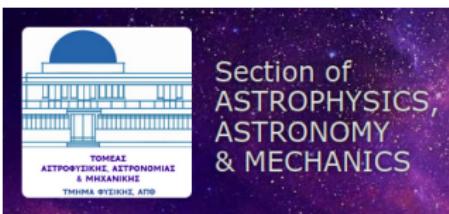


Universal Relations for Rapidly Rotating Neutron Stars Using Supervised ML Techniques

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ASTROPHYSICS,
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Overview

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- Rotating case
- Equation of state Models (EoSs)
- RNS code

2 Patterns & Supervised ML Methods

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- Cross Validation evaluation process
- Least Squares Regression

3 Regression Models

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- List of Regression models produced

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Rapidly Rotating NSs

Rotating case

- A rotating compact object is characterized by its mass M and its angular momentum J .
- ★ Stationary and axisymmetric spacetime in equilibrium:

$$ds^2 = -e^{(\gamma+\rho)} dt^2 + e^{(\gamma-\rho)} r^2 \sin^2 \theta (d\phi - \omega dt)^2 + e^{2\alpha} (dr^2 + r^2 d\theta^2), \quad (1)$$

where $\gamma, \rho, \omega, \alpha$ are metric functions of the quasi-isotropic coordinates (r, θ) .

- ★ Interior: we consider the stellar matter as a perfect fluid with local isotropy and energy-momentum tensor

$$T^{a\beta} = (\epsilon + P) u^a u^\beta + P g^{a\beta} \quad (2)$$

- ★ the matter inside the star is in hydrostatic equilibrium
- ★ Four-velocity of the star's fluid element: $u^a = u^t (t^a + \Omega \phi^a)$, where t^a, ϕ^a are the killing vectors.
- ★ $\Omega = \text{const}$ for uniformly rotating neutron stars
- Hydrostationary Equilibrium Equation (Euler's equation) for stationary and axisymmetric perfect fluid spacetime with $\Omega = \text{const}$:

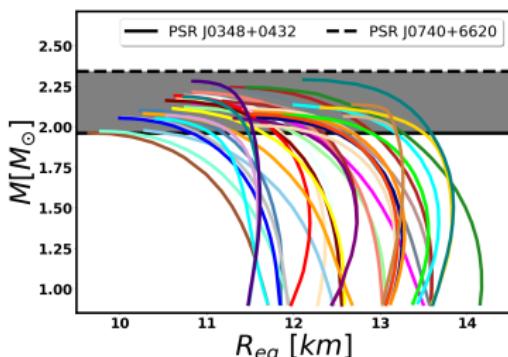
$$\frac{\nabla_a P}{\epsilon + P} = \nabla_a \ln u^t, \quad \text{where } u^t = \frac{e^{-(\rho+\gamma)/2}}{\sqrt{1 - (\Omega - \omega)^2 r^2 \sin^2(\theta) e^{-2\rho}}}$$

follows from the normalization condition $u^a u_a = -1$

- ★ Barotropic EoS $\epsilon = \epsilon(P)$ that correlates the thermodynamic variables $\epsilon(r)$ and $P(r)$

Equation of state Models (EoSs)

- We have used tabulated EoSs of cold ($T = 0$), ultradense nuclear matter from comPOSE (<https://compose.obspm.fr/home>) database.
- ★ Hadronic ($[n, p, e^-, \mu^-]$) (24), Hyperonic (n, p, e^-, H) (8) and Hybrid: Quark+Hadron (n, p, e^-, q) (6) models → (38 in total)



- Each EoS satisfies physical acceptability conditions which ensure β - equilibrium
- Constraints based on observational (E/M signals) are determined considering the masses of the two most massive radio pulsars known to date:
 - ★ PSR J0348+0432: $M = 2.01^{+0.04}_{-0.04} M_\odot$ (solid line, lower limit)
 - ★ PSR J0740+6620 ($M = 2.14^{+0.20}_{-0.18} M_\odot$) (dashed line, upper limit).
- Constraints based on GWs:
 - ★ GW170817: NS-NS merger analysis: $R_{M_{max}} \geq 9.60^{+0.14}_{-0.03}$ km
 - ★ GW170817: $M_{max} = 2.32 M_\odot$, (2 σ) bound, assuming that the final remnant was a BH.

