ENCODINGS OF BINARY VARIABLES FOR EFFICIENT USE ON A QUANTUM COMPUTER

GIACOMO FRANCESCHETTO, MÁRCIO TADDEI
ANTÓN MAKAROV, ENEKO ISABA OCEDO

ICE-8 SANTIAGO DE COMPOSTELA

NISQ DEVICES

NISQ DEVICES

- Noisy intermediate-scale quantum devices

NISQ DEVICES

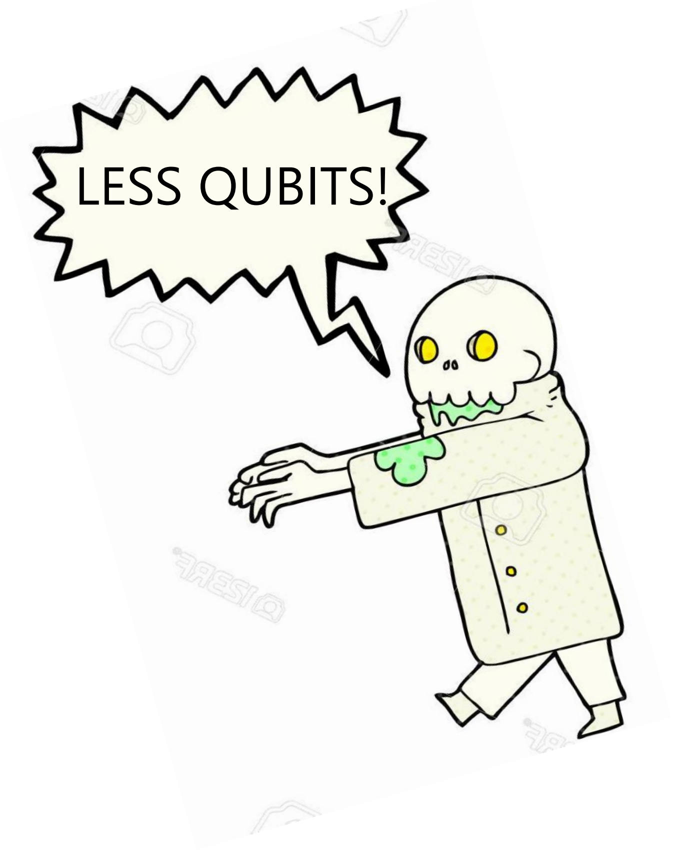
- Noisy intermediate-scale quantum devices
- Few qubits available

NISQ DEVICES

- Noisy intermediate-scale quantum devices
- Few qubits available

NISQ DEVICES

- Noisy intermediate-scale quantum devices
- Few qubits available
- More for *quantum annealers* – up to ~ 5000 *physical* qubits (D-Wave Advantage)



NISQ DEVICES

- Noisy intermediate-scale quantum devices
- Few qubits available
- More for *quantum annealers* – up to ~ 5000 *physical* qubits (D-Wave Advantage)
- Dedicated to QUBO problems: Quadratic Unconstrained Binary Optimization

NISQ DEVICES

- Noisy intermediate-scale quantum devices
- Few qubits available
- More for *quantum annealers* – up to ~ 5000 *physical* qubits (D-Wave Advantage)
- Dedicated to QUBO problems: Quadratic Unconstrained Binary Optimization
- How to make it work in a real-world problem?

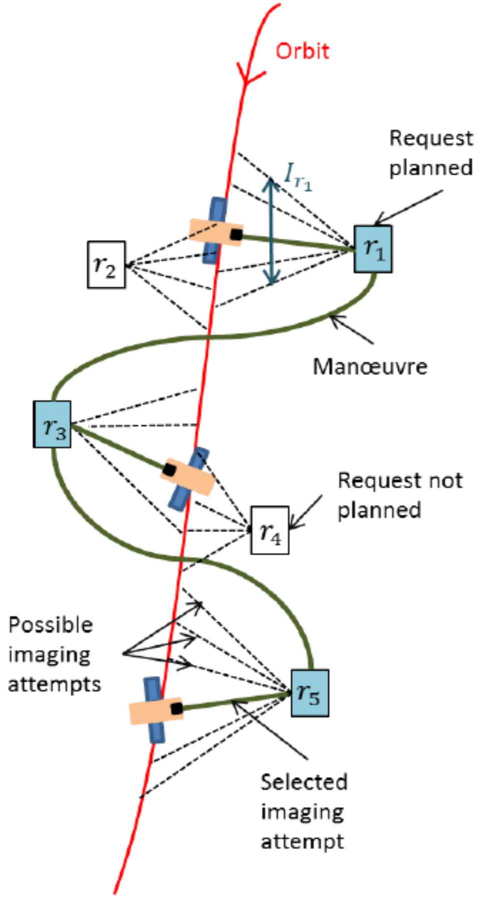
THE PROBLEM

THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)

