Classical simulation of short time many-body dynamics

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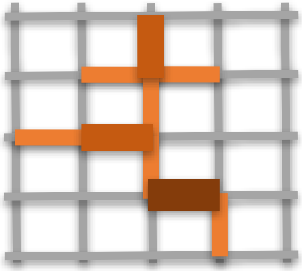
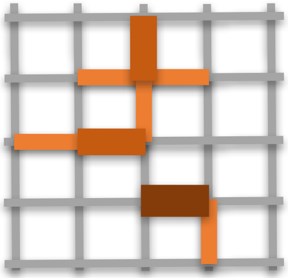
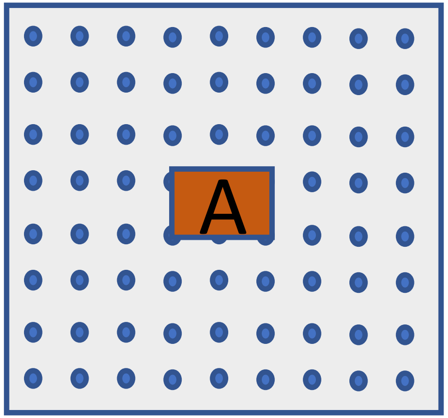
(arXiv:2210:11490)
Accepted in PRX Quantum





Simulating many-body dynamics

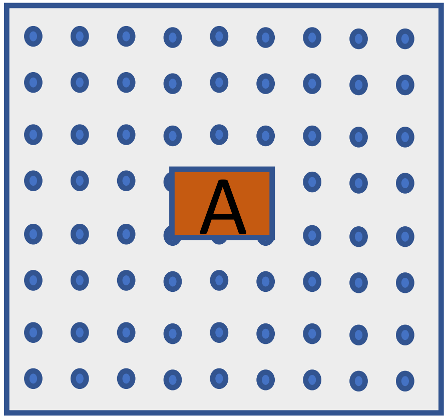
$$U = e^{-itH} \qquad H = \sum_i h_i$$



$$A(t) = e^{-iHt} A e^{iHt}$$

Simulating many-body dynamics

- Time evolution operator: $U = e^{-itH}$ $\quad H = \sum_i h_i$
- Observable: $A(t) = e^{-iHt} A e^{iHt}$



- Initial state: $|\Phi\rangle = \bigotimes_i |\phi_i\rangle$
- Goal:

$$|\langle\Phi|A(t)|\Phi\rangle - f(t)| \leq \epsilon$$

Computational problem

- Local Hamiltonian on N particles + few-body observable

$$|\langle\Phi|A(t)|\Phi\rangle - f(t)| \leq \epsilon$$

<u>Classically easy (P)</u>

$$t = \mathcal{O}(1)$$

$\cdots$

<u>Classically hard + quantum easy (BQP)</u>

$$t = \mathrm{poly}(N)$$

$\cdots$

<u>Quantum hard</u>

$$t \stackrel{??}{=} \exp(N)$$

time →

Computational problem

- Local Hamiltonian on N particles + few-body observable

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Classically easy (P)	$\cdots$	Classically hard + quantum easy (BQP)	$\cdots$	Quantum hard
$t = \mathcal{O}(1)$		$t = \text{poly}(N)$		$t \overset{??}{=} \exp(N)$

time →

How do we study this problem classically?

- Exact diagonalization (small systems)
- Tensor networks (short times, one dimension)
- Many other methods….(model-specific?)

$$A(t) = e^{-iHt} A e^{iHt}$$

- ……..
- This talk: cluster expansion ← short times, but very accurate and analytically tractable

Quantum dynamics: simple or not?

