# Optimal thermodynamic control for rapidly driven systems

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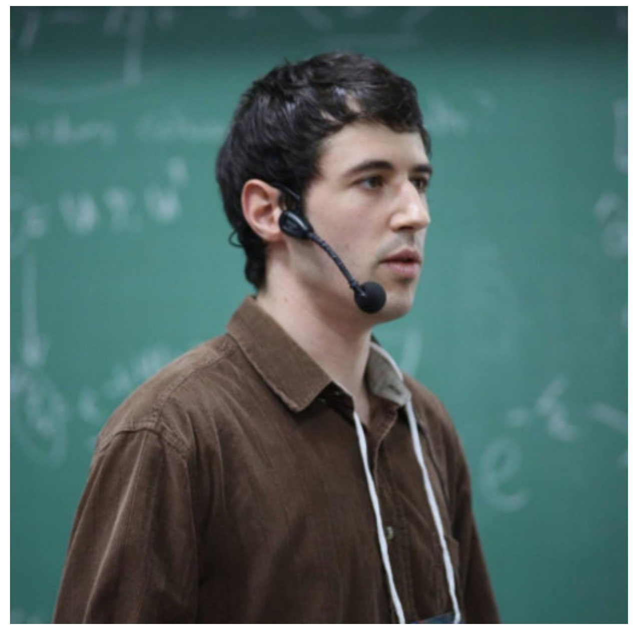


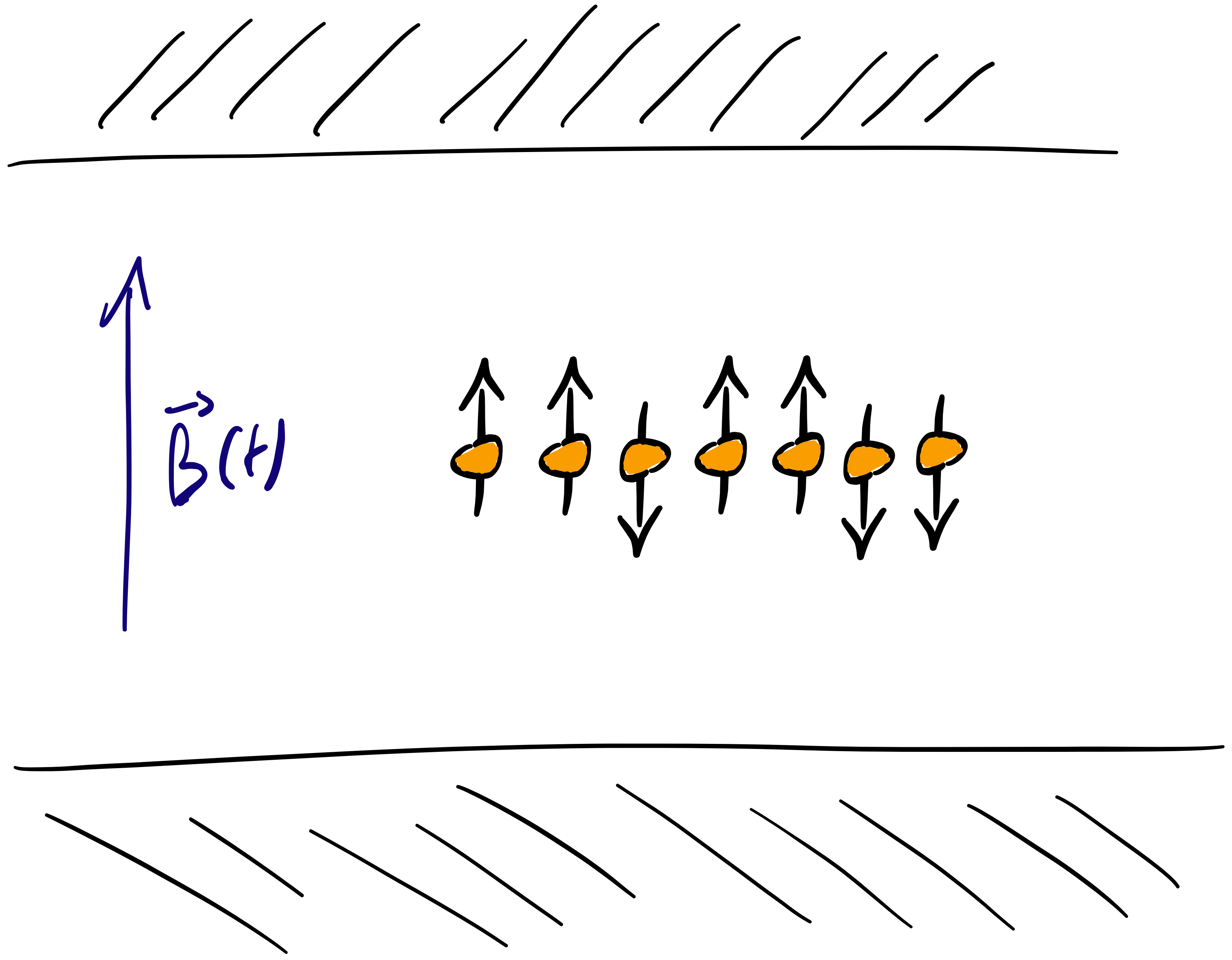
# Thermodynamics of protocols

$$\hat{\rho}_t \longrightarrow \mathcal{E}_{s,t}[\hat{\rho}_t] = \hat{\rho}_s$$

Unitary dynamics

Lindbladian dynamics

$$\frac{d}{dt}\hat{\rho}_t = i[\hat{\rho}_t, \hat{H}(t)] + \mathcal{D}_t[\hat{\rho}_t]$$