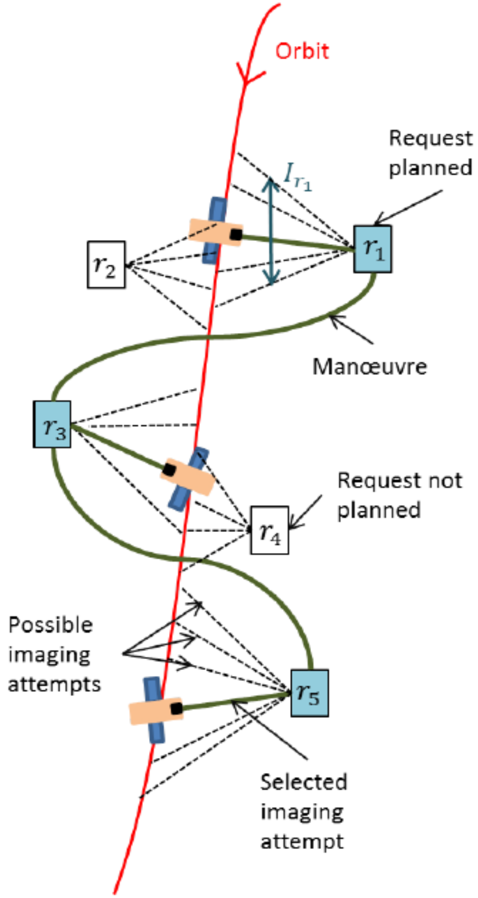
THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)



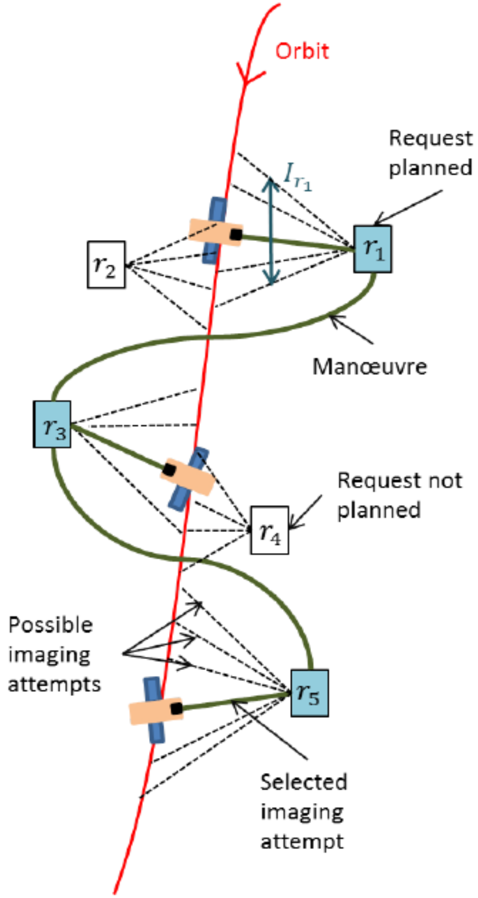
THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?



THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?
- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

