Aspects of Relativistic and Carrollian fluids

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Fluids and holography

- Relativistic fluids as states of finite (T, μ) of a bnr CFT in the hydrodynamic regime Hubeny, Minwalla, Rangamani '11
- The fluids at the horizon and the bnr are connected through the holographic RG Emparan, Hubeny, Rangamani '13; Kuperstein, Mukhopadhyay '13

Can we connect the bnr fluid with the bulk solution?

From bnr to bulk

- A radial Hamiltonian (ADM) evolution from the time-like bnr $\mathcal{B}$ at $r \longrightarrow \infty$
- Two data, the bnr metric g_{bnr} and the symmetric, traceless, conserved $\mathrm{T}_{\mu\nu}$

$$T_{\mu\nu} = T_{\nu\mu}\,, \quad T^{\mu}{}_{\mu} = 0\,, \quad \nabla_{\mu} T^{\mu}{}_{\nu} = 0$$

Introduction and motivation

From bulk to the bnr

1. Consider a 4d asymptotically (locally) AdS_4 space with $R = -12k^2$

2. We can obtain its bnr data through the Fefferman–Graham (FG) gauge

$$\mathrm{d}s^2_{\text{bulk}} = \frac{\mathrm{d}r^2}{k^2 r^2} + k^2 r^2 \mathrm{d}s^2_{\text{bnr}} + \cdots + \frac{16\pi G}{3k^2 r} \mathrm{T}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \cdots$$

Fefferman–Graham '85, '07

3. The holographic r-coordinate is read perturbatively in the FG coordinate gauge

4. The additional pieces depend on the bnr data, e.g. curvature terms

5. This is an on-shell expansion

A SIMPLE EXAMPLE

Consider the AdS-Schwarzschild space

$$\mathrm{d}s^2_{\mathrm{bulk}} = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)$$

where

$$f(r) = 1 + k^2r^2 - \frac{2GM}{r}$$

- The bnr metric reads: $\mathrm{d}s^2_{\mathrm{bnr}} = -k^2\mathrm{d}t^2 + \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)$
- The conformal energy–momentum tensor is of the perfect form:

$$\mathrm{T}^{\mathrm{perf}}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = p\left(3\mathrm{u}^2 + \mathrm{d}s^2_{\mathrm{bnr}}\right), \quad \mathrm{u} = -\mathrm{d}t, \quad \varepsilon = 2p = \frac{Mk^2}{4\pi G}$$

- The bulk metric obeys Einstein's equations $R_{\mu\nu} = -3k^2g_{\mu\nu}$
- k appears as the velocity of light in the boundary

Question can we take the limit of $k \longrightarrow 0$? Is it meaningful?

THE CARROLL LIMIT

Lorentzian transformations $(t, x) \mapsto (t', x')$

$$ct' = \gamma(ct - \frac{v}{c}x)\,, \quad x' = \gamma(x - vt)\,, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

- Galilean group is a contraction of the Poincare group, where $c \longrightarrow \infty$

$$\text{Absolute time} \quad t' = t \quad x' = x - vt$$

- Carrollian group is a contraction of the Poincare group, where $c \longrightarrow 0$
  Lévy-Leblond 65', Sen Gupta 66'

$$\text{Absolute space} \quad t' = t - \frac{x}{\tilde{v}}, \quad x' = x\,, \quad \text{where} \quad v = \frac{c^2}{\tilde{v}}$$

- Equivalently the Lorentz boosts $L_i = ct\partial_i + \frac{x_i}{c}\partial_t$ in these two limits equal to

$$G_i = \lim_{c\to\infty} \frac{L_i}{c} = t\partial_i \quad \text{and} \quad C_i = \lim_{c\to 0} cL_i = x_i\partial_t\,.$$

As the Galilean limit, its Carroll analogue is also of the non-relativistic type.