A HUGE matrix you cannot diagonalize

$$U \in (\mathbb{C}^d)^{\otimes N}$$

The exponential of a "simple" local operator

$$U = e^{-itH}$$

$$H = \sum_i h_i$$

Taylor expansion and classical computation

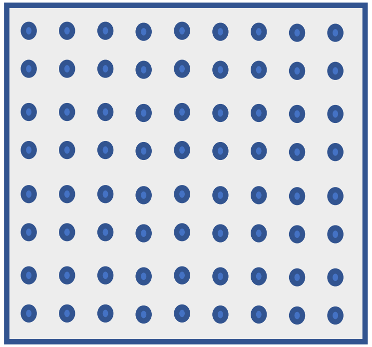
$$|F(t) - F_m(t)| \le e^{-\mathcal{O}(m)}$$

$$F(t) = \sum_{m=0}^{\infty} \frac{K_m}{m!} t^m$$

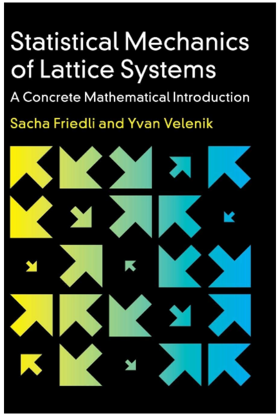
$$F(t)_M = \sum_{m=0}^{M} \frac{K_m}{m!} t^m$$

<u>Ingredients:</u>

- Prove convergence of Taylor series for high enough degree (analyticity).
- Estimate cost of calculating Taylor coefficients.
- Computed Taylor series gives approximation.



Cluster expansion: main idea



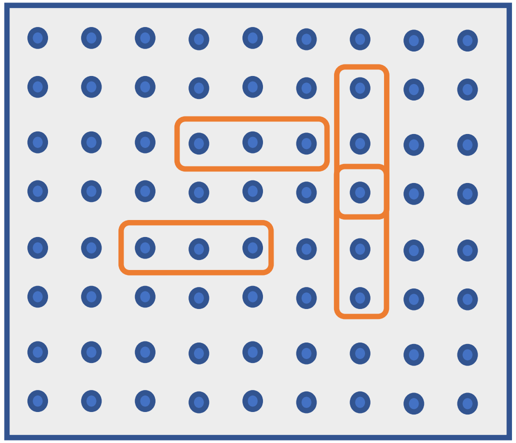
- Taylor series expansions for quantities defined on lattices.

$$\log Z \equiv \log \mathrm{Tr}[e^{-\beta H}] = \sum_m^\infty \frac{K_m^{(\beta)}}{m!}\beta^m \qquad e^{-\beta H} = \mathbb{I} - \beta H + \frac{\beta^2}{2}H^2 + \dots$$

- Find efficient way of writing & computing the Taylor moments in terms of *clusters*

$$H = \sum_X h_X \qquad K_m^{(\beta)} = \sum_{\mathbf{W}} \prod (\dots)\mathrm{Tr}[h_1 \dots . h_n]$$