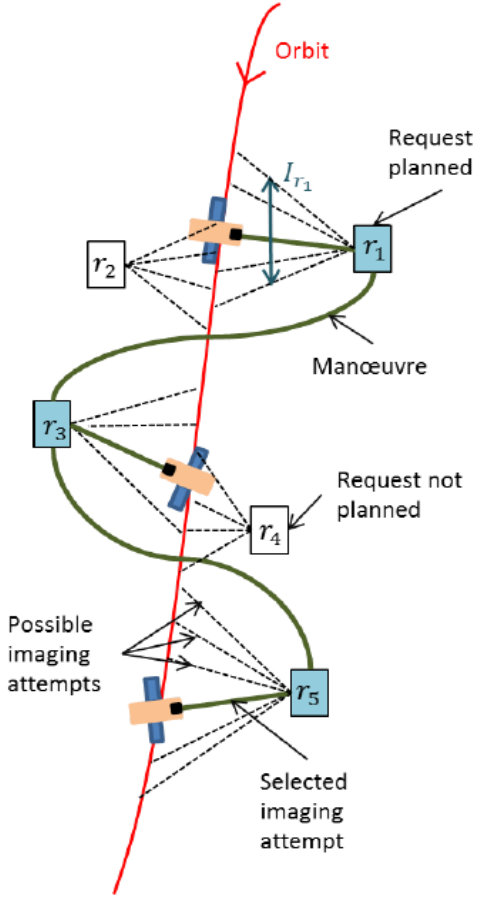


THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```



THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

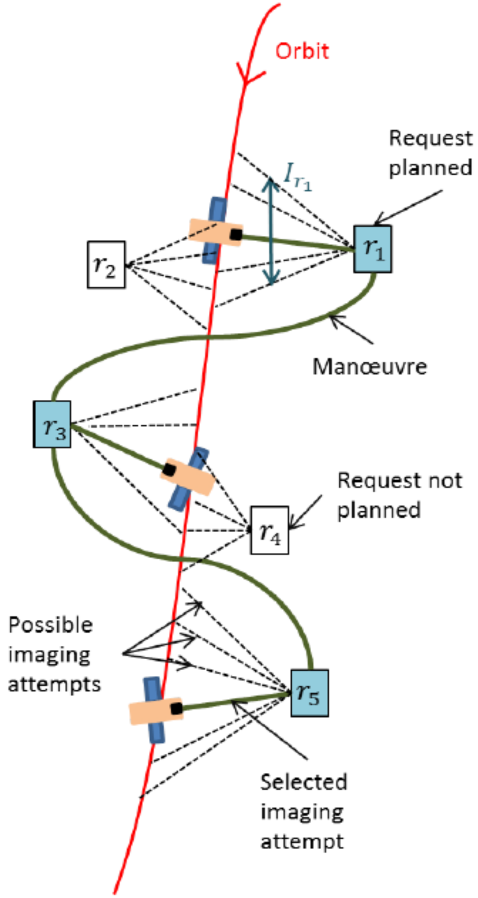
```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras



THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

8					
0	1	3	1	2	3
1	1	3	1	2	3
2	1	3	1	2	3
3	1	3	1	2	3
4	2	1	13		
5	2	1	13		
6	2	1	13		
7	2	1	13		

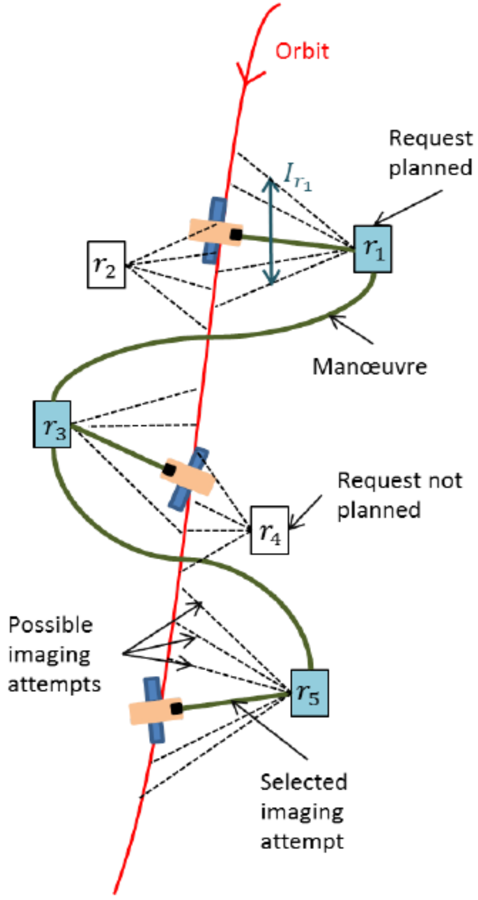
#requests

weight

#cameras

list of cameras

- Mono photos (3 possible cameras), stereo photos (unique option)



THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

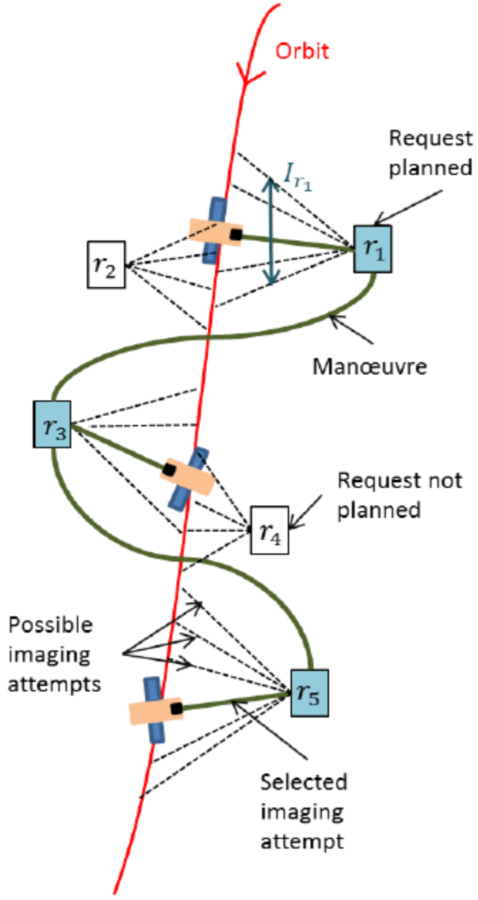
#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```



- Mono photos (3 possible cameras), stereo photos (unique option)

THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

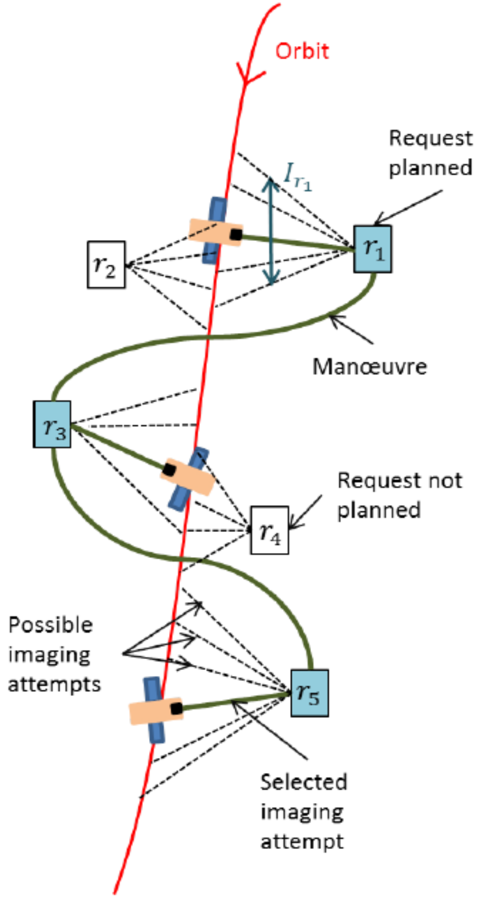
```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras



- Mono photos (3 possible cameras), stereo photos (unique option)

THE PROBLEM

- Satellite image-acquisition scheduling problem (SIASP)
- Satellite pass over Earth, too many photo requests – which to undertake?

