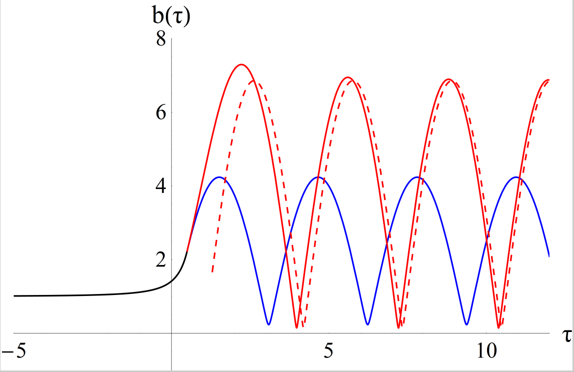

Figure: Left plot: The form of the function $\mathrm{b}(\tau)$ for $\omega_0 = 1$ and $\mathrm{H} = 2$, and $\tau_0 = 0.5$. The black line corresponds to the dS era, the blue line to a RD era, the red lines to the MD era.

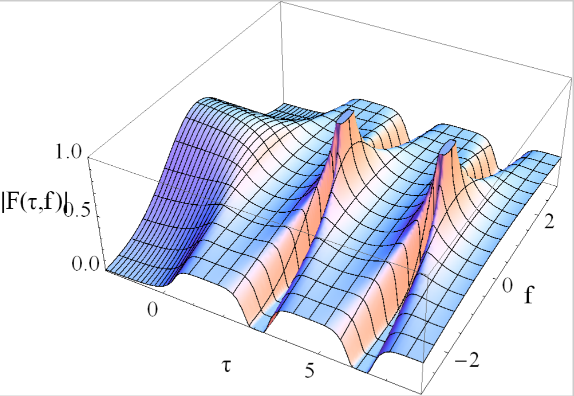

Figure: Left plot: The amplitude of the 'ground-state' wave function for the transition from a dS to a RD background at $\tau = 0.5$, for $\omega_0 = 1$, $H = 2$.

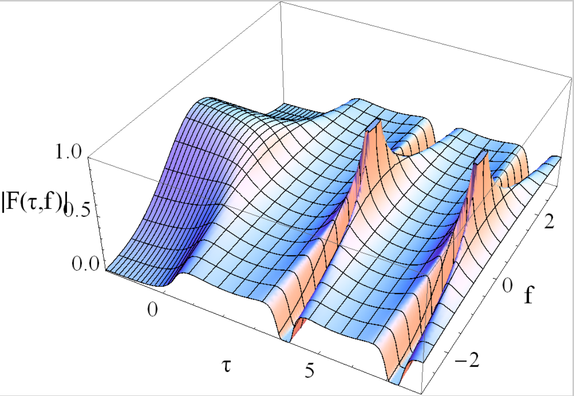

Figure: Left plot: The amplitude of the 'ground-state' wave function for the transition from a dS to a MD background at $\tau = 0.5$, for $\omega_0 = 1$, H = 2.

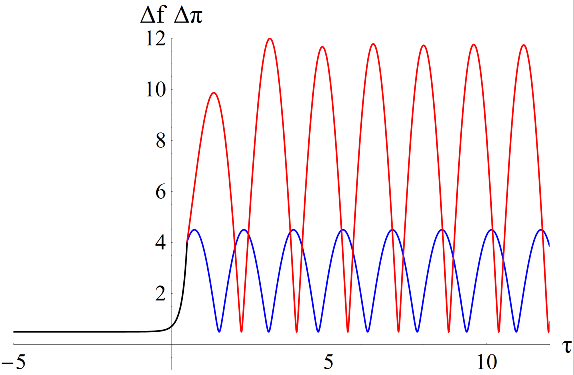

Figure: Left plot: The product of uncertainties $\Delta f \Delta \pi$ during the evolution of the wave function.

Entanglement entropy of two quantum oscillators

- Hamiltonian

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2 - \lambda(\tau)(x_1^2 + x_2^2)\right].$$

- For oscillators arising from a massive field in dS, $\lambda(\tau) = 2\kappa/\tau^2$. For a massless field in a general background, $\lambda(\tau) = a''/a$.
- The Hamiltonian can be rewritten as

$$H = \frac{1}{2}\left[p_+^2 + p_-^2 + w_+^2(\tau)x_+^2 + w_-^2(\tau)x_-^2\right],$$