Numerical Setup-Parameters

- RNS code: Is used to construct neutron star equilibrium model sequences (static, rotating, keplerian)
<https://github.com/cgca/rns>

- ★ The user must provide a tabulated (realistic) cold EoS as an input
- ★ EoS input quantities:

Energy density: ϵ [gr/cm ³]	Pressure: P [dyn/cm ²]	Pseudo-Enthalpy: $H(P)$ [cm ² /s ²]	Baryon number density: n_b [cm ⁻³]
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- A unique equilibrium model is characterized by specifying two parameters:
 - ★ central energy density $\epsilon_c \rightarrow$ related to the compactness of the star
 - ★ axes ratio (r_p/r_e) \rightarrow related to the amount of rotation
- Computational grid used in this work: MDIV \times SDIV = 151 \times 301
 - ★ MDIV: number of point in $\mu = \cos \theta$ -direction
 - ★ SDIV: number of point in $s = r/(r + r_{eq})$ -direction
- Extended ensemble of astrophysically relevant models with $\epsilon_c \sim (3 \times 10^{14} - 3.1 \times 10^{15}) \text{ gr/cm}^3$ and masses starting from $\sim 0.9M_\odot$ and up to the star's maximum mass M_{max} .
 - ★ In total, our entire sample includes 11983 models of rotating neutron stars from a few hundred Hz ($\simeq 268$ Hz) up to ~ 2 kHz close to the mass-shedding (keplerian) limit.
- Stellar parameters (output) used in this work:
 - ★ $M, R_{eq}, J_{geom}, \Omega = const, \mathcal{E} = T/|W|, I, M_2^{GH}, S_3^{GH}$
- From these parameters, we construct the following quantities:
 - Star's overall structure: $\mathcal{C} = M/R_{eq}, \mathcal{K} = \mathcal{C}^{-1}, \bar{I} = I_{geom}/M^3, \mathcal{D} = M \times \tilde{f}/\chi$
 - Spin parameterizations: $\chi = J/M^2, \sigma = \Omega^2 R_{eq}^3/GM, \mathcal{E} = T/|W|, M \times f/c, R_{eq} \times f/c$
 - Multipole moments: $\bar{Q} = -M_2^{GH}/M^3 \chi^2, \bar{S}_3 = -S_3^{GH}/M^3 \chi^4$

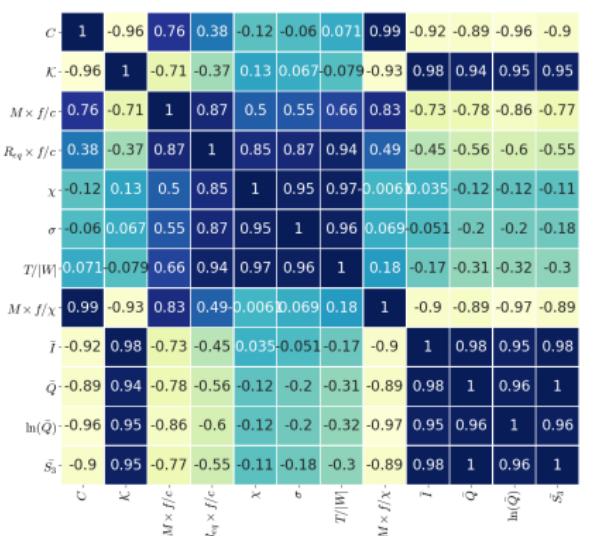
Universality Patterns & Supervised ML Techniques

Correlation Analysis

- Neutron star global parameters (M, R_{eq}, \dots) have a strong dependence on the EoS
- Primary Goal: To investigate EoS-insensitive (Universal) relations between the stellar parameters.
- In order to investigate more systematically the universality patterns that may exist between the various physical observable quantities, we decided to use tools from statistical analysis.
- Given the sample data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^N$ (training set) consisting of N pairs, the Pearson's correlation coefficient is a dimensionless statistical metric function given by

$$\rho_{x,y} = \frac{\text{Cov}[x,y]}{\sigma_x \sigma_y}, \quad -1 \leq \rho_{x,y} \leq 1$$

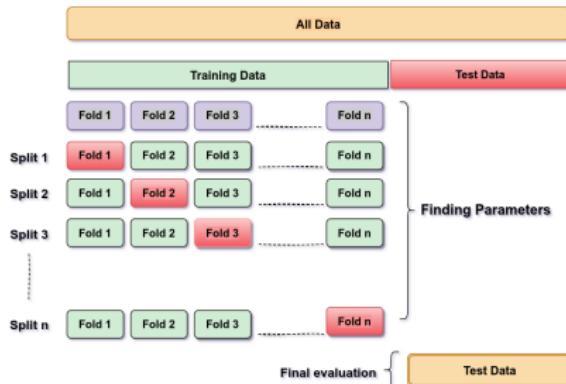
- ★ $\text{Cov}[x, y] = E[x, y] - \bar{x}\bar{y} = \sigma_{xy}$ and σ_x, σ_y standard deviations
- ★ $\rho_{x,y} \rightarrow 1$: x, y directly proportional
- ★ $\rho_{x,y} \rightarrow -1$: x, y inversely proportional
- ★ $\rho_{x,y} \rightarrow 0$: no linear dependence between x & y