- SPOT5 dataset (Bensana, Lemaitre, Verfaille, Constraints **4**, 293 (1999))

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

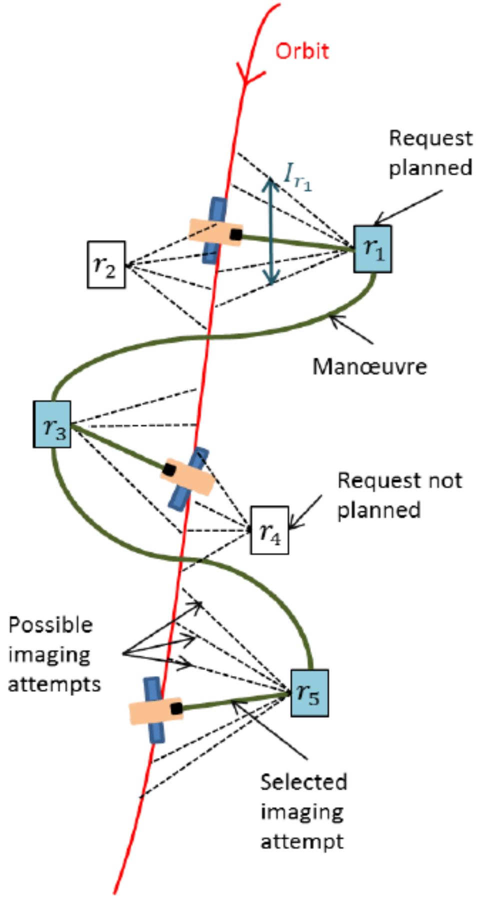
```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras



- Mono photos (3 possible cameras), stereo photos (unique option)
- Instances from 8 to 364 requests, from 7 to 9744 constraints

THE QUANTUM HARDWARE

THE QUANTUM HARDWARE

- Quantum annealers: ground state of

THE QUANTUM HARDWARE

- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

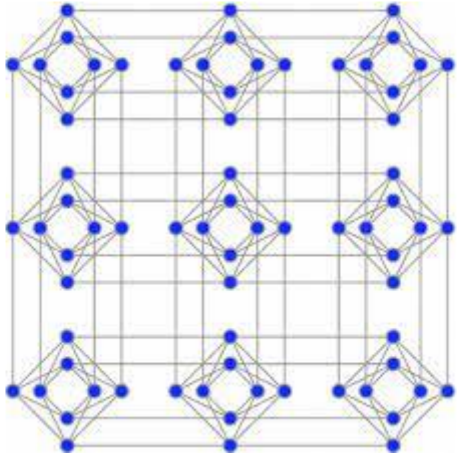
THE QUANTUM HARDWARE

- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

- Topology defines available couplings

THE QUANTUM HARDWARE

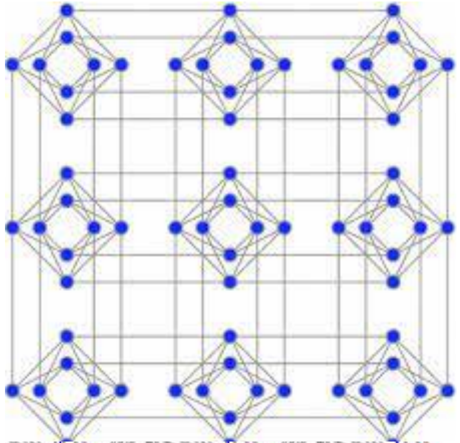


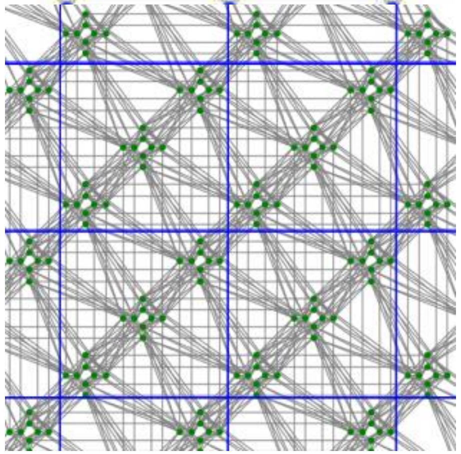
- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

- Topology defines available couplings

THE QUANTUM HARDWARE





- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