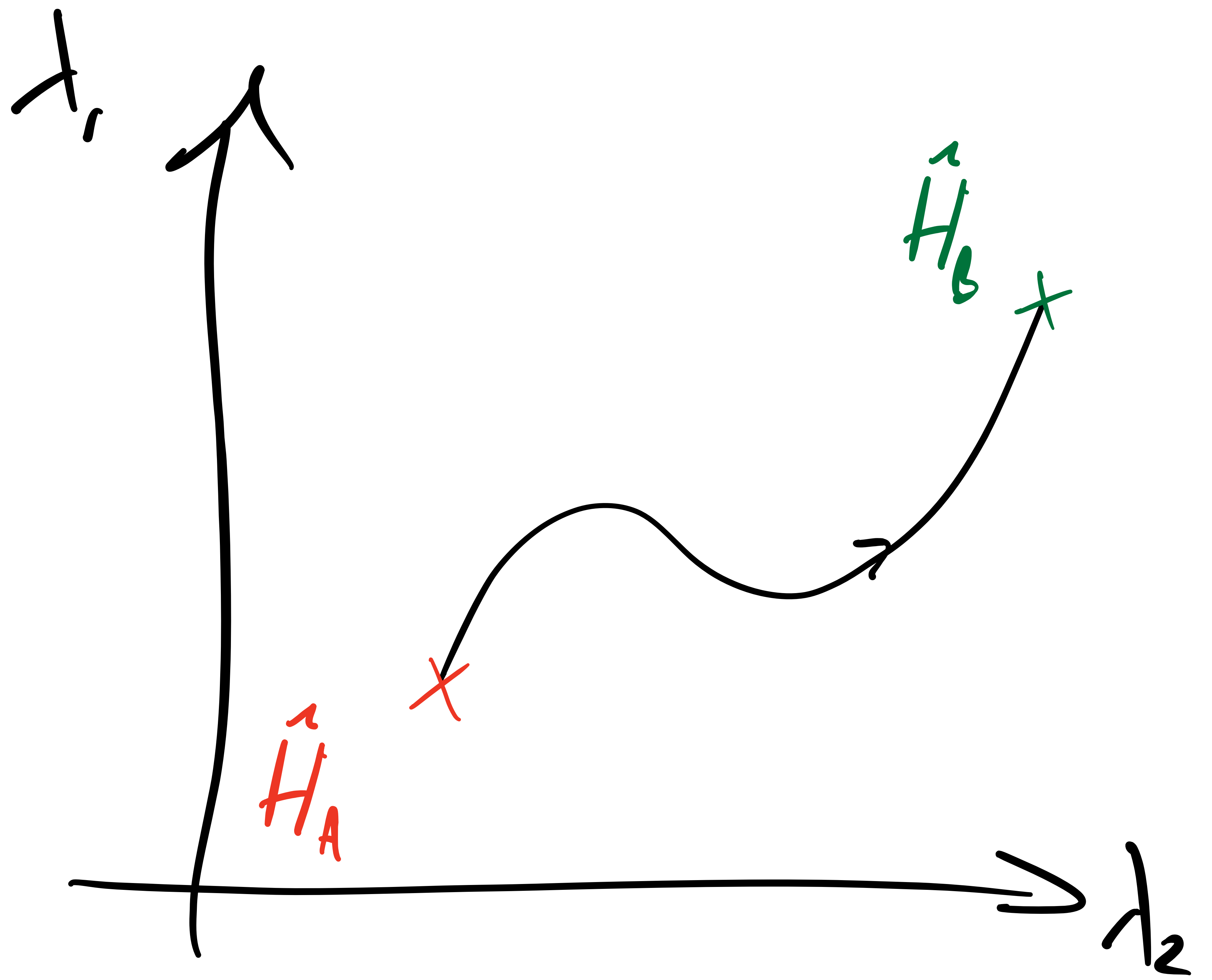


# Thermodynamics of protocols

$$\Delta E = W - Q$$

$$\langle W \rangle = \int_0^\tau dt \, \mathrm{Tr}[\hat{\rho}_t \hat{H}'(t)]$$

$$\langle W \rangle = \Delta F + \langle W_{ex} \rangle$$



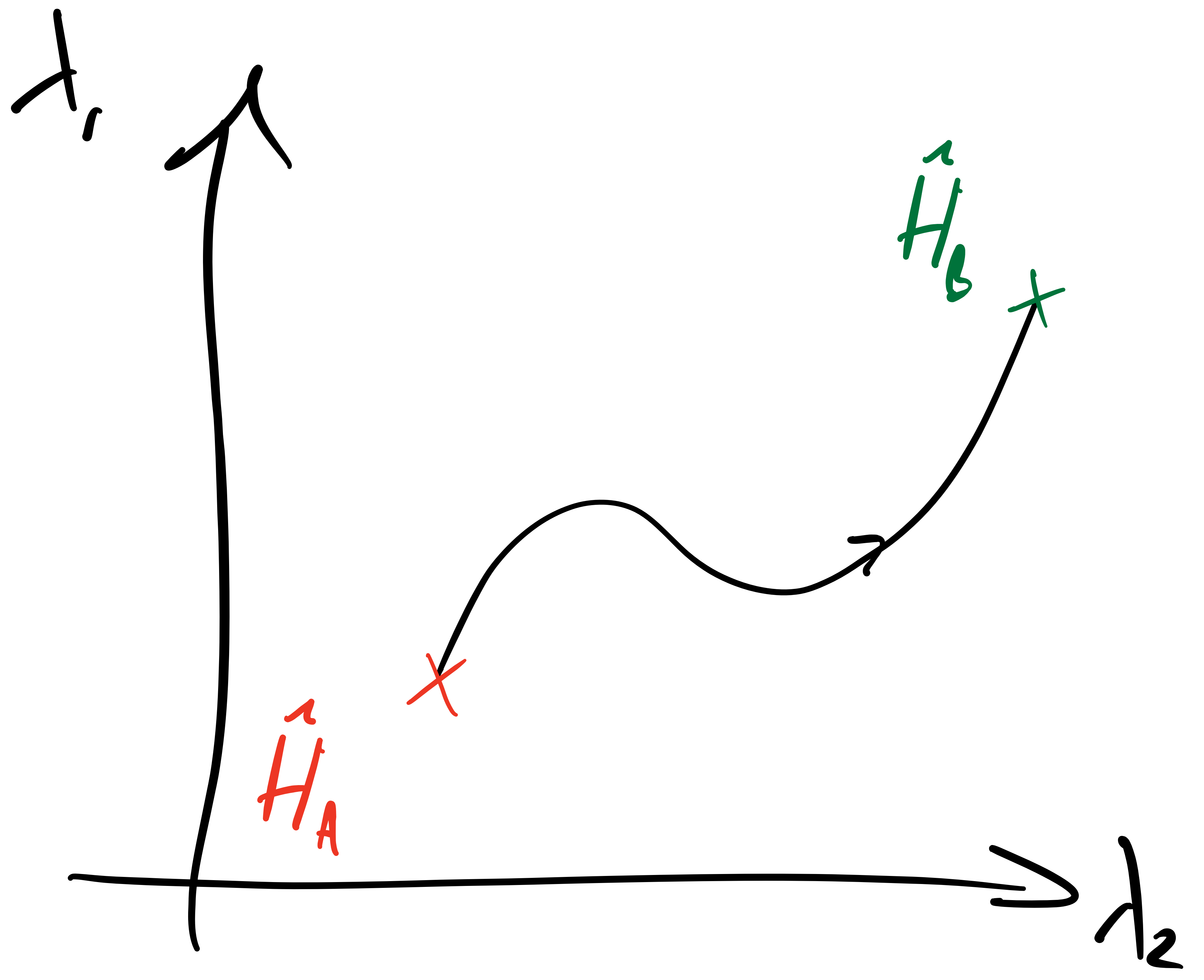

$$\hat{H}(t) = \hat{H}_0 + \sum_k \lambda_k(t) \hat{X}_k$$

# Thermodynamics of protocols

$$\Delta E = W - Q$$

$$\langle W \rangle = \int_0^\tau dt \, \mathrm{Tr}[\hat{\rho}_t \hat{H}'(t)]$$

$$\sigma_W^2 = 2\mathrm{Re} \int_0^\tau dt \int_0^s ds \, \mathrm{Tr}[\hat{H}'(t) \mathscr{E}_{t,s}[\Delta_{\hat{\rho}_s} \hat{H}'(s) \hat{\rho}_s]]$$