$$||h_X|| \leq 1$$

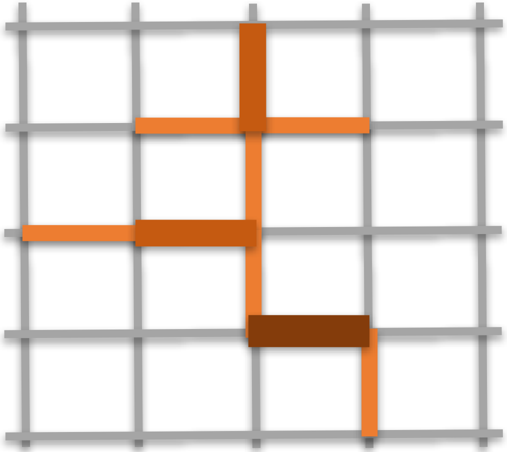


- *CLUSTER*: A multiset of Hamiltonian terms

$$\mathbf{W} = \{h_1, h_1, h_2, \dots, h_l, h_l\}$$

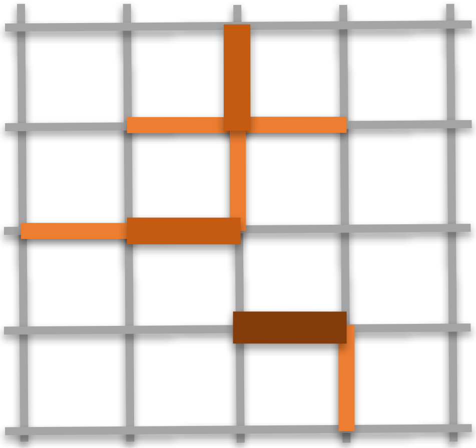
Types of clusters

- Connected



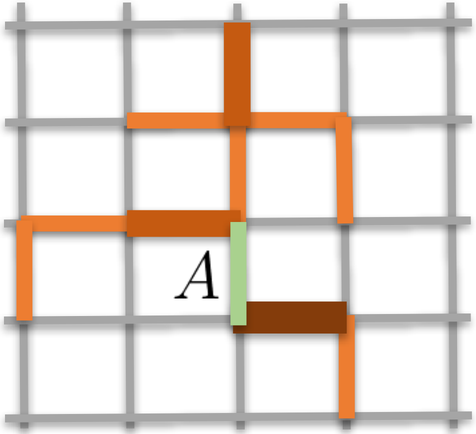
$$\mathbf{W} \in \mathcal{G}_m$$

- Disconnected



$$\mathbf{W} \in \mathcal{C}_m \setminus \mathcal{G}_m$$

- Connected to A



$$\mathbf{W} \in \mathcal{G}_m^A$$

$$\mathbf{W} = \{h_1, h_1, h_2, ..., h_l, h_l\}$$

Heisenberg time evolution

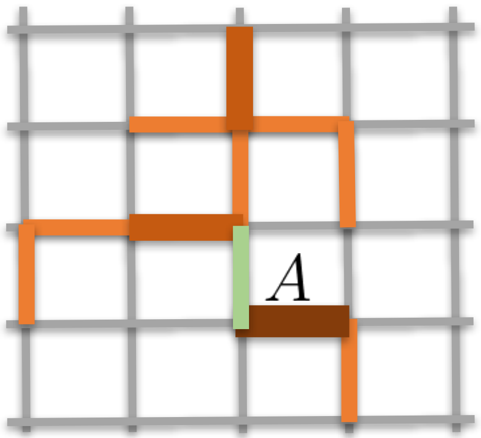
- Classical simulation of

$$\langle\Phi|e^{-itH}Ae^{itH}|\Phi\rangle = \sum_{m=0}^{\infty}\frac{(it)^m}{m!}\langle\Phi|[H,[H,...[H,A]]|\Phi\rangle$$

$$= \sum_{m=0}^{\infty}\frac{(it)^m}{m!}\sum_{\mathcal{G}_m^A}\langle\Phi|[h_{X_1},[h_{X_2},...[h_{X_m},A]]|\Phi\rangle$$

$$H = \sum_i h_i$$

- Taylor expansion in terms of *connected* clusters



Heisenberg time evolution

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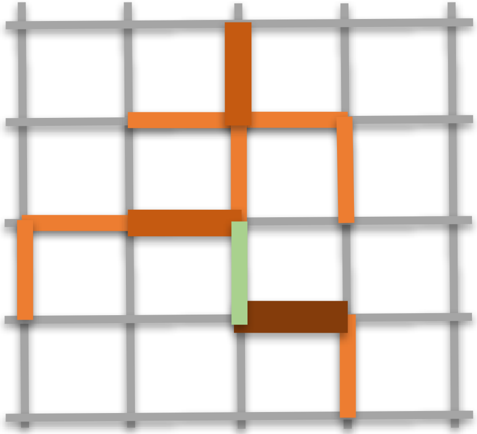
$$= \sum_{m=0}^{\infty}\frac{(it)^m}{m!}\sum_{\mathcal{G}_m^A}\langle\Phi|[h_{X_1},[h_{X_2},...[h_{X_m},A]]|\Phi\rangle$$

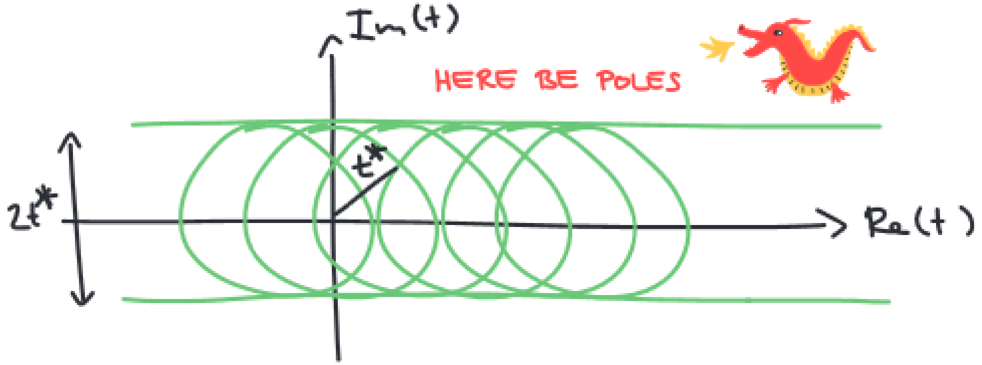
- Count connected clusters: $|\mathcal{G}_A^m| \leq e^{\mathcal{O}(m)}$
- Weight of each cluster: $m!2^m||A||$
- Taylor series (and algorithm) for short times