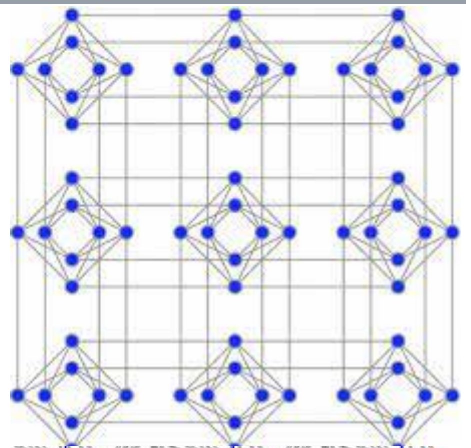
- Topology defines available couplings

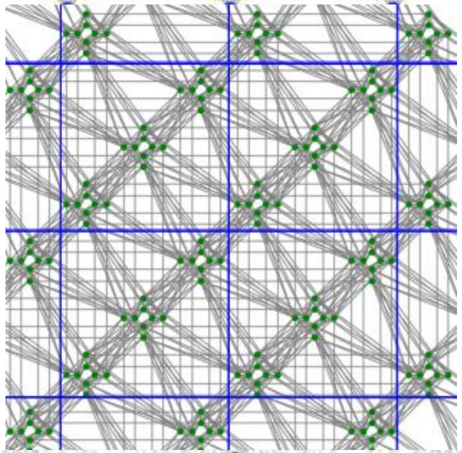
THE QUANTUM HARDWARE

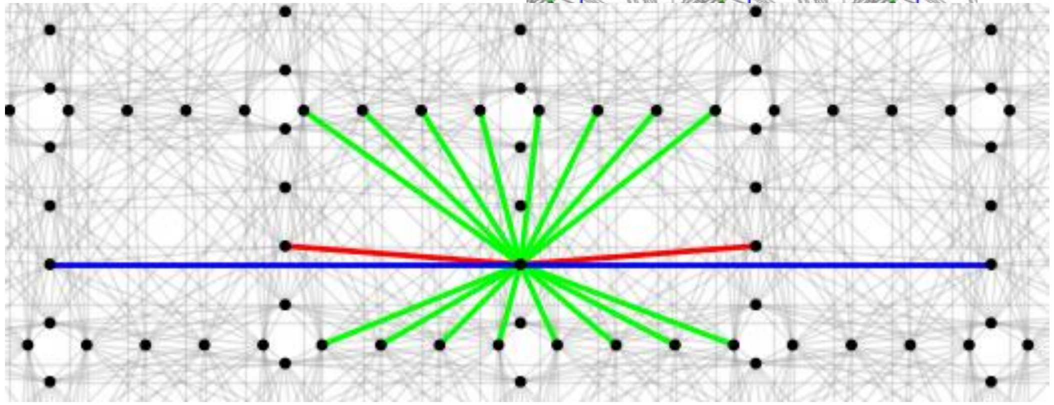
- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i \; + \; \sum_{ij} J_{ij} \; Z_i Z_j$$

- Topology defines available couplings





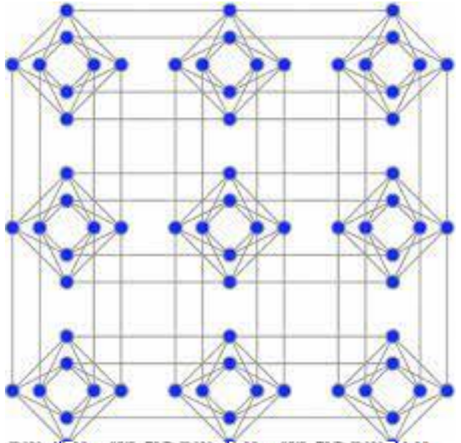


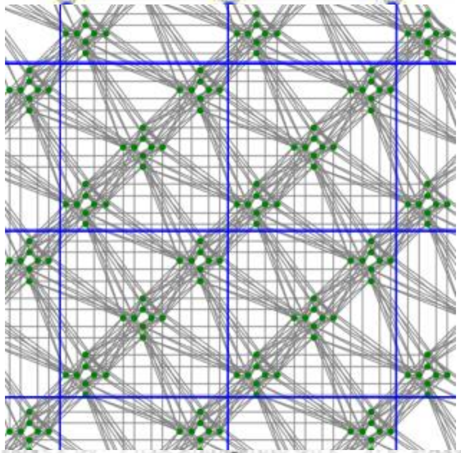
THE QUANTUM HARDWARE

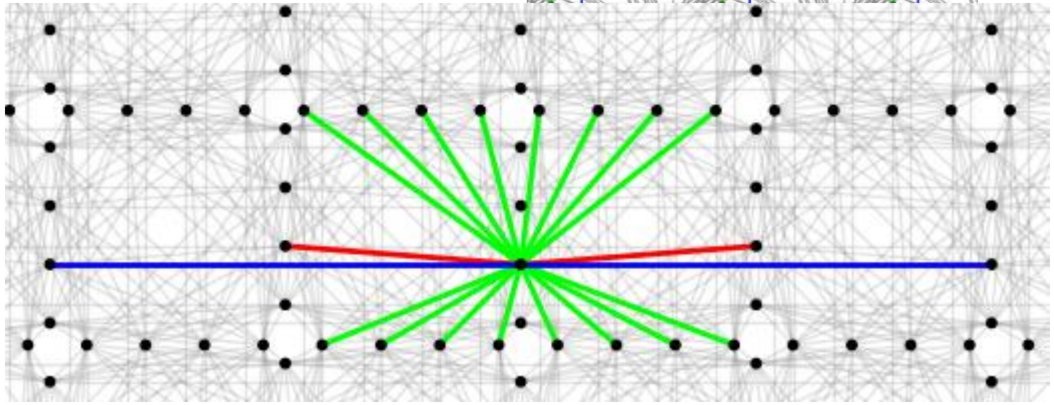
- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

- Topology defines available couplings
- Connection to QUBO







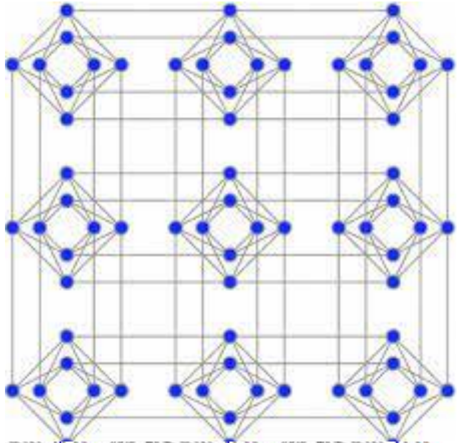
THE QUANTUM HARDWARE

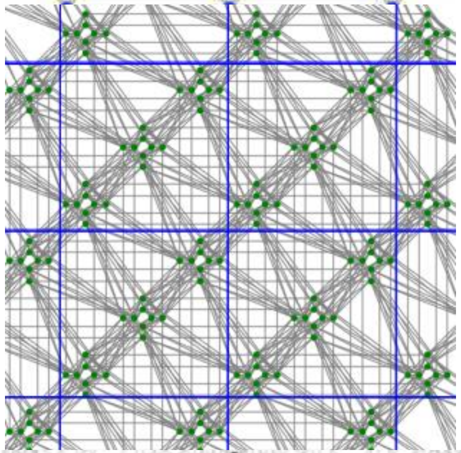
- Quantum annealers: ground state of

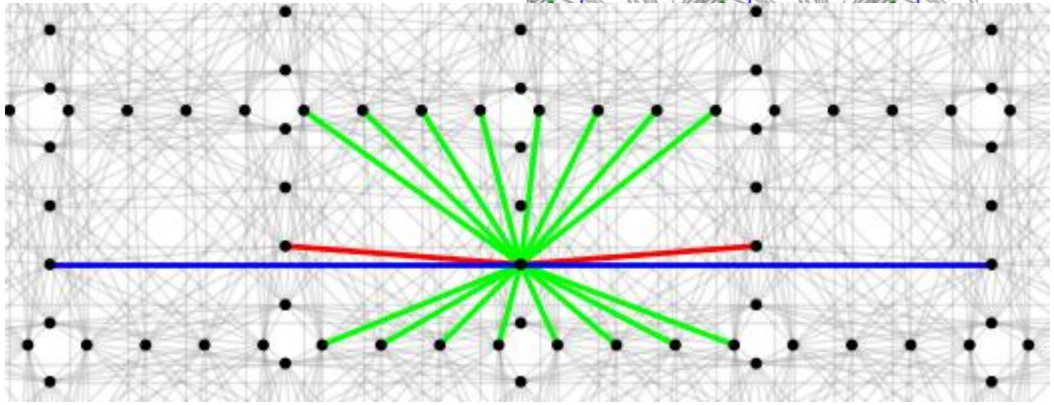
$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

- Topology defines available couplings
- Connection to QUBO