$x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}}$, $\omega_{0+}^2 = k_0, \omega_{0-}^2 = k_0 + 2k_1$, $w_\pm^2(\tau) = \omega_{0\pm}^2 - \lambda(\tau)$.
- The 'ground state' is the tensor product of the 'ground states' of the two decoupled normal modes:

$$\psi_0(x_+, x_-) = \left(\frac{\Omega_+\Omega_-}{\pi^2}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}\left(\Omega_+ x_+^2 + \Omega_- x_-^2\right) + \frac{i}{2}\left(G_+ x_+^2 + G_- x_-^2\right.\right.$$

$$\Omega_\pm(\tau) \equiv \frac{\omega_{0\pm}}{b^2(\tau;\omega_{0\pm})}, \quad G_\pm(\tau) \equiv \frac{b'(\tau;\omega_{0\pm})}{b(\tau;\omega_{0\pm})}.$$

- Express the wave function in terms of x_1, x_2.
- The reduced density matrix is given by

$$\rho(x_2, x_2') = \int_{-\infty}^{+\infty} dx_1 \psi_0(x_1, x_2)\psi_0^*(x_1, x_2').$$

- The Gaussian integration gives

$$\rho(x_2, x_2') = \sqrt{\frac{\gamma - \beta}{\pi}} \exp\left(-\frac{\gamma}{2}(x_2^2 + x_2'^2) + \beta x_2 x_2'\right) \exp\left(i\frac{\delta}{2}(x_2^2 - x_2'^2)\right),$$

where γ, β, δ are functions of $\Omega_\pm$, $G_\pm$.
- The eigenfunctions of the reduced density matrix satisfy

$$\int_{-\infty}^{+\infty} dx_2' \rho(x_2, x_2') f_n(x_2') = p_n f_n(x_2).$$

- One finds

$$f_n(x) = H_n(\sqrt{\alpha}x) \exp\left(-\frac{\alpha}{2}x^2\right) \exp\left(i\frac{\delta}{2}x^2\right),$$

where $\alpha = \sqrt{\gamma^2 - \beta^2}$ and H_n is a Hermite polynomial.

- The eigenvalues $\mathrm{p_n}$ are

$$\mathrm{p_n} = \sqrt{\frac{2(\gamma-\beta)}{\gamma+\alpha}}\left(\frac{\beta}{\gamma+\alpha}\right)^{\mathrm{n}} = (1-\xi)\xi^{\mathrm{n}},$$

where

$$\xi = \frac{\beta}{\gamma+\alpha}.$$

They satisfy

$$\sum_{\mathrm{n}=0}^{\infty} \mathrm{p_n} = (1-\xi)\sum_{\mathrm{n}=0}^{\infty}\xi^{\mathrm{n}} = 1.$$

- The entanglement entropy can be calculated as

$$\mathrm{S} = -\sum_{\mathrm{n}=0}^{\infty}(1-\xi)\xi^{\mathrm{n}} \ln\left[(1-\xi)\xi^{\mathrm{n}}\right] = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi.$$

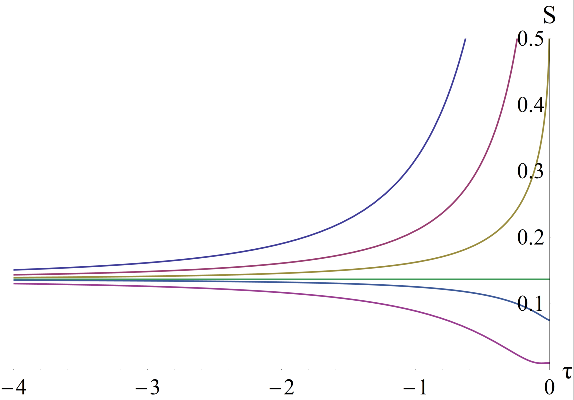

Figure: Left plot: The entanglement entropy in a dS background as a function of conformal time τ for $\omega_+ = 1$, $\omega_- = 2$ and κ =1, 0.5, 0.2, 0, −0.1, −0.5 (from top to bottom).

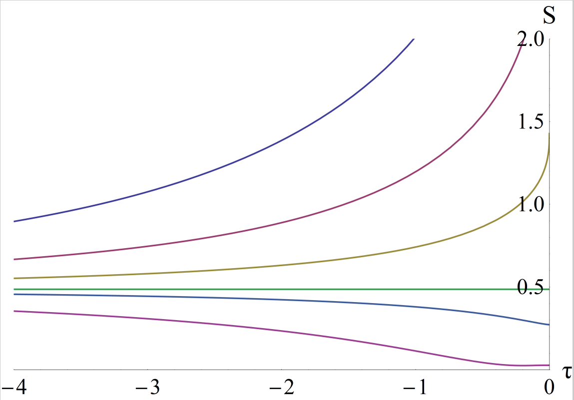

Figure: Left plot: The entanglement entropy in a dS background as a function of conformal time τ for $\omega_+ = 1$, $\omega_- = 0.2$ and κ =1, 0.5, 0.2, 0, −0.1, −0.5 (from top to bottom).