Cross-Validation

- Sample data $\mathcal{D} = \{(x_i, y_i, z_i)\}_{i=1}^n$ consisting of n pairs $\longrightarrow \mathcal{D}$ forms the training set.
 - ★ Main goal: To find the best functional form that links the data and avoid overfitting.
 - ★ Common practice: hold out a part of \mathcal{D} , hidden from the training process, known as test set $(X_{\text{test}}, Z_{\text{test}})$
 - ★ Evaluation process: For a model function $\hat{z} = F(x, y)$, we have to optimize a trial function called Loss function \mathcal{L} (training process)+ cross-check to the test set.
- Generalization through k-fold ('Leave-One-Out') Cross-Validation \rightarrow dataset splits
 - ★ A model is trained using $k - 1$ of the folds as training data.
 - ★ The resulting model is validated on the k -th fold of the data as a test set.
 - ★ The process is repeated iteratively for the whole dataset.
 - ★ for n samples, we have n different training sets and n different test sets, i.e., the number of test sets is the same as the number of samples.

Cross-Validation



- The reported performance estimation is the average of the values computed during the LOOCV.
- For estimating and evaluating the model's \hat{z} performance in the LOOCV test, we use the following statistical metric functions:
 - ★ $MAE(\mathbf{z}, \hat{\mathbf{z}}) = \frac{1}{n} \sum_{i=0}^{n-1} |z_i - \hat{z}_i|$, $Max_Error(\mathbf{z}, \hat{\mathbf{z}}) = \max(|z_i - \hat{z}_i|)$
 - Maximum deviation: $d_{\max}(\mathbf{z}, \hat{\mathbf{z}}) = \max\left(\frac{|z_i - \hat{z}_i|}{\max(\epsilon, |z_i|)}\right)$, $MSE(\mathbf{z}, \hat{\mathbf{z}}) = \frac{1}{n} \sum_{i=0}^{n-1} (z_i - \hat{z}_i)^2$
 - $MAE(\mathbf{z}, \hat{\mathbf{z}}) = \frac{1}{n} \sum_{i=0}^{n-1} \frac{|z_i - \hat{z}_i|}{\max(\epsilon, |z_i|)}$.
- From different functional forms \hat{z} examined →, the one with the optimal statistical metric evaluation functions is selected!
- We define a relation between some chosen parameters to be universal, when the relative errors in the validation set are $\lesssim \mathcal{O}(10\%)$.

Leasts Squares Regression

- When the best functional form $\hat{z} = \mathcal{F}(x, y; \hat{a})$ is determined \rightarrow How can we determine its coefficients, \hat{a} ?
- ★ For example, given a dataset that consists of (x_i, y_i, z_i) , $i = 1, \dots, n$ data where (x_i, y_i) are the independent variables and z_i are the dependent ones, the best-fit optimal parameters \hat{a} , also known as optimizers, can be found by optimizing the Loss function
- ★ Optimizing/minimizing the Loss function:

$$\mathcal{L}(\hat{a}) = \sum_{i=1}^n ||z_i - \mathcal{F}(x_i, y_i; \hat{a})||^2 = \sum_{i=1}^n r_i^2, \quad (3)$$

where $r_i = z_i - \mathcal{F}(x_i, y_i; \hat{a})$, and $\hat{z} = \mathcal{F}(x, y; \hat{a}) = \sum_{j=0}^m \hat{a}_j \mathcal{H}_j(x, y)$

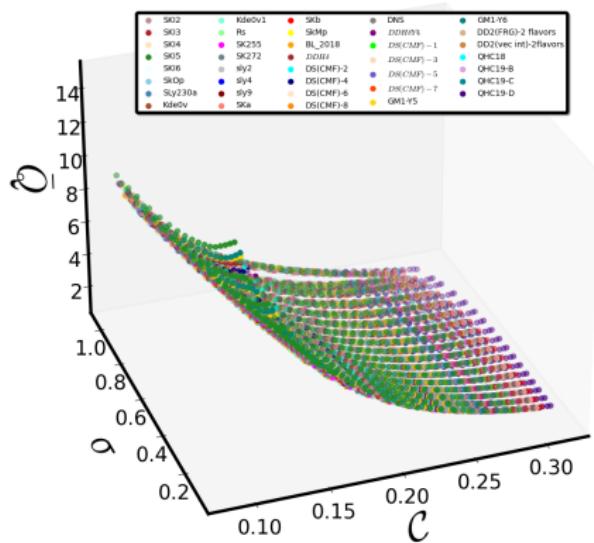
- m adjustable and uncorrelated optimizers/coefficients
- $\mathcal{H}_j(x, y)$ is a function of polynomial combinations of x and y
- The regression coefficients \hat{a} for the model that best fits the data should be given from the minimum of the Loss function by setting the corresponding partial derivative to zero. Since the model contains m uncorrelated optimizer parameters \hat{a} , there are m equations given by:

$$\frac{\partial \mathcal{L}}{\partial \hat{a}_j} = -2 \sum_{i=1}^n r_i \left(\frac{\partial r_i}{\partial \hat{a}_j} \right) = -2 \sum_{i=1}^n r_i \left(\frac{\partial \mathcal{F}}{\partial \hat{a}_j} \right) = 0, \quad j = 0, \dots, m \quad (4)$$