ground state of $\sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j \quad \rightarrow \quad \underset{\boldsymbol{x}}{\operatorname{argmin}} \; \boldsymbol{x}^T \cdot Q \cdot \boldsymbol{x}$







THE QUANTUM HARDWARE

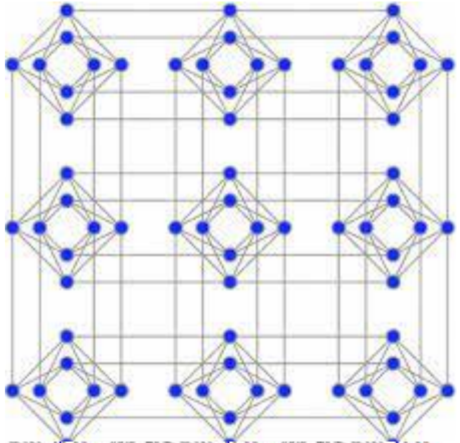
- Quantum annealers: ground state of

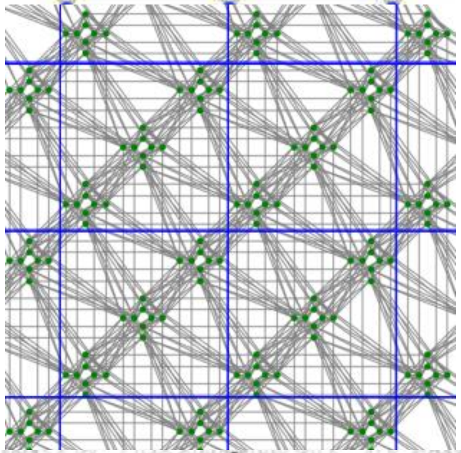
$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

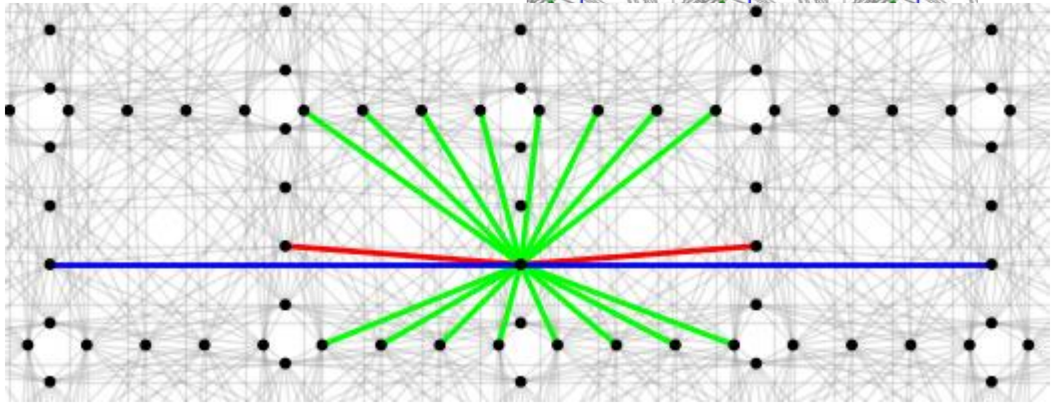
- Topology defines available couplings
- Connection to QUBO

ground state of $\sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j \quad \rightarrow \quad \underset{\boldsymbol{x}}{\operatorname{argmin}} \; \boldsymbol{x}^T \cdot Q \cdot \boldsymbol{x}$

- SIASP is not QUBO!



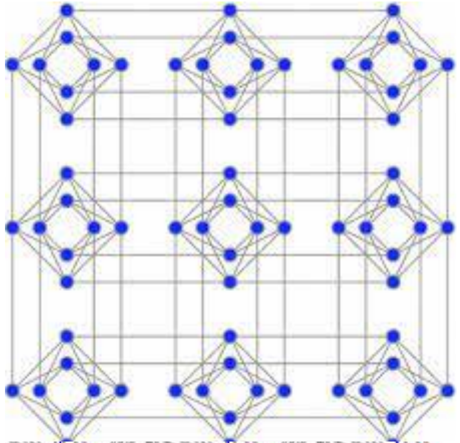


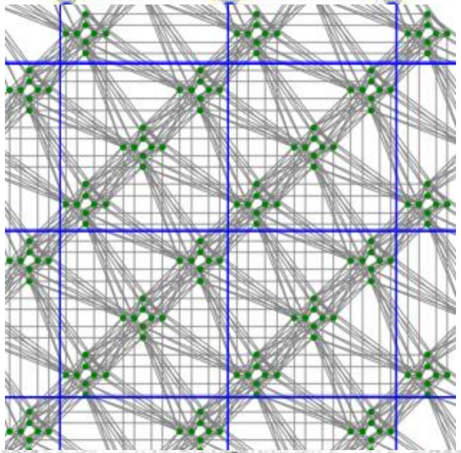


THE QUANTUM HARDWARE

- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$



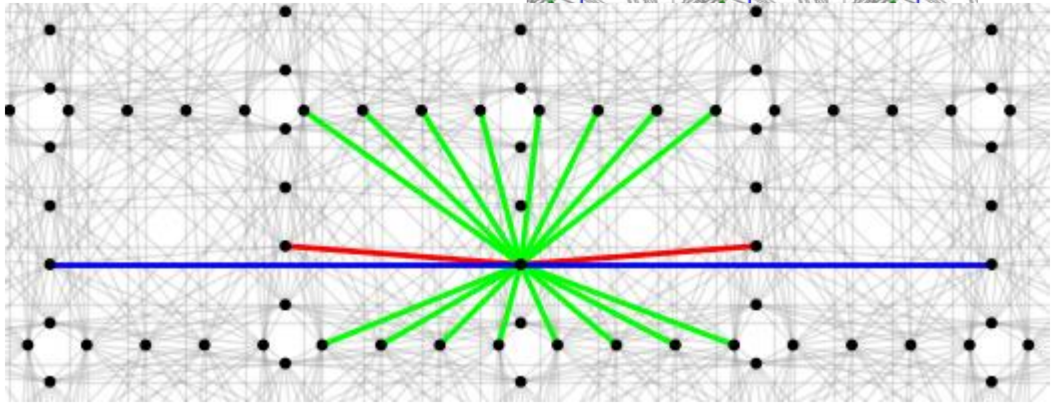


- Topology defines available couplings