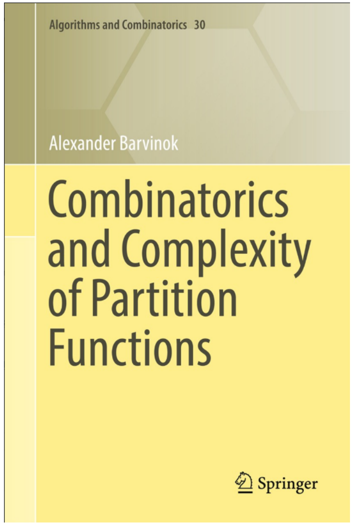
$t \leq t^*$ $$|\langle\Phi|A(t)|\Phi\rangle - \sum_{m=0}^{M}\frac{t^m}{m!}K_m| \leq \frac{(t/t^*)^M}{1-(t/t^*)}||A||$$



Arbitrary times: analytic continuation

- Function is analytic on a strip, not just a disk. $|\langle e^{-itH} A e^{itH} \rangle| \leq ||A||$
- Analytic continuation (Barvinok '16, Harrow et al. 1910.09071).
- We can use Taylor series at the origin to calculate any later point, with **overhead**.
- Idea: use series of a function that maps disk to rectangle.





Arbitrary times: main result

- Series converges, but much more slowly (exponentially badly in time)
- RESULT: for arbitrary times, there is a classical algorithm that outputs $f(t)$

$$|\langle\Phi|e^{-itH}Ae^{itH}|\Phi\rangle - f(t)| \leq \epsilon$$

- With runtime

$$\left(\exp(\mathcal{O}(t))\frac{1}{\epsilon}\right)^{\exp(\mathcal{O}(t))}$$

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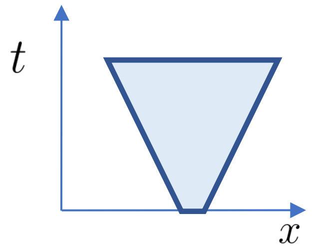
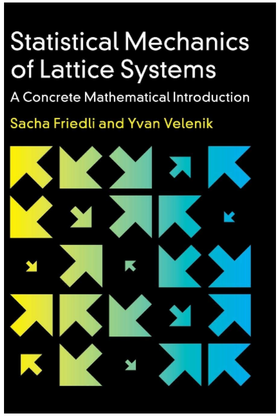
- Remark: for $t = \mathcal{O}(1)$, runtime is $\mathrm{poly}(\epsilon^{-1})$
- Previously: Lieb-Robinson lightcone method has runtime

$$e^{\mathcal{O}(l^D)} \sim e^{\mathcal{O}((vt+\log(1/\epsilon))^D)}$$



Fidelity / Loschmidt echo

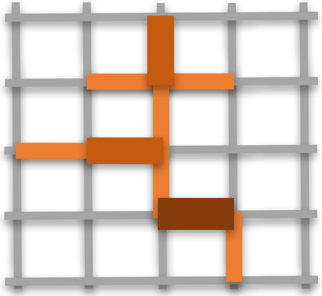


$$\log\langle\Phi|e^{-itH}|\Phi\rangle \quad \text{vs.} \quad \log[\mathrm{Tr}e^{-\beta H}]$$