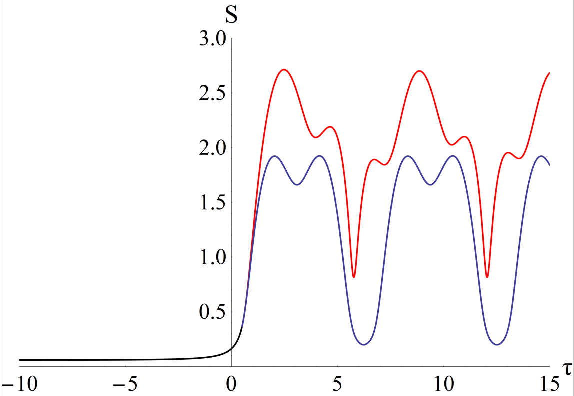

Figure: Left plot: The entanglement entropy as a function of conformal time τ for $\omega_+ = 1$, $\omega_- = 1.5$, H = 2 and $\tau_0 = 0.5$. The black line corresponds to a dS background, with a transition at τ_0 to either a RD era (blue line) or to a MD era (red line).

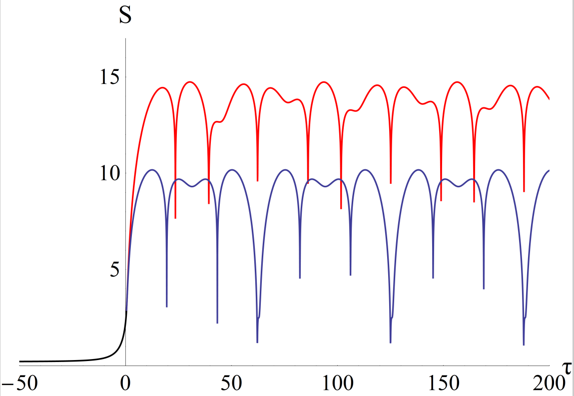

Figure: Similarly to the previous plot for $\omega_+ = 0.1$, $\omega_- = 0.25$, H = 2 and $\tau_0 = 0.5$.

- The generalization to a system of N coupled oscillators proceeds along the lines of the original work of Srednicki.
- The system is assumed to lie in the ground state of each canonical mode in the asymptotic past (Bunch-Davies vacuum).
- Later this becomes a squeezed state, with a wave function that reflects the horizon crossing and freezing of each mode.
- When n oscillators are traced out, the reduced density matrix is

$$\rho(\mathrm{x}_2, \mathrm{x}_2') = \left(\frac{\det \mathrm{Re}(\gamma - \beta)}{\pi^{\mathrm{N}-\mathrm{n}}}\right)^{1/2}$$
$$\times \exp\left(-\frac{1}{2}\mathrm{x}_2^{\mathrm{T}}\gamma\,\mathrm{x}_2 - \frac{1}{2}\mathrm{x}_2'^{\mathrm{T}}\gamma\,\mathrm{x}_2' + \mathrm{x}_2^{\mathrm{T}}\beta\,\mathrm{x}_2' + \frac{\mathrm{i}}{2}\mathrm{x}_2^{\mathrm{T}}\delta\,\mathrm{x}_2 - \frac{\mathrm{i}}{2}\mathrm{x}_2'^{\mathrm{T}}\delta\,\mathrm{x}_2'\right).$$

- γ and δ are $(\mathrm{N}-\mathrm{n}) \times (\mathrm{N}-\mathrm{n})$ real symmetric matrices, while β is a $(\mathrm{N}-\mathrm{n}) \times (\mathrm{N}-\mathrm{n})$ Hermitian matrix.
- The eigenvalues of the density matrix do not depend on δ.

- A major technical difficulty arises because the matrices γ and β cannot be diagonalized through real orthogonal transformations in order to identify the eigenvalues of the reduced density matrix.
- These are guaranteed to be real by the nature of the density matrix, but the determination of their exact values requires an extensive analysis.
- A method has been developed for their computation. A detailed presentation is given in the publications.