- Connection to QUBO

ground state of $\sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$ → $\underset{\boldsymbol{x}}{\operatorname{argmin}} \; \boldsymbol{x}^T \cdot Q \cdot \boldsymbol{x}$

- SIASP is not QUBO!

i) Constraints (not U)



THE QUANTUM HARDWARE

- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j$$

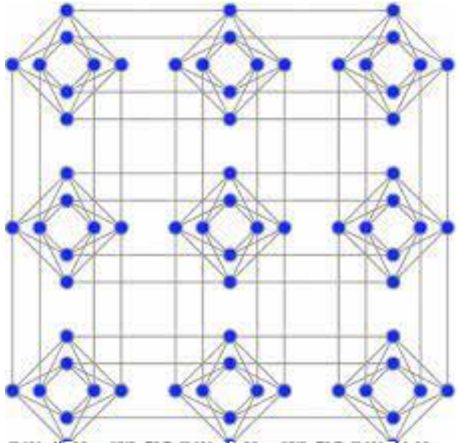
- Topology defines available couplings
- Connection to QUBO

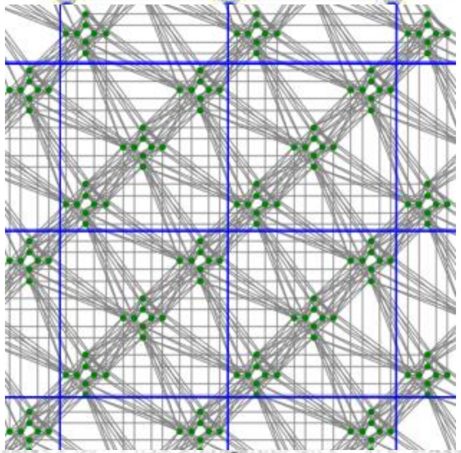
ground state of $\sum_i h_i Z_i + \sum_{ij} J_{ij} Z_i Z_j \quad \rightarrow \quad \underset{\boldsymbol{x}}{\operatorname{argmin}} \; \boldsymbol{x}^T \cdot Q \cdot \boldsymbol{x}$

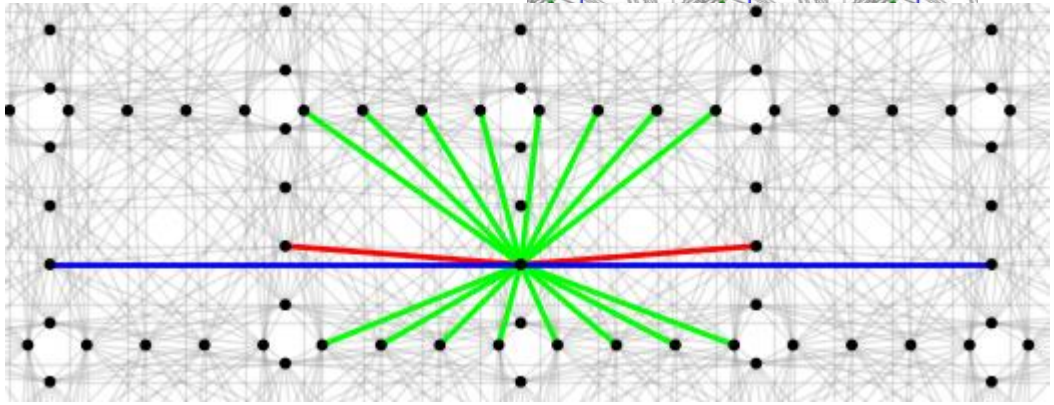
- SIASP is not QUBO!

i) Constraints (not U)

- Penalty terms







THE QUANTUM HARDWARE

- Quantum annealers: ground state of

$$H_P = \sum_i h_i Z_i \; + \; \sum_{ij} J_{ij} \; Z_i Z_j$$

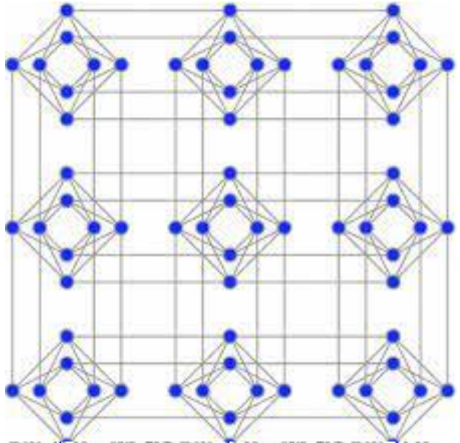
- Topology defines available couplings