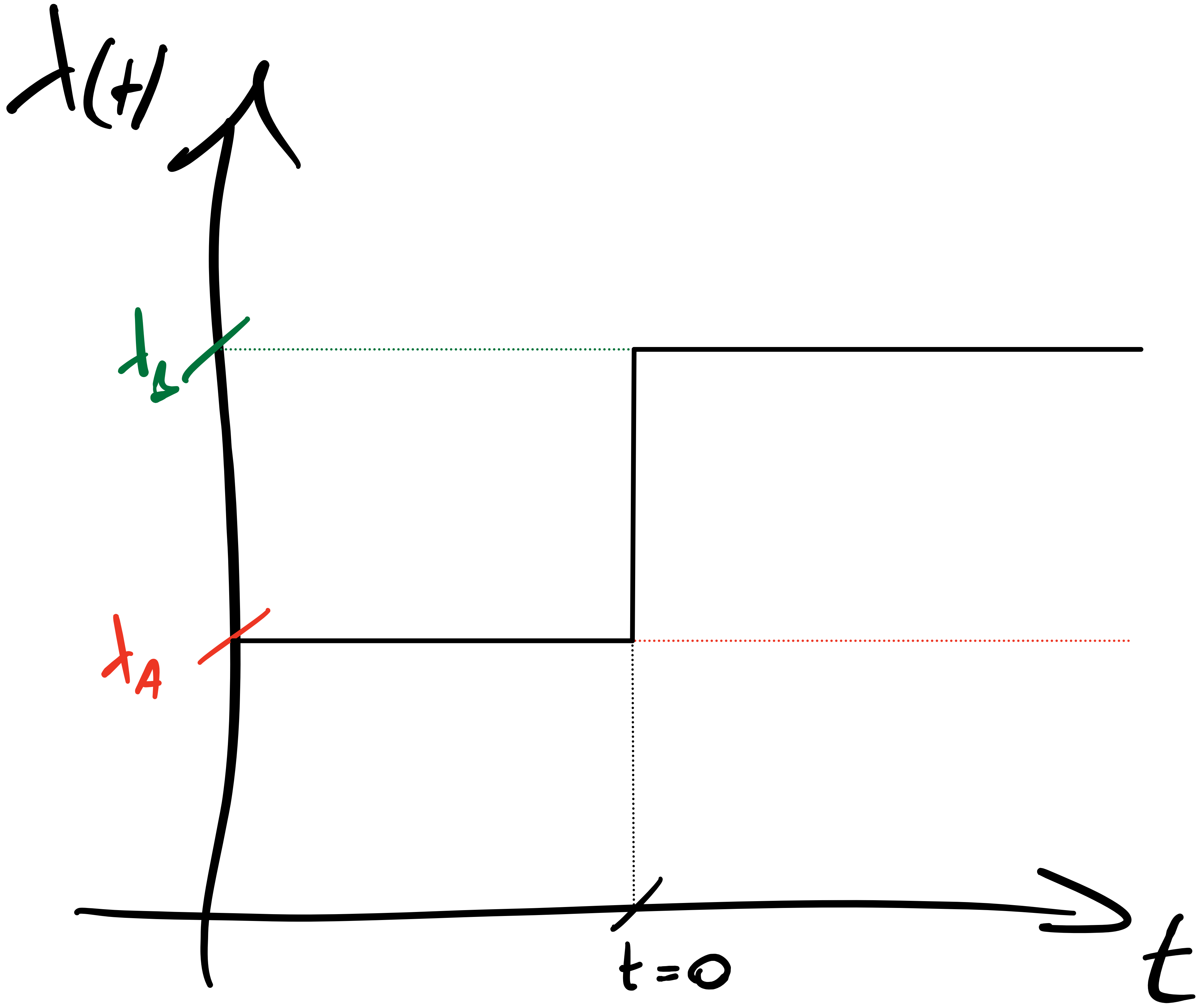
$$\hat{H}(t) = \hat{H}_0 + \sum_k \lambda_k(t) \hat{X}_k$$

# Infinitely fast protocols


$$\langle W_{ex} \rangle = k_B T \, S(\hat{\pi}_A \| \hat{\pi}_B)$$

$$\sigma_W^2 = k_B^2 T^2 V(\hat{\pi}_A \| \hat{\pi}_B)$$

$$\hat{\pi} = \frac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]} \qquad \hat{\rho}_0 = \hat{\pi}_A$$