Regression Models

EoS-Insensitive Relations: $\bar{Q} = \bar{Q}(\mathcal{C}, \sigma)$

- Relativistic multipole moments depend on the neutron star's internal structure, which is determined by the unknown equation of state (EoS) → exploring EoS-insensitive relations (high importance)
- Investigation of the $\bar{Q}(\mathcal{C}, \sigma)$ parameterization
 - ★ $\bar{Q} \equiv -M_2 M / J^2 = -M_2 / (M^3 \chi^2)$, $\chi = J/M^2$, $\mathcal{C} = M/R_{eq}$, and $\sigma = \Omega^2 R_{eq}^3 / GM$ in the respective ranges $0.085 \leq \mathcal{C} \leq 0.313$ and $0.067 \leq \sigma \leq 1.033$.



EoS-Insensitive Relations: $\bar{Q} = \bar{Q}(\mathcal{C}, \sigma)$

- Surface $\bar{Q} = \bar{Q}(\mathcal{C}, \sigma)$: We used indicative functions of the form:

$$\boxed{\bar{Q}(\mathcal{C}, \sigma) = \sum_{n=0}^{\kappa} \sum_{m=0}^{\kappa-n} \hat{a}_{nm} \mathcal{C}^n \sigma^m.} \quad (5)$$

Table: Indicative list of LOOCV evaluation metrics for the $\bar{Q}(\mathcal{C}, \sigma)$ (5) parameterization with different values of κ .

MAE	Max Error	MSE	$d_{\max}(\%)$	MAPE (%)	κ
0.2265	2.549	0.0896	81.853	7.321	2
0.0918	1.093	0.0174	36.463	2.689	3
0.0657	0.771	0.0104	13.011	1.585	4
0.0632	0.744	0.0097	6.899	1.491	5
0.0649	0.786	0.099	9.538	1.575	6
0.0621	0.758	0.0095	7.173	1.455	7

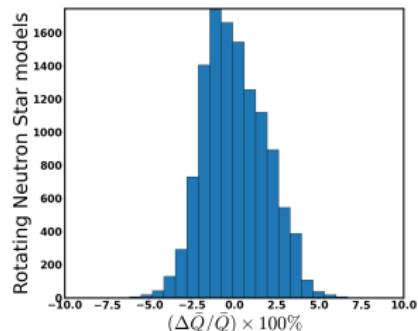
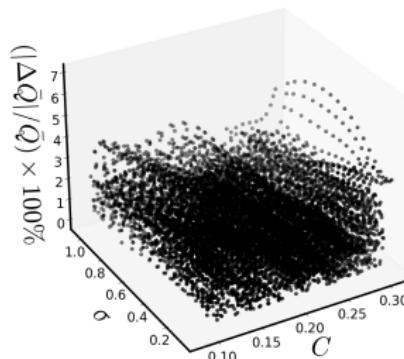
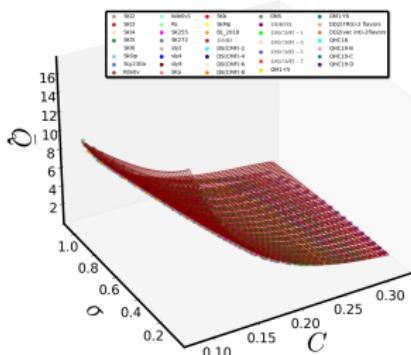
- ★ Best functional form for $\kappa = 5$ (optimal evaluation metrics). More or less complicated forms of \mathcal{C} , and σ do not improve the fit quality.

Table: \hat{a}_{nm} regression optimizers for the $\bar{Q}(\mathcal{C}, \sigma)$ parameterization (5) with $\kappa = 5$.