Asymptotic flatness

- BMS group is the asymptotic isometry group of asymptotically flat spacetimes
  H. Bondi, M. van der Burg, A. Metzner, R. Sachs '62
- It can be viewed that $\mathrm{BMS}_d \equiv \mathrm{ccarr}(d-1)$ conformal isometry group
  C. Duval, G. W. Gibbons & P. A. Horváthy 14'
- If flat holography exists, the dual FT should lie on a Carrollian spacetime
- Carrollian dynamics emerges in asymptotically flat spacetimes
  L. Ciambelli, C. Marteau, A. C. Petkou, P. M. Petropoulos & KS '18; Penna '18; A. Bagchi, S. Chakrabortty, D. Grumiller, B. Radhakrishnan, M. Riegler & A. Sinha '21

Focal Points

In this talk we will focus on:

1. Revisit of the relativistic hydrodynamics.
2. Carroll fluids from covariance.
3. But also from a non-relativistic limit.
4. Conclusions and Outlook

Plan of the talk

Revisit of the relativistic hydrodynamics

Carrolian fluid dynamics

Energy–momentum tensor

Without external forces the (neutral) fluid equations are:

$$\nabla_\mu T^{\mu\nu} = 0 \,, \quad \mu = 0, 1, \ldots, d$$

accompanied by a metric $g_{\mu\nu}$.

It is decomposed along a velocity field u^μ with $u_\mu u^\mu = -c^2$ as

$$T^{\mu\nu} = \frac{\varepsilon + p}{c^2} u^\mu u^\nu + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{1}{c^2}(u^\mu q^\nu + u^\nu q^\mu) \,,$$

where the viscous tensor $\tau^{\mu\nu}$ & the heat current q^μ (non-perfect e-m) obey

$$u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu \,, \quad \varepsilon = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu \,, \quad u^\mu q_\mu = 0 \,, \quad u^\mu \tau_{\mu\nu} = 0$$

How we derive the energy–momentum tensor

Energy–momentum conservation

Let's start from the action

$$S = \int d^{d+1}x \sqrt{-g}\,\mathcal{L}$$

the energy–momentum tensor can be defined as usual

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \,.$$

Demanding invariance under diffeomorphisms $x^\mu \to x^\mu - \xi^\mu$

$$\begin{aligned}
\delta_\xi S = \int d^{d+1}x \sqrt{-g}\, \delta_\xi g_{\mu\nu} T^{\mu\nu} &= -2 \int d^{d+1}x \sqrt{-g}\, \nabla_\mu \xi_\nu T^{\mu\nu} \\
&= -2 \int d^{d+1}x \sqrt{-g}\, \left(\nabla_\mu (\xi_\nu T^{\mu\nu}) - \xi_\nu \nabla_\mu T^{\mu\nu} \right) = 0
\end{aligned}$$

leading to the conservation of $T^{\mu\nu}$

If ξ^μ is a Killing vector ($\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$), we can define the conserved current

$$\nabla_\mu (\xi_\nu T^{\mu\nu}) = 0$$

also for a conformal Killing vector ($\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \propto g_{\mu\nu}$) provided that $T_\mu{}^\mu = 0$

We can extend the above introducing Weyl covariance: $g_{\mu\nu} \to \Omega^{-2} g_{\mu\nu}$ Appendix

Randers–Papapetrou coordinates

In a pseudo-Riemannian spacetime we can always write it in the form

$$ds^2 = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij}dx^i dx^j\,, \quad i = 1, 2, \ldots, d\,,$$

with (Ω, b_i, a_{ij}) functions of $(t, \mathbf{x})$.

These coordinates are well adapted for the Carrollian limit $c \to 0$, since

$$\boxed{t' = t'(t, \mathbf{x})\,, \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})}$$

reduce to

$$\Omega' = \frac{\Omega}{J}, \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}{}_k\,, \quad a'^{ij} = J^i{}_k J^j{}_l a^{kl}\,,$$

with Jacobian factors

$$J = \frac{\partial t'}{\partial t}\,, \quad j_i = \frac{\partial t'}{\partial x^i}\,, \quad J^i{}_j = \frac{\partial x'^i}{\partial x^j}\,.$$

Carrollian diffeomorphisms are generated such that $\xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$

Plan of the talk

Revisit of the relativistic hydrodynamics

Carrolian fluid dynamics

Carrolian covariance

Let us define a manifold $\mathcal{M} = \mathbb{R} \times S$ with coordinates $(t, \mathbf{x})$ equipped with

$$ds^2 = a_{ij} dx^i dx^j, \quad i = 1, \ldots, d$$

as well as the field of observers $\mathrm{e}_{\hat{t}}$ and the clock form $\theta^{\hat{t}}$

$$\mathrm{e}_{\hat{t}} = \frac{1}{\Omega}\partial_t\,, \quad \theta^{\hat{t}} = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$$

Properties:

- Invariance of $\mathrm{e}_{\hat{t}}, ds^2$ and $\theta^{\hat{t}}$ under Carrollian diffs: $t' = t'(t, \mathbf{x})\,,\ \mathbf{x}' = \mathbf{x}'(\mathbf{x})$.
- We introduce a new spatial derivative which transforms as usual

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega}\partial_t\,, \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j$$

  where we may also define the Carrollian vorticity and acceleration through

$$\left[\hat{\partial}_i, \hat{\partial}_j\right] = \frac{2}{\Omega}\varpi_{ij}\partial_t\,, \quad \varpi_{ij} = \partial_{[i} b_{j]} + b_{[i}\varphi_{j]}\,, \quad \varphi_i = \frac{1}{\Omega}\left(\partial_t b_i + \partial_i \Omega\right)$$

- Also its covariant version $\hat{\nabla}_i a_{jk} = 0\,, \quad \hat{\gamma}^i{}_{jk} = \frac{a^{il}}{2}\left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}\right)$
- Finally, we introduce a temporal one

$$\hat{D}_t \Phi = \partial_t \Phi\,, \quad \frac{1}{\Omega}\hat{D}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i{}_j V^j, \quad \hat{\gamma}^i{}_j = \frac{1}{2\Omega}a^{ik}\partial_t a_{kj}$$

Carrollian diffeomorphisms

Let us consider the action functional of Ω, b_i and a_{ij}

$$S = \int d^{d+1}x\, \Omega\sqrt{a}\mathcal{L}$$

and define the Carrollian momenta (energy, current and stress-tensor)

$$\Pi = -\frac{1}{\Omega\sqrt{a}}\left(\Omega\frac{\delta S}{\delta\Omega} + b_i\frac{\delta S}{\delta b_i}\right), \quad \Pi_i = \frac{1}{\Omega\sqrt{a}}\frac{\delta S}{\delta b_i}, \quad \Pi^{ij} = \frac{2}{\Omega\sqrt{a}}\frac{\delta S}{\delta a_{ij}}$$

Varying the action under Carrollian diffeomorphisms $\xi = \xi^t(t,\mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$

A. Petkou, P. Petropoulos, D. Rivera-Betancour, KS '22; L. Ciambelli, C. Marteau '19

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\Pi + \left(\hat{\nabla}_i + 2\varphi_i\right)\Pi^i + \Pi^{ij}\hat{\gamma}_{ij} = 0\,,$$

$$\left(\hat{\nabla}_j + \varphi_j\right)\Pi^j{}_i + 2\Pi^j\varpi_{ji} + \Pi\varphi_i + \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)P_j + P^i\hat{\gamma}_{ij} = 0\,, \quad \theta = \frac{1}{\Omega}\partial_t\ln\sqrt{a}$$

Remarks:

- The P_i's result as a temporal boundary term, where we recall that $\xi^i = \xi^i(\mathbf{x})$

$$\Omega\sqrt{a}\xi^i \times \text{red term} = \partial_t(\sqrt{a}\xi^i P_i).$$

- A Carrollian Killing does not imply generically conservation Appendix

Carrollian hydro as a non-relativistic limit

Energy–momentum tensor admits a small-c expansion (Randers–Papapetrou frame)

$$\begin{cases} \frac{1}{\Omega^2} T_{00} = \varepsilon_{\mathrm{r}} = \Pi + \mathcal{O}\left(c^2\right), \\ -\frac{c}{\Omega} T_0{}^i = q_{\mathrm{r}}^i = \Pi^i + c^2 P^i + \mathcal{O}\left(c^4\right), \\ T^{ij} = p_{\mathrm{r}} a^{ij} + \tau_{\mathrm{r}}^{ij} = \Pi^{ij} + \mathcal{O}\left(c^2\right). \end{cases}$$

Inserting the above into the conservation equations $\nabla_\mu T^{\mu\nu} = 0$, leads to

$$\begin{cases} \frac{c}{\Omega} \nabla_\mu T^\mu{}_0 = \mathcal{E} + \mathcal{O}\left(c^2\right) = 0, \\ \nabla_\mu T^{\mu i} = \frac{1}{c^2} \left(\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi^i + \Pi^j \hat{\gamma}_j{}^i \right) + \mathcal{G}^i + \mathcal{O}\left(c^2\right) = 0, \end{cases}$$

where

$$\mathcal{E} = -\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi - \left(\hat{\nabla}_i + 2\varphi_i \right) \Pi^i - \Pi^{ij} \hat{\gamma}_{ij},$$

$$\mathcal{G}_j = \left(\hat{\nabla}_i + \varphi_i \right) \Pi^i{}_j + 2\Pi^i \varpi_{ij} + \Pi \varphi_j + \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) P_j + P^i \hat{\gamma}_{ij}.$$

- As the term on P_i, so the constraint on Π^i is a bnr term from diff perspective

$$\sqrt{a}\Omega\, \eta_i(\mathbf{x}) \times \text{blue term} = \partial_t \left(\sqrt{a}\, a_{ij} \eta^i \Pi^j \right).$$

- The limit is richer in comparison with invariance under Carrollian diffs.

