Compact Objects in Gravity Theories

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Outline

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In collaboration with:

Theodoros Nakas

Black Holes in General Relativity and the No-Hair Theorem

- Black holes in General Relativity may be described only by three physical quantities: Mass, E/M charge and Angular Momentum.

- Black holes are very special objects: Two stars with the same mass are, in general, very different, but two black holes with the same characteristics (M, Q and J) will be identical.

- *No hair theorems:* Uniqueness theorems which state that in General Relativity only four possible solutions for black holes may exist.

No-Scalar Hair Theorem

Adding new matter/energy forms in the theory could lead to new black holes solutions?

The simplest form is a Scalar field coupled to the gravitational field:

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \right].$$

Assumptions:

- Asymptotically flatness,
- The scalar field has the same symmetries with the spacetime,
- $V(\Phi) > 0$.
- Minimal coupling.

Under these assumptions black holes with scalar hair do not exist [1].

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[1] J. D. Bekenstein, Phys. Rev. Lett. **28** (1972) 452

J. D. Bekenstein, Phys. Rev. D **51** (1995) no.12 R6608

Black hole solutions in Modified Gravity

If we break the assumptions we may find black hole solutions

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) + \alpha \mathcal{L}_i \left(g_{\mu\nu}, \Phi \right) \right],$$

The $\mathcal{L}_i$ term usually contains non-minimal couplings

For example the EsGB theory accepts asymptotically flat black hole solutions for $V(\Phi) < 0$. [2]

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) + \alpha f(\Phi) R^2_{GB} \right],$$

If we switch off the $\mathcal{L}_i$ term ($\alpha \to 0$), the background solution is not the Schwarzschild but instead depends on the potential $V(\Phi)$.

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[2] A. B, P. Kanti and N. Pappas, Phys. Rev. D **101** (2020) no.8, 084059 (arXiv:2003.02473).

The field equations

We assume a spherically symmetric form for the line-element:

$$ds^2 = -e^{A(r)} B(r) \, dt^2 + \frac{dr^2}{B(r)} + r^2 \big(d\theta^2 + \sin^2 \theta \, d\varphi^2 \big) \,,$$

The field equations are

$$A'(r) = \frac{r}{2} \left[\Phi'(r) \right]^2 \,,$$

$$B''(r) + \frac{3}{2} A'(r) B'(r) + \left\{ A''(r) + \frac{A'(r)}{r} + \frac{[A'(r)]^2}{2} - \frac{2}{r^2} \right\} B(r) = -\frac{2}{r^2} \,,$$

$$V(\Phi) = \frac{2}{r^2} - \frac{2}{r} A'(r) B(r) - \frac{2B(r)}{r^2} + \frac{1}{2} \left[\Phi'(r) \right]^2 B(r) - \frac{2B'(r)}{r} \,.$$
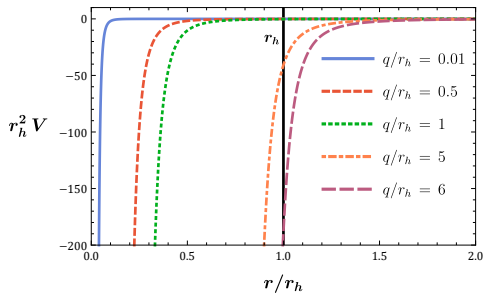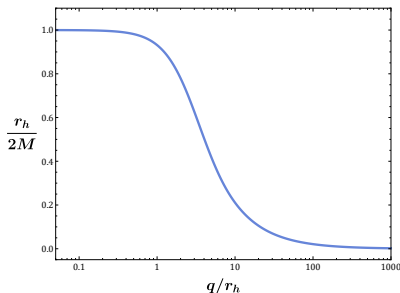
Black holes with a Coulombic scalar field

We assume a Coulombic form for the scalar field $\Phi(r) = \dfrac{q}{r}$ and we find the solution[3]:

$$A(r) = -\frac{q^2}{4r^2}, \qquad B(r) = 1 - \frac{2m(r)}{r}$$

$$m(r) = \frac{r}{2} + \frac{4r^3}{q^2} + \frac{e^{\frac{q^2}{8r^2}} r^2}{q^2}\left[-12M + \sqrt{2\pi}\, q \operatorname{erf}\left(\frac{q}{2\sqrt{2}\, r}\right)\right]$$