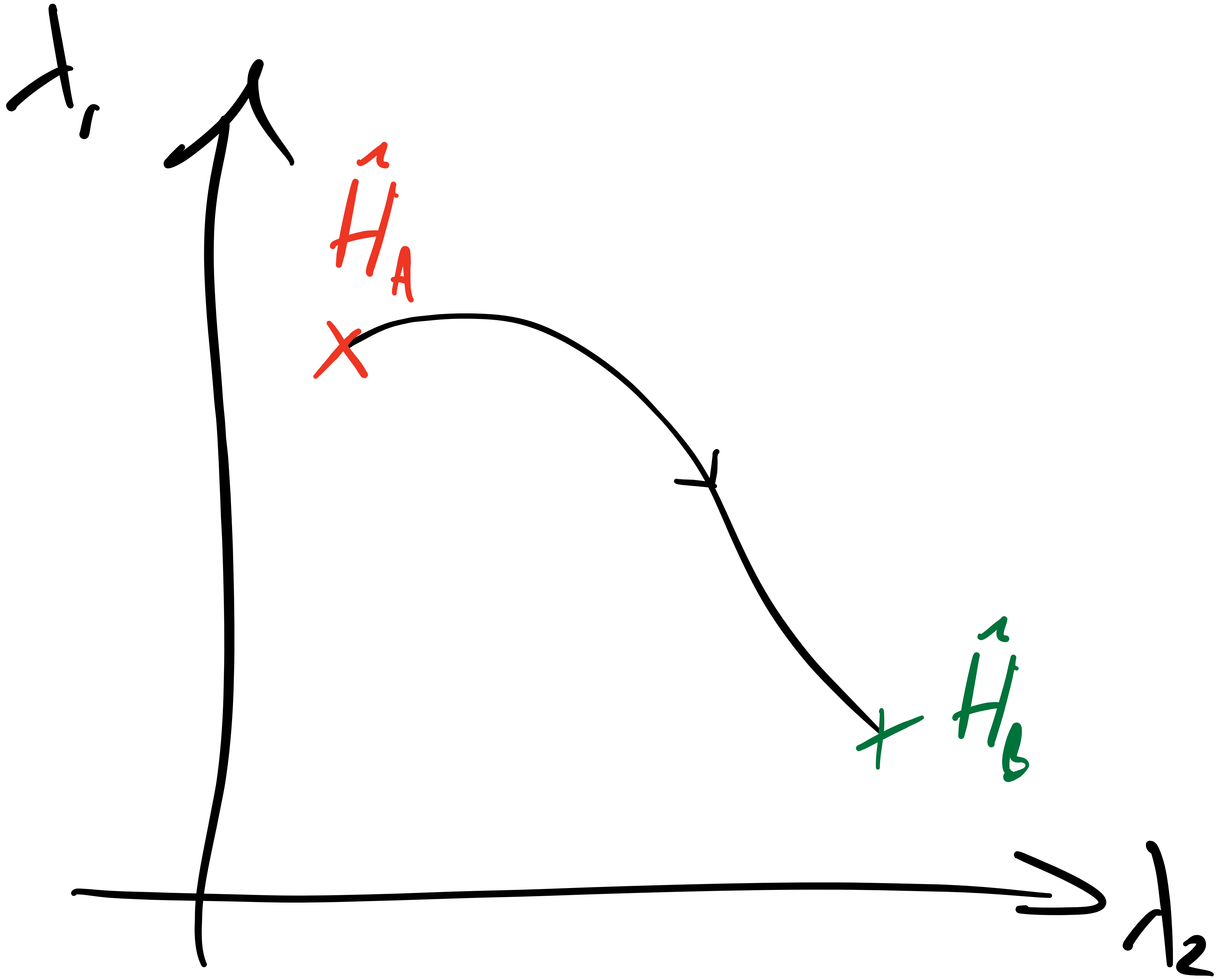


# Short time scales

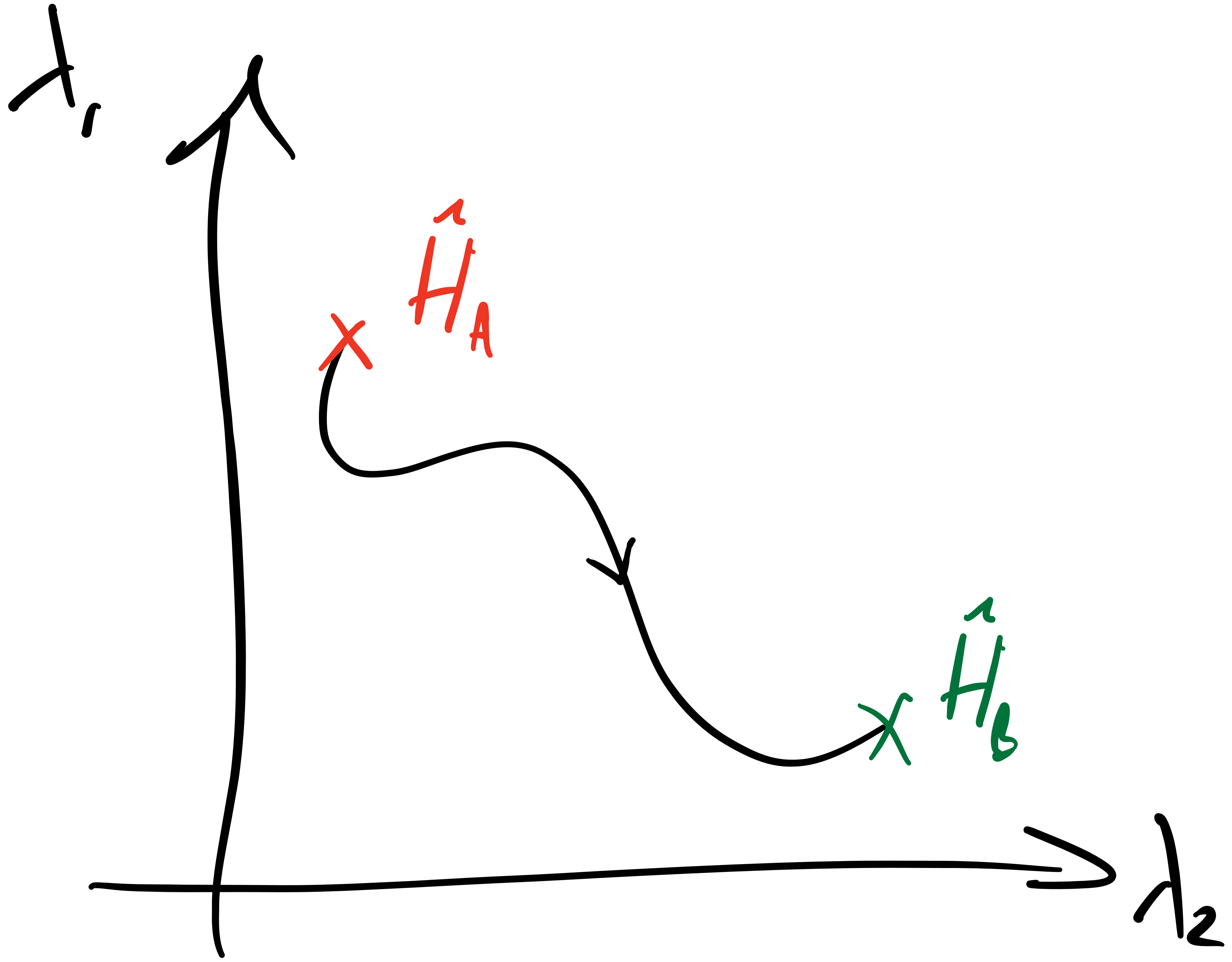
$$\frac{d}{dt}\hat{\rho}_t = \mathscr{L}_{\lambda_t}[\hat{\rho}_t]$$

$$\hat{\rho}_t = \hat{\pi}_A + \int_0^t ds\, \mathscr{L}_{\lambda_s}[\hat{\pi}_A] + \mathcal{O}(t^2/\tau_c^2)$$