Conclusion & Outlook

As we have seen

- Relativistic fluid: Important property in reconstructing Einstein's spaces $\Lambda \neq 0$
  Bulk diffs: bnr diffs, Weyl transformations and local Lorentz transformations.
  A. Mukhopadhyay, A. C. Petkou, P. M. Petropoulos, V. Pozzoli & K.S. 13';

  J. Gath, A. Mukhopadhyay, A.C. Petkou, P. M. Petropoulos and K.S. '15
- Similarly for the Carrollian fluid for reconstructing Ricci-flat spaces $\Lambda = 0$
  Bulk diffs: bnr diffs, Weyl transformations and local Carroll transformations.
  L. Ciambelli, C. Marteau, A.C. Petkou, P. M. Petropoulos, KS 18';

  A. Campoleoni, L. Ciambelli, C. Marteau, P. M. Petropoulos, KS 18'

We studied Carrollian hydrodynamics on arbitrary backgrounds:

- Our approach was based on covariance and diffeomorphism invariance.
- The limiting procedure is richer, further variables and equations.
- Compatible with diffeomorphism invariance, conjugate to new momenta.
- A richer structure is needed to make connection with flat holography.

In progress: Analyzing the Carrollian relative of the topological massive gravity

O. Mickovic, R. Olea, P.M. Petropoulos, D. Rivera-Betancour, KS

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{w_{\text{CS}}\, c^2} C_{\mu\nu} = 0$$

WEYL INVARIANCE

The system may be also invariant under Weyl transformations

$$ds^2 \to \Omega^{-2} ds^2\,, \quad u^\mu \to \Omega u^\mu\,,$$

if we introduce a Weyl connection

$$A_\mu = \frac{1}{c^2}\left(a_\mu - \frac{\Theta}{d} u_\mu\right)\,, \quad a_\mu = u^\nu \nabla_\nu u_\mu\,, \quad \Theta = \nabla_\mu u^\mu\,.$$

The Weyl covariant derivative is also metric compatible

$$\mathcal{D}_\rho g_{\mu\nu} = 0\,, \quad \mathcal{D}_\kappa f = (\partial_\kappa + w A_\kappa) f\,, \quad [\mathcal{D}_\kappa, \mathcal{D}_\lambda] f = w\, F_{\kappa\lambda} f\,, \quad F_{\kappa\lambda} = \partial_\kappa A_\lambda - \partial_\lambda A_\kappa\,.$$

The fluid dynamics is Weyl invariant provided that

$$\mathcal{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu}\,,$$

where $T^{\mu\nu}$ has conformal weight $d-1$ and it is traceless $T^\mu{}_\mu = 0$.

This leads to $\varepsilon = d\, p + \tau^\mu{}_\mu$ and conformal weights

weight	**observables**
$d+1$	ε, p
d	q_μ
$d-1$	$\tau_{\mu\nu}$

Isometries and the (non)-conservation

Killing fields of the Carrollian type satisfy

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi \mathrm{e}_{\hat{t}} = 0 \quad \Longrightarrow \quad \hat{\nabla}_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\hat{\gamma}_{ij} = 0, \quad \frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0$$

whereas the clock form $\theta^{\hat{t}}$ is not invariant.