Entanglement entropy of a quantum field in $1+1$ dimensions

- Consider a toy model of a massless scalar field in $1+1$ dimensions. The field is canonically normalized.
- Assume a background given by the FRW metric, neglecting the angular part. The curvature scalar R is equal to $-2\mathrm{H}^2$.
- The de Sitter era can be mimicked by including an effective mass term arising from a non-minimal coupling to gravity $\xi \mathrm{R}\phi^2$ with $\xi = -1/2$.
- The Hamiltonian of the discretized system is

$$\mathrm{H} = \frac{1}{2\epsilon}\sum_{\mathrm{j}=2}^{\mathrm{N}-1}\left[\pi_\mathrm{j}^2 + (\mathrm{f}_{\mathrm{j}+1} - \mathrm{f}_\mathrm{j})^2 - \frac{2\kappa}{(\tau/\epsilon)^2}\mathrm{f}_\mathrm{j}^2\right] + \frac{1}{2\epsilon}\sum_{\mathrm{j}=1,\mathrm{N}}\left[\pi_\mathrm{j}^2 + \mathrm{f}_\mathrm{j}^2 - \frac{2\kappa}{(\tau/\epsilon)^2}\mathrm{f}_\mathrm{j}^2\right]$$

  with $\kappa = 1$. We have modified the action for the oscillators at the ends of the chain, so as to impose boundary conditions corresponding to a vanishing field at the endpoints.
- The radiation dominated era with $\kappa = 0$ can be mimicked by assuming a transition to a flat background with $\mathrm{R} = 0$ at some time τ_0.

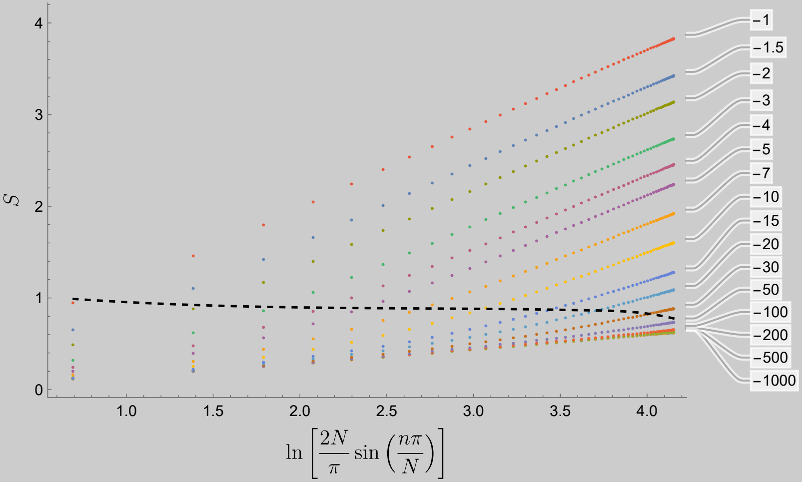

Figure: The entanglement entropy resulting from tracing out the part $n < k \leq N$ of a one-dimensional chain at various times, for a dS background.

- For $\tau \to -\infty$, the entanglement entropy can be described very well by the expression

$$S = \frac{c}{6} \ln\left(\frac{2L}{\pi\epsilon} \sin\frac{\pi\ell}{L}\right) + \bar{c}_1', \tag{2}$$

with $c = 1$, in agreement with Cardy, Calabrese 2004.

- For $\tau \to 0^-$ the entanglement entropy can be described very well by the expression

$$S = \ln\left(\frac{2L\, a(\tau)}{\pi\epsilon} \sin\frac{\pi\ell}{L}\right) + d, \tag{3}$$

where $a(\tau) = -1/(H\tau)$.

- The entropy grows with the number of efoldings $\mathcal{N} = \ln a(\tau)$.

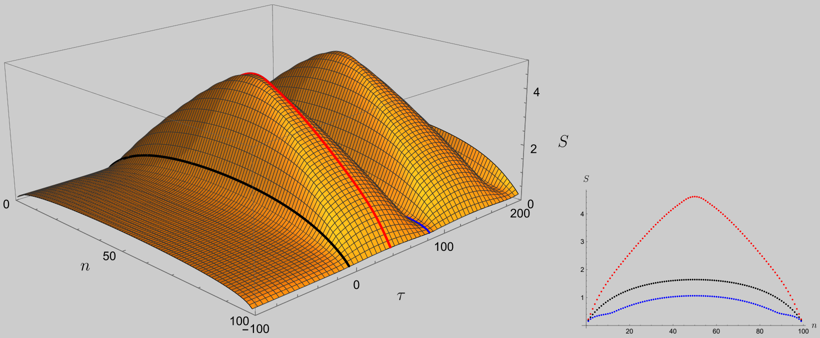

Figure: The entanglement entropy resulting from tracing out the part n < k ≤ N of a one-dimensional chain at various times. The transition from a dS to a RD background (black line) occurs at $\tau_0 = -5$. During the RD era, the maximal entanglement entropy (red line) is first achieved at $\tau = 85$, and the minimal entanglement entropy (blue line) at $\tau = 40$. For clarity, we also display the entanglement entropy at these times in the right plot.