$$- \frac{e^{\frac{q^2}{4r^2}} r^3}{q^3}\left\{ 4q - 12\sqrt{2\pi}\, M \operatorname{erf}\left(\frac{q}{2\sqrt{2}\, r}\right) + \pi q \left[\operatorname{erf}\left(\frac{q}{2\sqrt{2}\, r}\right)\right]^2\right\},$$

$$V(\Phi) = \frac{2(24 + \Phi^2)}{q^2} - \frac{12\,\Phi\, e^{\Phi^2/8}}{q^3}\left[12M - \sqrt{2\pi}\, q \operatorname{erf}\left(\frac{\Phi}{2\sqrt{2}}\right)\right]$$

$$+ \frac{(\Phi^2 - 12)e^{\Phi^2/4}}{q^3}\left\{ 4q - 12\sqrt{2\pi}\, M \operatorname{erf}\left(\frac{\Phi}{2\sqrt{2}}\right) + \pi q \left[\operatorname{erf}\left(\frac{\Phi}{2\sqrt{2}}\right)\right]^2\right\}.$$

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[3]A. B. and T. Nakas, JHEP **04** (2022), 096 (arXiv:2107.05656).

Black holes with Coulombic scalar field

For small values of the ratio q/r_h, the fraction $r_h/(2M)$ is equal to unity and therefore $r_h = 2M$ as in the Schwarzschild geometry.

As the value of q/r_h increases, the value of $r_h/(2M)$ decreases leading to ultra-compact black holes.

Black holes with normal and phantom scalar field

We may assume a form for the metric function A and by solving the first field equation we may find scalar field:

$$A(r) = -\xi \, \ln\left(\frac{1 + r^2/q^2}{r^2/q^2}\right) \,, \qquad \Phi(r) = 2\sqrt{\xi} \, \ln\left(\frac{1 + \sqrt{1 + r^2/q^2}}{r/q}\right) \,.$$

$$B(r) = (r/q)^{2(1-\xi)} \left(1 + r^2/q^2\right)^{\xi} \left[C_1 + 2 \int (r/q)^{3\xi-2} \left(1 + r^2/q^2\right)^{-3\xi/2} H(r) \, d(r/q)\right]$$
$$+ H(r) \left[C_2 - 2 \int (r/q)^{\xi} \left(1 + r^2/q^2\right)^{-\xi/2} \, d(r/q)\right] \,,$$

where

$$H(r) \equiv \left\{ \begin{array}{ll} (r/q)^{2(1-\xi)} \left(1 + r^2/q^2\right)^{\xi} \int (r/q)^{\xi-4} \left(1 + r^2/q^2\right)^{-\xi/2} \, d\left(r/q\right), & \xi \in \mathbb{Z} \\[3mm] (r/q)^{-(1+\xi)} \dfrac{\left(1 + r^2/q^2\right)^{\xi}}{\xi - 3} \, {}_2F_1\left(\dfrac{\xi}{2}, \dfrac{\xi-3}{2}; \dfrac{\xi-1}{2}; -\dfrac{r^2}{q^2}\right), & \xi \notin \mathbb{Z} \end{array} \right\} \,.$$
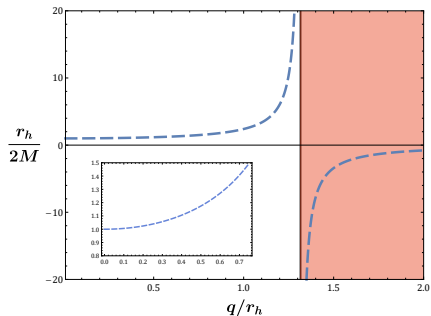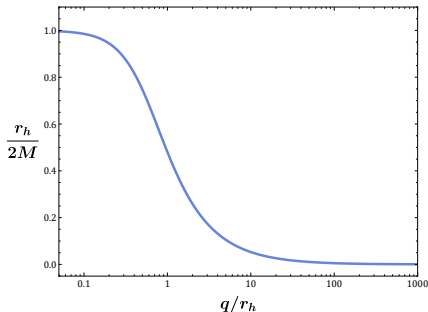
For $\xi = 5$ we find a normal solution

$$B(r) = \left(1 + \frac{r^2}{q^2}\right)^2 \left[\frac{q^4}{r^4} + \frac{3q^6}{r^6} + \frac{17}{9}\frac{q^8}{r^8} - \frac{2M}{q}\frac{q^8}{r^8}\left(1 + \frac{r^2}{q^2}\right)^{3/2}\right],$$

$$V(\Phi) = \frac{\sinh^6\left(\frac{\Phi}{2\sqrt{5}}\right)}{18q^3}\left\{121q + \cosh\left(\frac{2\Phi}{\sqrt{5}}\right)\left[17q - 18M\left|\coth\left(\frac{\Phi}{2\sqrt{5}}\right)\right|\right]\right.$$
$$\left. - 54M\left|\coth\left(\frac{\Phi}{2\sqrt{5}}\right)\right| + 6\cosh\left(\frac{\Phi}{\sqrt{5}}\right)\left[17q - 12M\left|\coth\left(\frac{\Phi}{2\sqrt{5}}\right)\right|\right]\right\}.$$