- <u>KEY INSIGHT</u>: with product states also only connected clusters contribute

$$|\Phi\rangle = |\phi\rangle^{\otimes N}$$

- <u>Result</u>: classical algorithm for Loschmidt echo



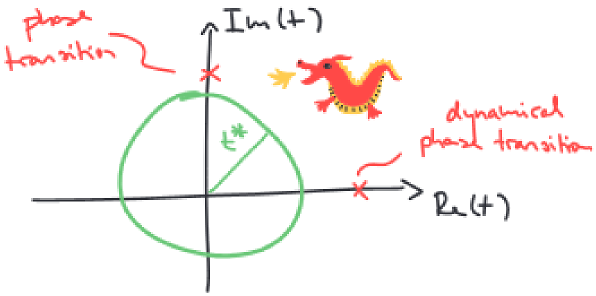
$$|\log\langle\Phi|e^{iHt}|\Phi\rangle - \sum_{m=0}^{M}\frac{t^m}{m!}K_m| \leq \epsilon \qquad t \leq t^*$$

$$M = \mathcal{O}(\log(N/\epsilon)) \qquad \boxed{\mathrm{runtime} = \mathrm{poly}(N/\epsilon)}$$

Some physical consequences:

- Result: analyticity and efficient algorithm for

$$\log\langle\Phi|e^{-itH}|\Phi\rangle \qquad t \leq t^*$$



- So it takes at least t^* to become orthogonal to initial state
- Strengthening over previous Quantum Speed Limits (Mandeltam-Tamm, Margolus-Levitin)

$$t_{QSL} \geq t^* = \mathcal{O}(1) \qquad \text{Vs.} \qquad t_{QSL} \geq \mathcal{O}(1/\sqrt{N})$$

- Dynamical phase transitions (Heyl 1709.07461): t^* is a lower bound to how fast they occur. (in analogy with thermal phase transitions)
- **BONUS**: concentration bounds (Chernoff) for short-time evolved states.

Computational complexity of quantum dynamics

Evolution time	$\leq t^*$	$\mathcal{O}(1)$	$\mathcal{O}(\mathrm{polylog}(N))$	$\mathcal{O}(\mathrm{poly}(N))$
$\langle A(t) \rangle$	**P**	**P**	**??**	**BQP-complete**
$\log\langle e^{-itH} \rangle$	**P**	**#P-hard**	**#P-hard**	**#P-hard**
$\langle e^{-itH} \rangle$	**P**	**??**	**??**	**BQP-complete**

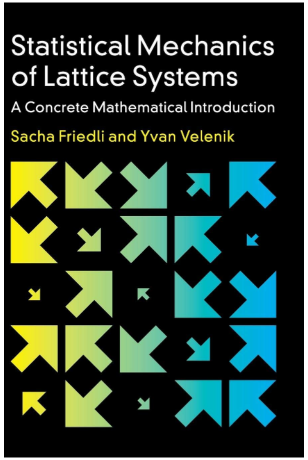
- Complexity of simulating to small additive error $\epsilon = 1/\mathrm{poly}(N)$
- #P hard -> Galanis et al 2005.01076
- BQP hardness: standard arguments + de las Cuevas (1104.2517)

Conclusions

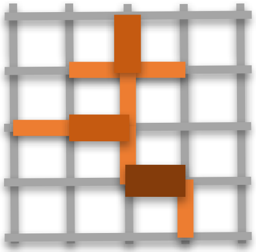
- <u>Cluster expansion</u>: versatile and well-studied tool for partition functions + related problems.

  -Shows convergence of Taylor approximation and yields efficient algorithms.

  -Works for many different interaction graphs.



- <u>Here</u>: it also works for problems of **quantum dynamics**.



- Versatile technique for classical simulation of many quantum problems.

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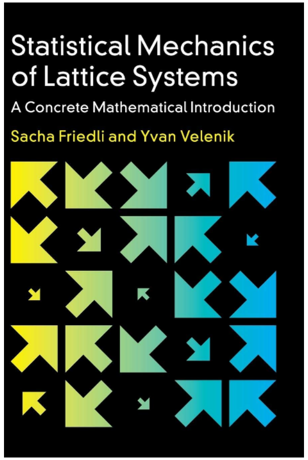

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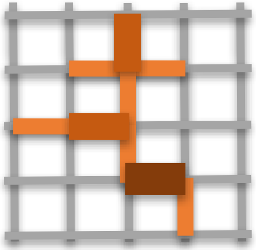

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Thanks!!