$$\tau_c^{-1} = \max_{0<t<1} \|\mathscr{L}_{\lambda_t}\|$$



# Excess work at short time scales

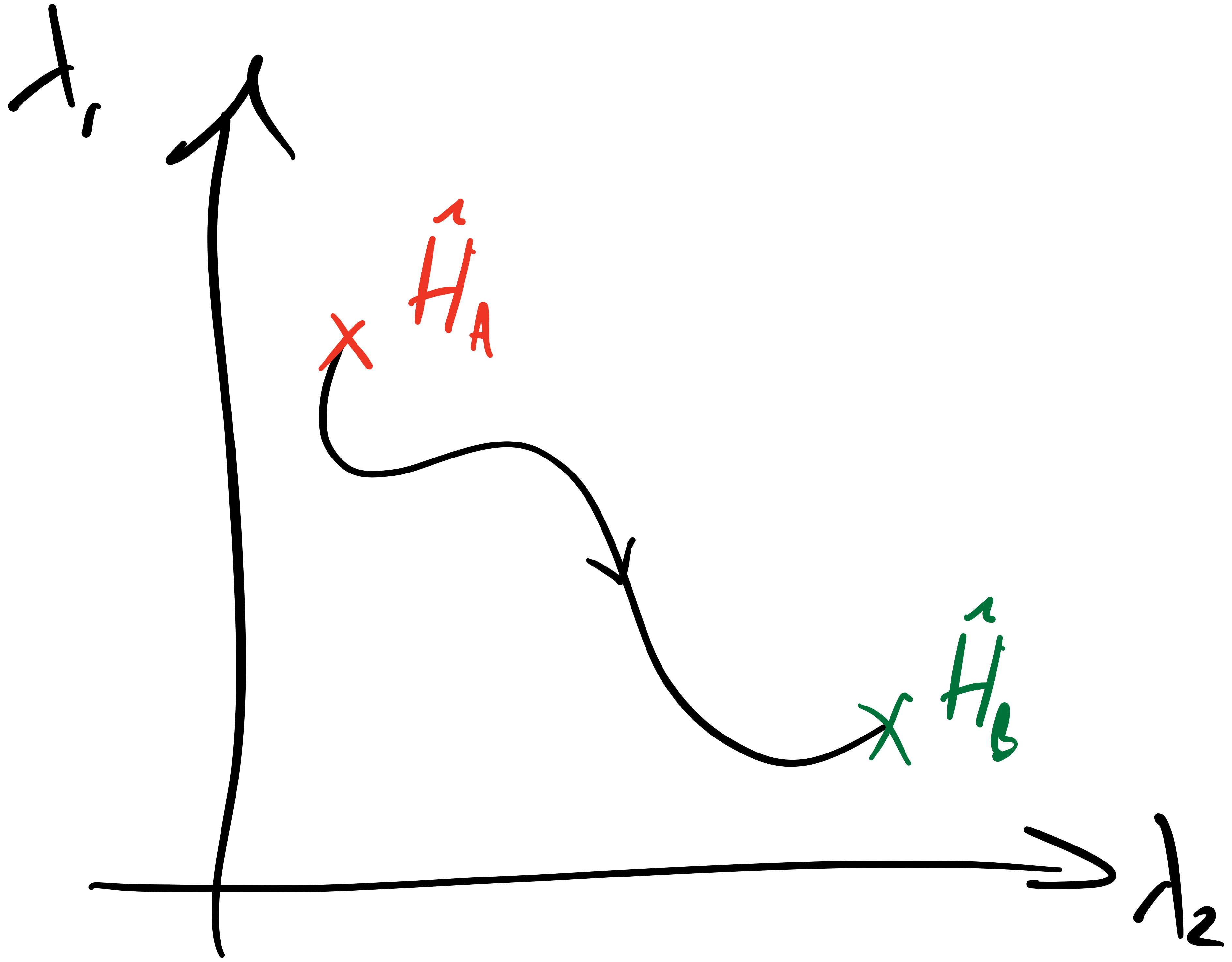



$$\hat{H}(t) = \hat{H}_0 + \vec{\lambda}(t) \cdot \vec{X}$$

$$\vec{R}(\lambda) = \left\langle \mathscr{L}_\lambda^\dagger [\vec{X}] \right\rangle_A$$

$$\langle W_{ex} \rangle = k_B T\, S(\hat{\pi}_A \| \hat{\pi}_B) + \int_0^\tau dt\, (\vec{\lambda}_B - \vec{\lambda}_t) \cdot \vec{R}(\lambda_t)$$

# Work fluctuations at short time scales




$$\Delta X = X - \langle X \rangle_A \qquad \hat{H}(t) = \hat{H}_0 + \vec{\lambda}(t) \cdot \vec{X}$$

$$[G_\lambda]_{ij} = \frac{1}{2} \left\langle \mathscr{L}_\lambda^\dagger \left[ \{ \Delta X_i, \Delta X_j \} \right] \right\rangle_A$$

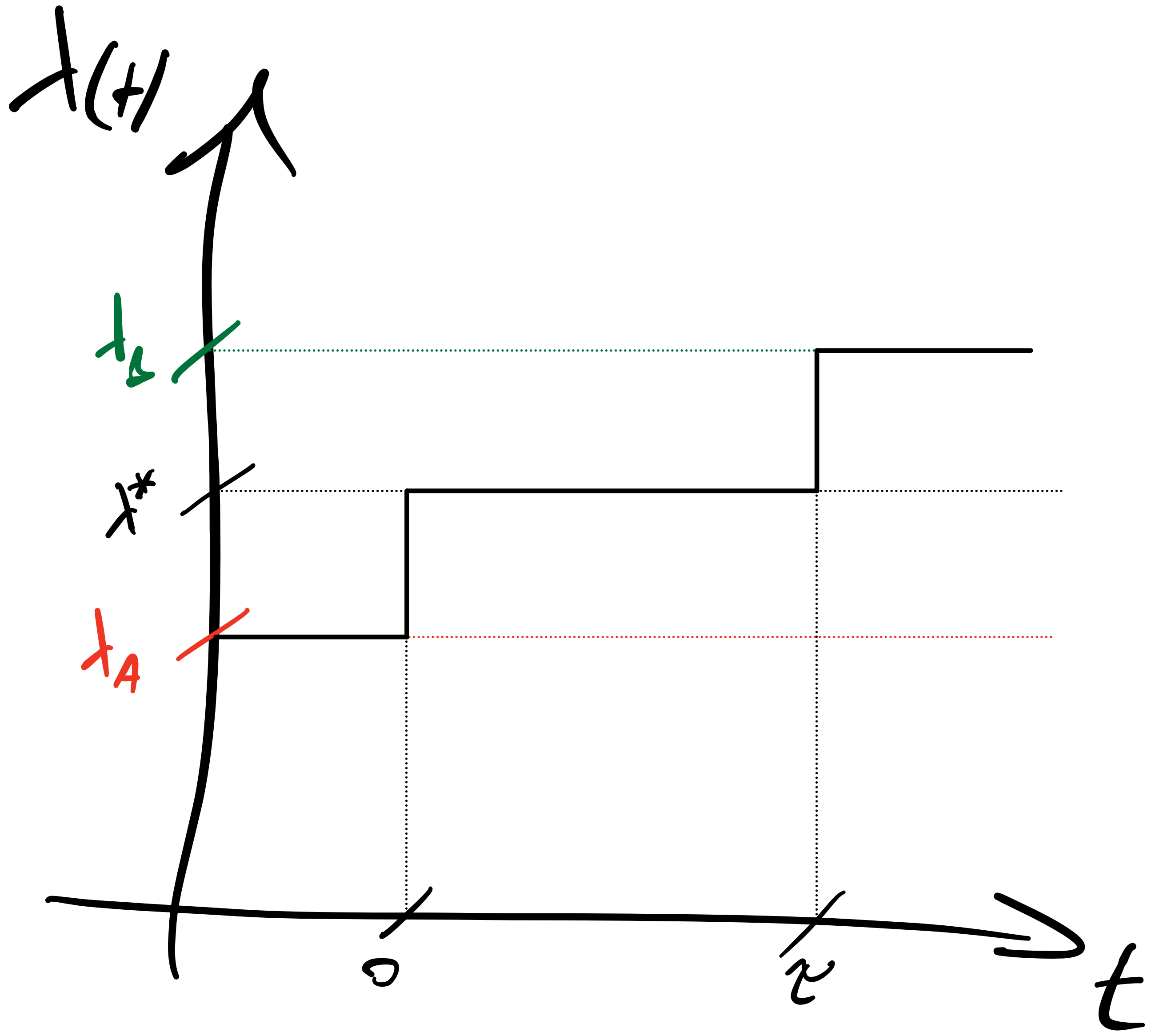
$$[B_\lambda]_{ij} = \left\langle \{ \mathscr{L}_\lambda^\dagger [\Delta X_i], \Delta X_j \} \right\rangle_A$$