$\hat{a}_{00} \cdot 10^2$ 0.6802751	$\hat{a}_{01} \cdot 10^1$ -4.3993096	$\hat{a}_{02} \cdot 10^1$ 2.7137482	$\hat{a}_{03} \cdot 10^1$ -1.0955132
$\hat{a}_{04} \cdot 10^{-2}$ -9.6154370	\hat{a}_{05} 1.445554	$\hat{a}_{10} \cdot 10^3$ -1.0699002	$\hat{a}_{11} \cdot 10^2$ 5.0094955
$\hat{a}_{12} \cdot 10^{-2}$ -2.2554846	$\hat{a}_{13} \cdot 10^1$ 7.6965321	$\hat{a}_{14} \cdot 10^1$ -1.3503803	$\hat{a}_{20} \cdot 10^3$ 7.7681018
$\hat{a}_{21} \cdot 10^{-2}$ 5.0094955	$\hat{a}_{22} \cdot 10^2$ 6.1857273	$\hat{a}_{23} \cdot 10^1$ -9.3568604	$\hat{a}_{30} \cdot 10^4$ -3.0342856
$\hat{a}_{31} \cdot 10^{-3}$ 5.5645259	$\hat{a}_{32} \cdot 10^2$ -6.0425422	$\hat{a}_{40} \cdot 10^4$ 6.1273085	$\hat{a}_{41} \cdot 10^3$ -5.1861927
$\hat{a}_{50} \cdot 10^{-4}$ -5.0058667			

EoS-Insensitive Relations: $\hat{Q} = \hat{Q}(\mathcal{C}, \sigma)$

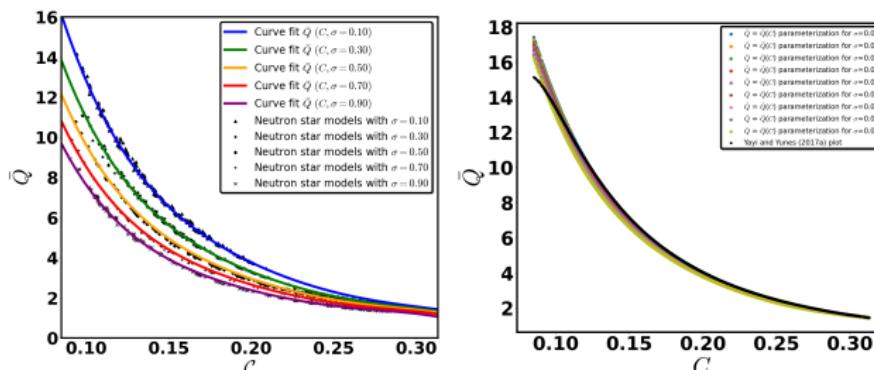
- Surface fit, Relative Deviations plot, and rotating models' data distribution concerning the relative errors.



- The relative errors between the formula (5) for $\kappa = 5$, and the observed \bar{Q} are $\lesssim 6.866\%$ for all EoSs and neutron star models considered.
 - Deviations ($\geq 5\%$) are due to the less compact stars with ($0.104 \leq \mathcal{C} \leq 0.162$, 48 models) and the most compact ones with ($\mathcal{C} \geq 0.264$, 13 models) regardless of the spin parameter σ .

EoS-Insensitive Relations: $\bar{Q} = \bar{Q}(C, \sigma)$

■ Curve fit for indicative values of σ



- ★ Left plot: The black dots correspond to the data values for the total ensemble of EoSs considered, while the colored curves to the theoretical prediction according to the formula (5) with $\kappa = 5$.
- ★ Right plot: Prediction of (5) for the slowly rotating limit using indicative values of $\sigma \in [0.01, 0.09]$.
- ★ Black-curve: Yagi and Yunes fitting formula
- ★ Curves approximately match, excluding the less compact stellar objects!
- The same procedure is followed for other indicative observable quantities such as $R_{eq}/M, \mathcal{D} = M \times \tilde{f}/\chi, \bar{I}, \bar{S}_3 \dots$

Indicative list of Universal Relations produced

- Regression models of the form:

$$Z(X, Y) = \sum_{n=0}^{\kappa} \sum_{m=0}^{\kappa-n} a_{nm} X^n Y^m$$

Z	X	Y	MAE	Max Error	MSE	$d_{\max}(\%)$	MAPE (%)	κ
Q	C	σ	0.063	0.7440	0.0097	6.899	1.491	5
K	χ	Q	0.070	0.6380	0.010	6.480	1.177	5
E	χ	$\ln Q$	1.34×10^{-4}	1.48×10^{-3}	5×10^{-8}	3.020	0.269	5
D	χ	$\ln Q$	5.7×10^{-5}	4.39×10^{-4}	7×10^{-9}	5.306	0.322	4
I	χ	Q	0.035	1.5220	0.004	5.613	0.360	4
I	σ	Q	0.117	1.6080	0.046	6.662	0.864	4
I	E	Q	0.032	1.5470	3.6×10^{-3}	5.507	0.334	4
\bar{I}	$R_{eq} \times \tilde{f}$	Q	0.152	1.8680	0.065	7.309	1.275	2
S_3	χ	I	0.085	1.8460	0.016	9.381	1.303	4
S_3	$\ln Q$	-	0.090	0.8930	0.017	4.847	1.308	4
S_3	$M \times \tilde{f}$	Q	0.060	0.5760	8×10^{-3}	3.161	0.857	3
S_3	σ	Q	0.056	0.6470	0.007	3.208	0.812	3
S_3	χ	Q	0.059	0.6612	0.008	3.289	0.865	3
S_3	E	Q	0.060	0.6402	0.008	3.253	0.866	2