An example $a^{ij} = \delta^{ij}, \Omega = 1$ and $b_i =$ constant with Carroll algebra $\mathfrak{carr}(d+1)$

$$\xi = \left(\Omega_i{}^j x^i + X^j\right)\partial_j + (T - B_i x^i)\partial_t \quad \Longrightarrow \quad \delta_\xi \theta^{\hat{t}} = \left(B_i + \Omega_i{}^j b_j\right)\mathrm{d}x^i \neq 0$$

Assuming an isometry, we have on-shell vanishing scalar (continuity equation)

$$\left(\frac{1}{\Omega}\partial_t + \theta\right)\kappa + \left(\hat{\nabla}_i + \varphi_i\right)K^i = 0\,, \quad \kappa = \xi^i P_i - \xi^{\hat{t}}\Pi\,, \quad K^i = \xi^j \Pi_j{}^i - \xi^{\hat{t}}\Pi^i$$

Using the energy & momentum on-shell conservation we find

$$\mathcal{K} = -\Pi^i\left(\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} - 2\xi^j \varpi_{ji}\right)$$

Comments:

- Even in flat space $\mathcal{K} = \Pi^i\left(B_i + \Omega_i{}^j b_j\right) \neq 0$, is not associated with a bnr term.
- The above construction also extends for conformal isometries

$$\mathcal{L}_\xi a_{ij} = \lambda\, a_{ij}\,, \quad \mathcal{L}_\xi \mathrm{e}_{\hat{t}} = \mu\, \mathrm{e}_{\hat{t}}\,, \quad 2\mu + \lambda = 0$$

More equations in the Carrollian limit

Let us expand the energy–momentum tensor as

$$\begin{cases} \frac{1}{\Omega^2}T_{00} = \varepsilon_{\mathrm{r}} = \frac{\tilde{\Pi}}{c^2} + \Pi + \mathcal{O}\left(c^2\right), \\ -\frac{c}{\Omega}T_0{}^i = q_{\mathrm{r}}^i = \frac{\tilde{\Pi}^i}{c^2} + \Pi^i + c^2P^i + \mathcal{O}\left(c^4\right), \\ T^{ij} = p_{\mathrm{r}}a^{ij} + \tau_{\mathrm{r}}^{ij} = \frac{\tilde{\Pi}^{ij}}{c^2} + \Pi^{ij} + \mathcal{O}\left(c^2\right) \end{cases}$$

yielding the additional equations

$$-\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi} - \left(\hat{\nabla}_i + 2\varphi_i\right)\tilde{\Pi}^i - \tilde{\Pi}^{ij}\hat{\gamma}_{ij} = 0,$$

$$\left(\hat{\nabla}_i + \varphi_i\right)\tilde{\Pi}^i{}_j + 2\tilde{\Pi}^i\varpi_{ij} + \tilde{\Pi}\varphi_j + \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi_j + \Pi^i\hat{\gamma}_{ij} = 0,$$

$$\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi}_j + \tilde{\Pi}^i\hat{\gamma}_{ij} = 0.$$

Comments:

1. The degrees of freedom are multiplied.
2. These equations can be derived using diffs by incorporating additional fields.

Hydrodynamic frame invariance

In the relativistic case the frame transformations (local Lorentz) are given through

$$\begin{aligned}
\delta\varepsilon &= -2\frac{q^i\delta\beta_i}{\sqrt{1-c^2\boldsymbol{\beta}^2}},\\
\delta q^i &= \frac{c^2\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}}\left(\frac{q^k\beta^i}{\sqrt{1-c^2\boldsymbol{\beta}^2}} - wh^{ki} - \tau^{ki}\right),\\
\delta\left(ph^{ij}+\tau^{ij}\right) &= \frac{c^2\delta\beta_k}{1-c^2\boldsymbol{\beta}^2}\left(\beta^i\left(ph^{jk}+\tau^{jk}\right)+\beta^j\left(ph^{ik}+\tau^{ik}\right)\right) - \frac{\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}}\left(q^ih^{jk}+q^jh^{ik}\right).
\end{aligned}$$

Leaving $T_{\mu\nu}$ invariant.

In the Carrollian case we find

$$\varepsilon = \eta + \mathcal{O}\left(c^2\right), \quad p = \varpi + \mathcal{O}\left(c^2\right), \quad q^i = Q^i + c^2\pi^i + \mathcal{O}\left(c^4\right), \quad \tau^{ij} = -\Xi^{ij} + \mathcal{O}\left(c^2\right),$$

with transformations

$$\delta\eta = -2\delta\beta_iQ^i, \quad \delta Q^i = 0,, \quad \delta\pi^i = \delta\beta_j\left(\Xi^{ij} - (\eta+\varpi)a^{ij} + \beta^iQ^j\right), \quad \delta\left(\Xi^{ij} - \varpi a^{ij}\right) = \delta\beta_k\left(Q^ia^{jk} + Q^ja^{ik}\right).$$

Leaving Π, P_i, Π^i and Π^{ij} invariant.