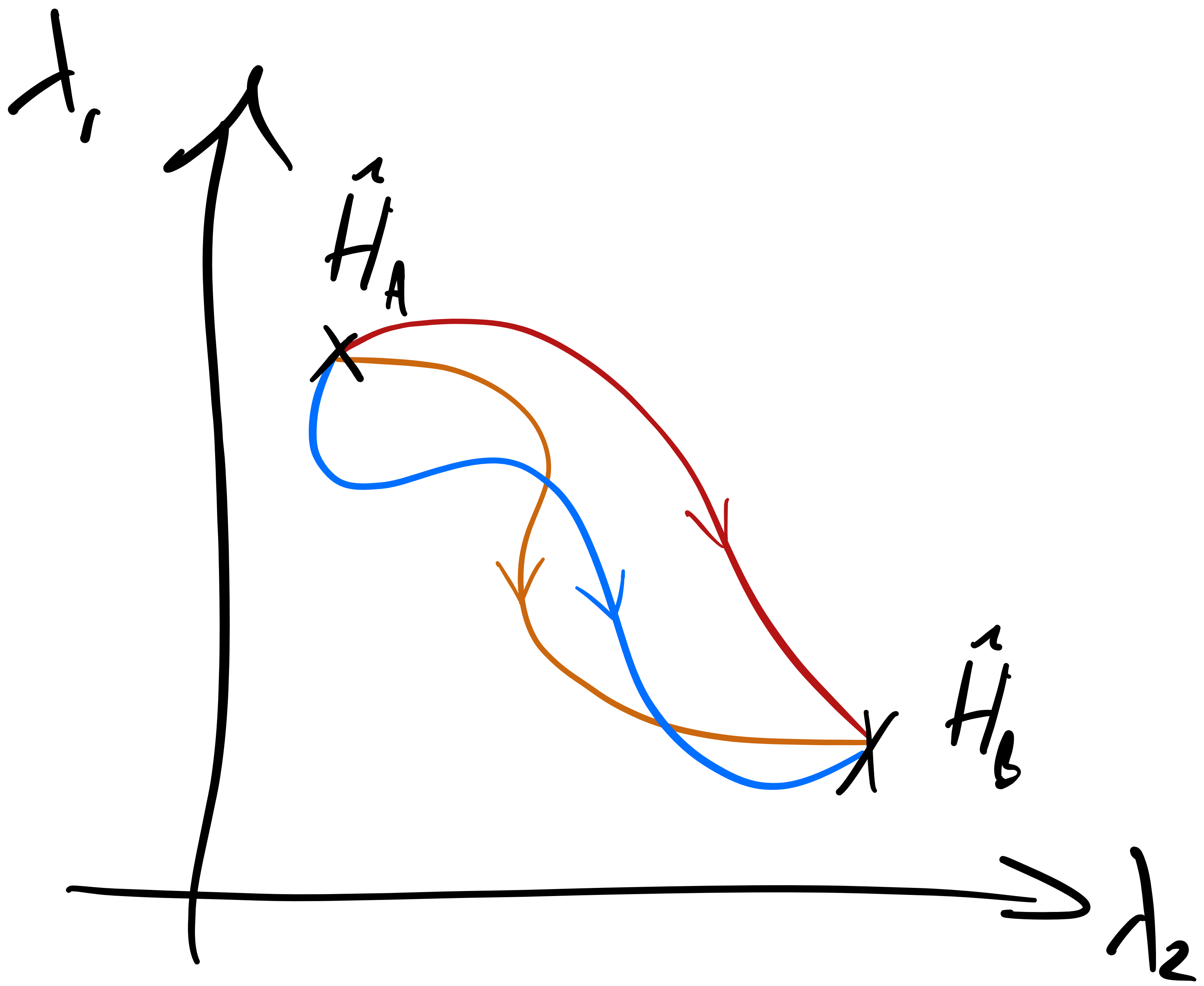
$$\sigma_W^2 = k_B^2 T^2 V(\hat{\pi}_A \| \hat{\pi}_B) + \int_0^\tau dt \, (\vec{\lambda}_B - \vec{\lambda}_t) \cdot G_{\lambda_t} (\vec{\lambda}_B - \vec{\lambda}_t) + (\vec{\lambda}_B - \vec{\lambda}_t) \cdot B_{\lambda_t} (\vec{\lambda}_t - \vec{\lambda}_A)$$