- For more information, you can follow that link to see the paper <https://arxiv.org/abs/2303.04273>

Conclusions

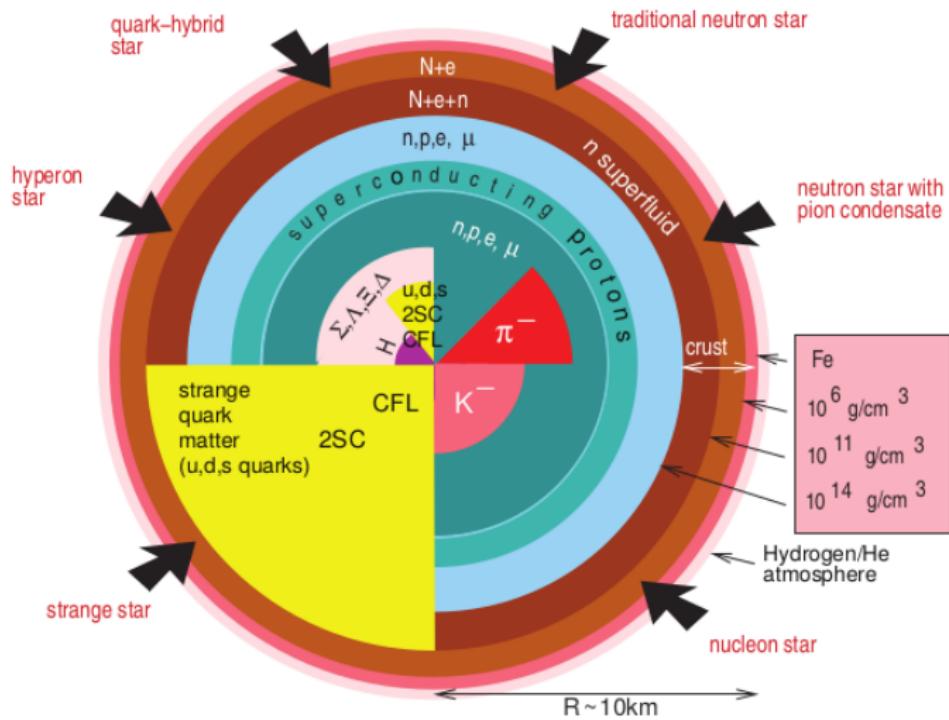
Concluding Remarks

- We have used an ensemble of 38 tabulated EoSs of cold, dense nuclear matter in equilibrium
- Each EoS satisfies physical acceptability conditions as well as constraints coming from E/M signals and GWs
- We used a sample of 11983 rapidly rotating stellar models
- We used correlation analysis to unveil universality patterns
- We used supervised ML methods such as LOOCV & Least Squares Regression to evaluate and produce the best possible fitting formulae
- We suggested well-behaved EoS-insensitive relations for the star's global parameters



Backup Slides

Star's Cross Section



Non rotating case

- We consider Static, isolated, and spherically symmetric Neutron Star with mass M and radius R_{eq}
- ★ Exterior vacuum metric \rightarrow Schwarzschild
- ★ Interior: we consider the stellar matter as a perfect fluid with local isotropy and energy-momentum tensor

$$T^{a\beta} = (\epsilon + P) u^a u^\beta + P g^{a\beta} \quad (6)$$

- ★ Line element:

$$ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (7)$$

Independent of time: the matter inside the star is in hydrostatic equilibrium.

- Einstein's Field equations \rightarrow Hydrostatic equilibrium equations (TOV):

$$G_{tt} = 8\pi T_{tt} \implies \frac{dm}{dr} = 4\pi r^2 \epsilon(r), \quad (8a)$$

$$G_{rr} = 8\pi T_{rr} \implies \frac{d\nu}{dr} = \frac{m(r) + 4\pi r^3 P(r)}{r(r - 2m(r))}, \quad (8b)$$

$$\nabla^\beta T_{a\beta} = 0 \implies \frac{dP}{dr} = -(\epsilon(r) + P(r)) \frac{d\nu}{dr} \quad (8c)$$

- ★ Three ODEs - Four unknowns: $m(r) = \frac{r}{2} (1 - e^{-2\lambda(r)})$, $\epsilon(r)$, $P(r)$, $\nu(r)$
- ★ Supplementary equation? Barotropic EoS $\epsilon = \epsilon(P)$ that correlates the thermodynamic variables $\epsilon(r)$ and $P(r)$
- Numerical solution $\rightarrow M - R_{eq}$ curve

Tables of Cold EoS models

Table: Hadronic cold EoS models.