For $\xi = -2$ we find a phantom solution

$$B_{\mathfrak{p}}(r) = \left(1 + \frac{r^2}{q^2}\right)^{-2}\left[\frac{r^4}{q^4} - \frac{2M}{q}\frac{r}{q}\left(\frac{3}{5} + \frac{r^2}{q^2}\right) - \frac{1}{3}\right],$$

$$V_{\mathfrak{p}}(\tilde{\Phi}) = \frac{\tanh^5\left(\frac{\tilde{\Phi}}{2\sqrt{2}}\right)}{15q^3\cosh\left(\frac{\tilde{\Phi}}{2\sqrt{2}}\right)}\left[48M + 45q\sinh\left(\frac{\tilde{\Phi}}{2\sqrt{2}}\right) + 5q\sinh\left(\frac{3\tilde{\Phi}}{2\sqrt{2}}\right)\right].$$

In both cases as q/r_h approaches zero, the ratio $r_h/(2M)$ goes to unity and therefore $r_h = 2M$ as in the Schwarzschild case.

For the normal solutions as the value of q/r_h increases, the value of $r_h/(2M)$ decreases leading to ultra-compact black holes.

For the phantom solutions as the value of q/r_h increases, so does the value of $r_h/(2M)$.

Unlike ultra-compact solutions, phantom scalar fields result in black holes which are less dense compared to the corresponding Schwarzschild black holes of the same mass.

Solutions with slow rotation

In order to find slow rotating solutions we use the following metric

$$ds^2 = -e^{A(r)} B(r)\, dt^2 + \frac{dr^2}{B(r)} + r^2 \left\{ d\theta^2 + \sin^2\theta\, [d\varphi - \varepsilon\, \omega(r)\, dt]^2 \right\}.$$

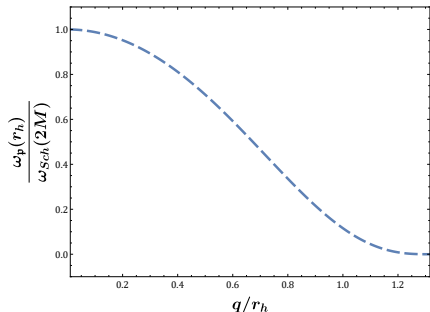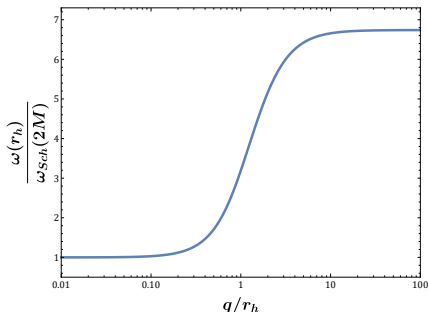By using the background static solution we find

$$\omega(r) = \omega_0 - \frac{6Jr^{\xi-3}}{q^\xi(\xi-3)}\, {}_2F_1\left(\frac{\xi-3}{2}, \frac{\xi}{2}; \frac{\xi-1}{2}; -\frac{r^2}{q^2}\right).$$

- For $\xi = 5$

$$\omega(r) = \frac{2J}{r^3}\left(1 + \frac{q^2}{r^2}\right)^{-3/2}.$$

- For $\xi = -2$

$$\omega_{\mathfrak{p}}(r) = \frac{2J}{r^3}\left(1 + \frac{3q^2}{5\,r^2}\right).$$

In both cases as q/r_h approaches zero, the angular velocity ratio goes to unity and therefore the angular velocities are equal to the Schwarzschild case.

Normal black holes rotate faster that the corresponding Schwarzschild black holes.

On the contrary the angular velocity of a given phantom black hole is consistently smaller than the corresponding angular velocity of a Schwarzschild black hole with the same mass.

On a Classical Mechanics point of view an increase in the horizon radius corresponds to an increase in its moment of inertia, which in its turn makes the rotation of such an object harder.