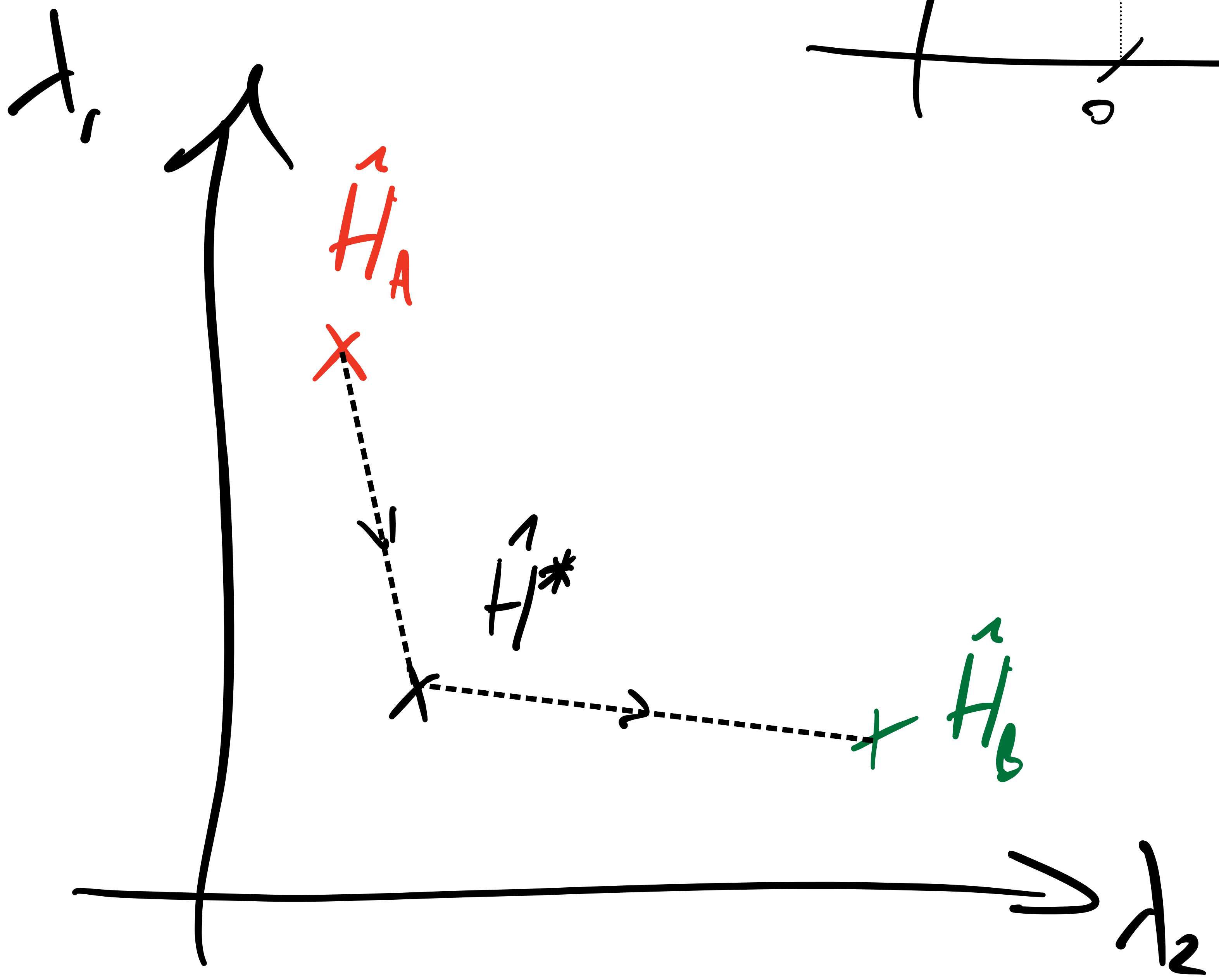
# Optimizing work and fluctuations

$$C_{save} = -\frac{1}{\tau}\int_0^\tau dt\,(\vec{\lambda}_B - \vec{\lambda}_t)\cdot G_{\lambda_t}(\vec{\lambda}_B - \vec{\lambda}_t) + (\vec{\lambda}_B - \vec{\lambda}_t)\cdot B_{\lambda_t}(\vec{\lambda}_t - \vec{\lambda}_A)$$

$$P_{save} = -\frac{1}{\tau}\int_0^\tau dt\,(\vec{\lambda}_B - \vec{\lambda}_t)\cdot \overrightarrow{R}(\lambda_t)$$



# Closed quantum systems

$$\frac{d}{dt}\hat{\rho}_t = i[\hat{\rho}_t, \hat{H}_{\lambda_t}] \qquad \hat{H}_{\lambda_t} = \hat{H}_0 + \vec{\lambda}_t \cdot \vec{X}$$

$$P_{save} = i\langle[\hat{H}_B, \hat{H}_\lambda]\rangle_A$$

$$C_{save} = i\Big(\langle[\hat{H}_B^2, \hat{H}_\lambda]\rangle_A - \langle\{\hat{H}_A, [\hat{H}_B, \hat{H}_\lambda]\}\rangle_A - 2\langle\hat{H}_B - \hat{H}_A\rangle_A\langle[\hat{H}_B, \hat{H}_\lambda]\rangle_A\Big)$$