EoS	Model	Matter	M_{\max}	M_{\odot}	$R_{M_{\max}}$ [km]	$R_{1.4M_{\odot}}$ [km]
RG(SLY2)	EI-CEF-Schurme	n, p, e, μ	2.06		10.06	11.79
RG(SKb)	EI-CEF-scyrme	n, p, e, μ	2.20		10.58	12.21
RG(SkMp)	EI-CEF-scyrme	n, p, e, μ	2.11		10.60	12.50
RG(SLY9)	EI-CEF-scyrme	n, p, e, μ	2.16		10.65	12.47
RG(Skl3)	EI-CEF-scyrme	n, p, e, μ	2.25		11.34	13.55
RG(KDE0v)	EI-CEF-scyrme	n, p, e, μ	1.97		9.62	11.42
RG(SK255)	EI-CEF-scyrme	n, p, e, μ	2.15		10.84	13.15
RG(Rs)	EI-CEF-scyrme	n, p, e, μ	2.12		10.76	12.93
RG(Skl5)	EI-CEF-scyrme	n, p, e, μ	2.25		11.47	14.08
RG(SKa)	EI-CEF-scyrme	n, p, e, μ	2.22		10.82	12.92
RG(SkOp)	EI-CEF-scyrme	n, p, e, μ	1.98		10.16	12.13
RG(SLY230a)	EI-CEF-scyrme	n, p, e, μ	2.11		10.18	11.83
RG(SKI2)	EI-CEF-scyrme	n, p, e, μ	2.17		11.25	13.48
RG(Skl4)	EI-CEF-scyrme	n, p, e, μ	2.18		10.66	12.38
RG(Skl6)	EI-CEF-scyrme	n, p, e, μ	2.20		10.71	12.49
RG(KDE0v1)	EI-CEF-scyrme	n, p, e, μ	1.98		9.71	11.63
RG(SK272)	EI-CEF-scyrme	n, p, e, μ	2.24		11.20	13.32
RG(SLY4)	EI-CEF-scyrme	n, p, e, μ	2.06		10.02	11.70
GDTB(DDH δ)	RMF	n, p, e	2.16		11.19	12.58
DS(CMF)-2	SU(3)-RMF	n, p, e	2.13		11.96	13.70
DS(CMF)-4	SU(3)-RMF	n, p, e	2.05		11.60	13.26
DS(CMF)-6	SU(3)-RMF	n, p, e	2.11		11.58	13.30
DS(CMF)-8	SU(3)-RMF	n, p, e, Δ^-	2.09		11.59	13.30
BL(chiral)_2018	chPT-BBG-BHF	n, p, e, μ	2.08		10.26	12.31

Tables of Cold EoS models

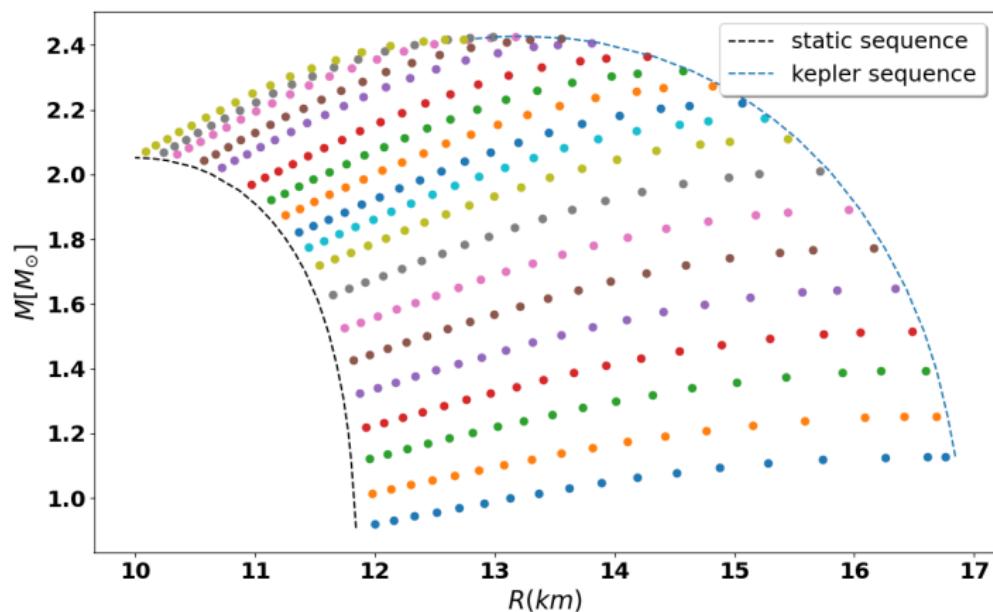
Table: Hyperonic cold EoS models.