Entanglement entropy of a quantum field in $3+1$ dimensions

- Massless scalar field in $3+1$ dimensions.
- Hamiltonian:

$$
\begin{aligned}
H = \frac{1}{2\epsilon} \sum_{l,m} \sum_{j=1}^{N} \Bigg[\pi_{lm,j}^2 \quad &+ \quad \left(j+\frac{1}{2}\right)^2 \left(\frac{f_{lm,j+1}}{j+1} - \frac{f_{lm,j}}{j}\right)^2 \\
&+ \quad \left(\frac{l(l+1)}{j^2} - \frac{2\kappa}{(\tau/\epsilon)^2}\right) f_{lm,j}^2 \Bigg],
\end{aligned}
$$

with $\kappa = 1$.
- Trace out the oscillators with $j\epsilon < R$.
- Sum over l, m.
- Fit the result with a function ($\epsilon = 1$)

$$
S = s(\tau)\, R^2 + c(\tau)\, R^3 + d(\tau).
$$

The logarithmic correction is assumed to be subleading.

Preliminary results

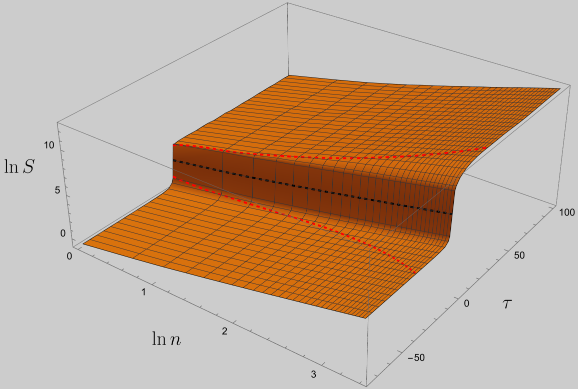

Figure: Entanglement entropy as a function of entangling radius and time. The red lines indicate the location of the horizon at various times. The black line indicates the entropy at the time $\tau_0 = -5$ of the transition from the de Sitter era with $\mathrm{H} = 10$ to the radiation dominated era.

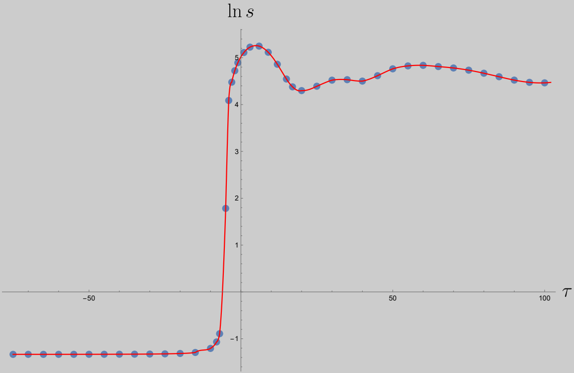

Figure: The coefficient of the quadratic term in the entanglement entropy.

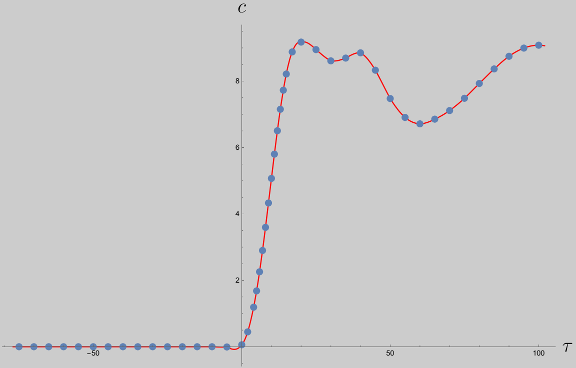

Figure: The coefficient of the cubic term in the entanglement entropy.

- Momentum modes that start as pure quantum fluctuations in the Bunch-Davies vacuum during inflation are expected to freeze when they exit the horizon and transmute into classical stochastic fluctuations.
- This is only part of the picture. Even though its classical features are dominant, the field never loses its quantum nature.
- The various modes evolve into squeezed states.
- The squeezing triggers a strong enhancement of quantum entanglement. The effect is clearly visible in the entanglement entropy.
- The enhancement is proportional to the number of efoldings during the inflationary era.
- The entanglement entropy survives during the eras of radiation or matter domination. A volume effect appears during these eras.
- Quantum mechanical picture of reheating?
- Observable consequences?