→ Linear dependence on $\lambda_t$

→ No advantage for 1 and 2 control parameters

→ Diverging optimum: error scales as $\mathcal{O}(\|\vec{\lambda}_t\|^3\tau^2)$

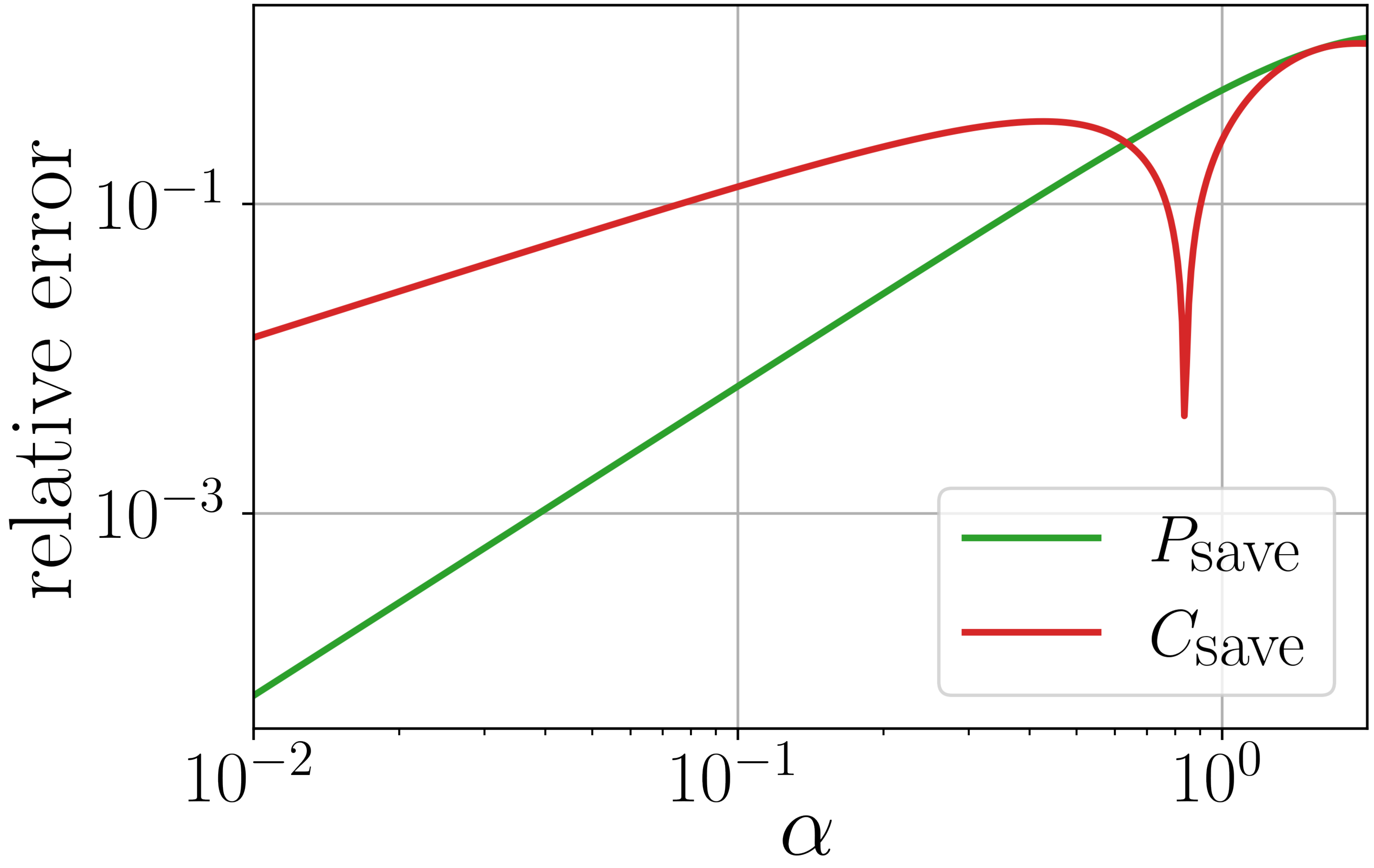
# Unitary evolution on a qubit

$$\hat{H}_\lambda = J\vec{\lambda}\cdot\vec{\sigma} \qquad \hat{H}_A = J\hat{\sigma}^x \longrightarrow \hat{H}_B = J\hat{\sigma}^z$$

$$\hat{H}^* = \alpha J \hat{\sigma}^y$$

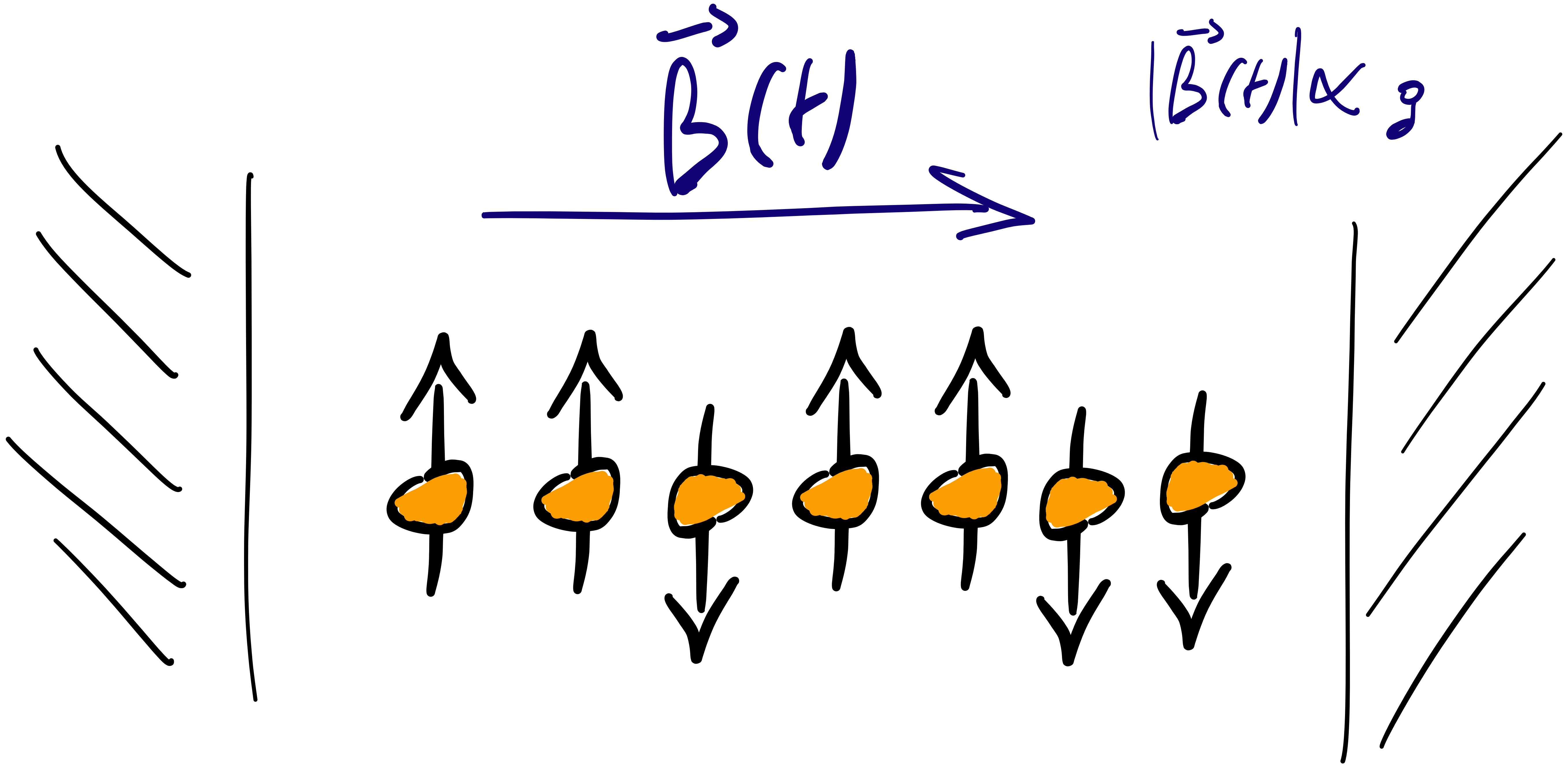
$$P_{save} = 2\alpha J^2 \tanh(\beta J)$$

$$C_{save} = 4\alpha J^3 \cosh(\beta J)^{-2}$$




Validity $\longrightarrow$ $\alpha \ll \dfrac{1}{J\tau}$

# Open quantum systems — Ising chain




$$\hat{H} = -J\sum_i (\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x)$$

$$\tau_{eq}\frac{d}{dt}\hat{\rho}_t = \hat{\pi}_{\lambda_t} - \hat{\rho}_t$$

→ phase transition at low temperatures at $g = 1$

$$Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$$

$$\lim_{n\to\infty}\frac{1}{n}\log Z = \int_0^{2\pi} dk \,\log\cosh \beta J\sqrt{1+g^2-2g\cos k}$$

# Open quantum systems — Ising chain


high temperature

low temperature

# Conclusion

- Optimal fast protocols consist of two jumps
  - "naïve" protocols do significantly worse
  - Optimizing work & fluctuations at the same time is not always possible
- Optimum for unitary evolution is out of the fast regime
- Full work statistics?
- Thermodynamic uncertainty relations?

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