EoS	Model	Matter	M_{\max}	M_{\odot}	$R_{M_{\max}}$ [km]	$R_{1.4M_{\odot}}$ [km]
OPGR(DDH δ Y4)	RMF	$n, p, e, H = [\Lambda, \Xi^-]$	2.05		11.26	12.58
OPGR(GM1Y5)	RMF	$n, p, e, H = [\Lambda, \Xi^-, \Xi^0]$	2.12		12.31	13.78
OPGR(GM1Y6)	RMF	$n, p, e, H = [\Lambda, \Xi^-, \Xi^0]$	2.29		12.13	13.78
DNS	SU(3)-CMF	$n, p, e, \mu, H = [\Lambda, \Sigma^-]$	2.10		12.00	13.58
DS(CMF)-1	SU(3)-CMF	$n, p, e, H = [\Lambda, \Sigma^-]$	2.07		11.88	13.57
DS(CMF)-3	SU(3)-CMF	$n, p, e, H = [\Lambda, \Sigma^-]$	2.00		11.56	13.15
DS(CMF)-5	SU(3)-CMF	$n, p, e, H = [\Lambda, \Sigma^-]$	2.07		11.43	13.20
DS(CMF)-7	SU(3)-CMF	$n, p, e, H = [\Lambda, \Sigma^-, \Delta^-]$	2.07		11.43	13.20

Table: Hybrid: Quark-Hadron cold EoS models.

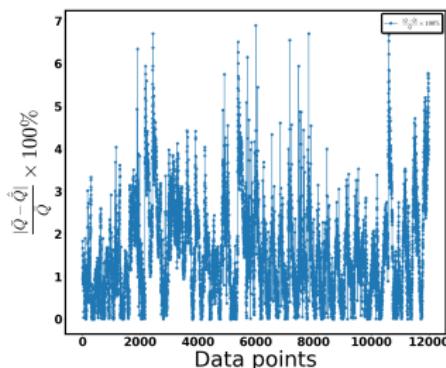
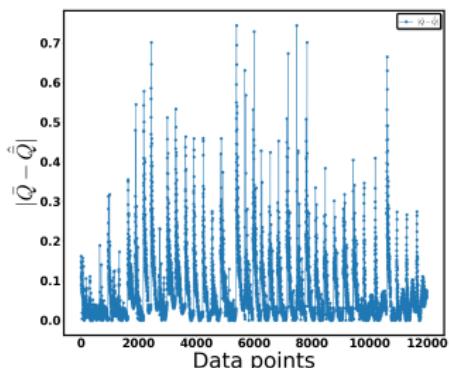
EoS	Model	Matter	M_{\max}	M_{\odot}	$R_{M_{\max}}$ [km]	$R_{1.4M_{\odot}}$ [km]
OOS(DD2-FRG) (2) flavors	NP-FRG	n, p, e, q	2.05		12.55	13.20
OOS(DD2-FRG) vec int-(2) flavors	NP-FRG	n, p, e, q	2.14		12.70	13.20
BHK(QHC18)	NJL-MF	n, p, e, q	2.05		10.41	11.49
BFH(QHC19-B)	NJL-MF	n, p, e, q	2.07		10.60	11.60
BFH(QHC19-C)	NJL-MF	n, p, e, q	2.18		10.80	11.60
BFH(QHC19-D)	NJL-MF	n, p, e, q	2.28		10.90	11.60

EoS SLy4: non-rotating vs rotating sequences vs Keplerian



LOOCV metrics and $\bar{Q}(\mathcal{C}, \sigma)$ analytical form

- Indicative plots for evaluation metrics at LOOCV



- Best functional form that describes the data:

$$\begin{aligned}\bar{Q}(\mathcal{C}, \sigma) = & -50058.667189\mathcal{C}^5 + \mathcal{C}^4(61273.085058 - 5186.192707\sigma) \\ & - \mathcal{C}^3(-604.254218\sigma^2 + 5564.525930\sigma - 30342.856057) \\ & - \mathcal{C}^2(-93.568604\sigma^3 + 618.572725\sigma^2 - 2387.15185\sigma + 7768.101816) \\ & + C(-13.503803\sigma^4 + 76.965321\sigma^3 - 225.548462\sigma^2 + 500.949545\sigma - 1069.900205) \\ & + 1.4456\sigma^5 - 0.096154\sigma^4 - 10.955132\sigma^3 + 27.137482\sigma^2 - 43.993096\sigma + 68.027506\end{aligned}$$