Stability under axial perturbations

We consider small perturbations under the background of the static black hole

$$g_{\mu\nu}^{tot} = g_{\mu\nu} + h_{\mu\nu}\,, \qquad \Phi^{tot} = \Phi + \delta\Phi\,.$$

The perturbations are classified into two distinct categories based on their parity properties: axial perturbations with *odd* parity $(-1)^{L+1}$, and polar perturbations exhibiting *even* parity $(-1)^L$.
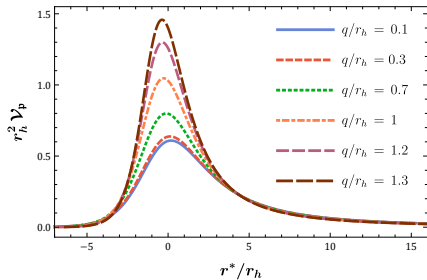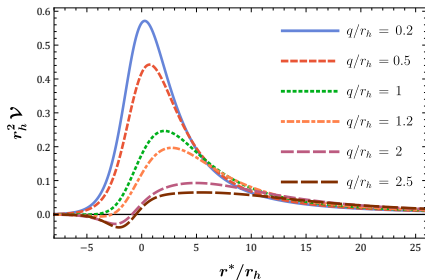
For the axial perturbations $\delta\Phi = 0$.

The perturbations are described by a Schrödinger-like differential equation

$$\frac{d^2\Psi(r^*)}{dr^{*2}} + \left[k^2 - \mathcal{V}(r)\right]\Psi(r^*) = 0\,,$$

where the potential is determined by

$$\mathcal{V} = \frac{e^A B}{2r}\left\{B'\left(3rA' - 2\right) + B\left[2r\left(A'' + \Phi'^2\right) + rA'^2 - 3A'\right] + 2rB'' + \frac{2L(L+1)}{r}\right\}\,.$$

Unstable modes correspond to bound states in the Schrödinger-like equation .

In a region where the potential is negative-definite and has the shape of a well there is nothing which prevents a bound state to exist.

By purely observing the form of the graphs, one can promptly deduce that normal solutions with $\xi = 5$ are stable for $q/r_h \lesssim 1$.

The phantom solutions with $\xi = -2$ proves to be stable $\forall \frac{q}{r_h} \in (0, \sqrt[4]{3})$.

Conclusions

- Black holes may be constructed using a potential engineering method.

- Depending on the type of the scalar field the resulting solutions may describe either ultra-compact or ultra-sparse black holes.

- The solutions may be always generalized into slowly rotating.

- The solutions are characterized by the mass of the black hole M, the angular momentum J and a scalar charge q of secondary type.

- For every solution the sign of the kinetic term is opposite than the sing of the potential.

Thank You!

The stability equation

In the Regge-Wheeler gauge the axial perturbations are

$$
h_{\mu\nu}^{\text{odd}} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{bmatrix} e^{-ikt} \sin\theta\, \partial_\theta P_L(\cos\theta)\,, \qquad \delta\Phi = 0\,.
$$

We may show that perturbation h_0 is not independent.

Under the redefinition

$$
h_1(r) = \frac{r\,\Psi(r)}{B(r)\,e^{A(r)/2}}\,,
$$

and by using the tortoise coordinate

$$
dr^* = dr\,e^{-A(r)/2}/B(r)
$$

we may easily write the equation for h_1 in a a Schrödinger-like form

$$
\frac{d^2\Psi(r^*)}{dr^{*2}} + \left[k^2 - \mathcal{V}(r)\right]\Psi(r^*) = 0\,.
$$

The Horndeski Theory

The more general scalar-tensor theory in four dimensions with second order equations of motion:

$$\mathcal{S} = \int d^4x \sqrt{-g}\,[\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5]$$

with:

$$\mathcal{L}_2 = G_2(\phi, X),$$

$$\mathcal{L}_3 = G_3(\phi, X)\nabla^2\phi, \qquad\qquad\qquad X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi,$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}\left[(\nabla^2\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right],$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}\left[(\nabla^2\phi)^3 - 3(\nabla^2\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].$$

If we make the choices:

$$G_2 = -X + 8f^{(4)}X^2(3 - \ln X), \quad G_3 = -4f^{(3)}X(7 - 3\ln X),$$

$$G_4 = 1 + 4\ddot{f}X(7 - \ln X), \quad G_5 = -4f\ln X,$$

We get the Einstein Scalar Gauss-Bonnet theory