- Connection to QUBO

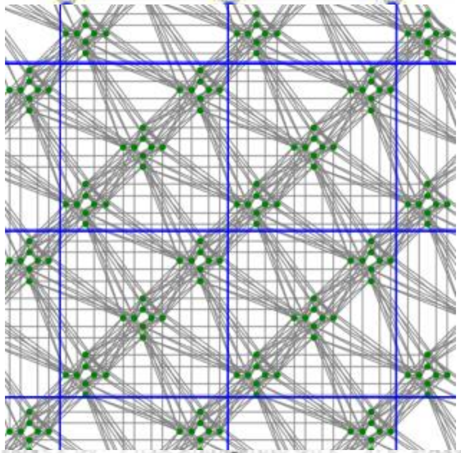
ground state of $\sum_i h_i Z_i \; + \sum_{ij} J_{ij} \; Z_i Z_j$ → $\underset{\boldsymbol{x}}{\operatorname{argmin}} \; \boldsymbol{x}^T \cdot Q \cdot \boldsymbol{x}$

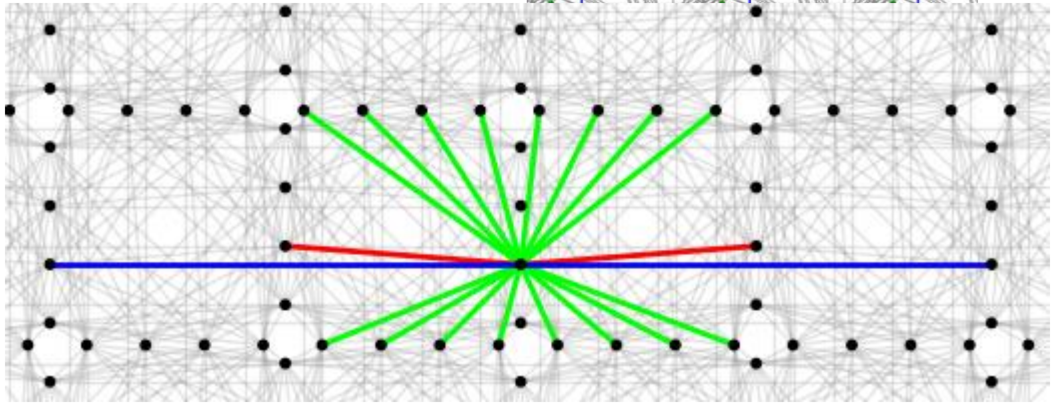
- SIASP is not QUBO!

i) Constraints (not U)

- Penalty terms
- E.g.: constraint $x_1 + x_2 \leq 1$, extra term $P \, x_1 x_2$







THE FIXED-LENGTH ENCODING

THE FIXED-LENGTH ENCODING

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

- #requests
- weight
- #cameras
- list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

- #constraints
- multiplicity
- photos involved
- constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"

8

0	1	3	1	2	3
1	1	3	1	2	3
2	1	3	1	2	3
3	1	3	1	2	3
4	2	1	13		
5	2	1	13		
6	2	1	13		
7	2	1	13		

#requests

weight

#cameras

list of cameras

7

2	1	0	3	3	2	2	1	1
2	2	0	3	3	2	2	1	1
2	3	0	3	3	2	2	1	1
2	5	4	13	13				
2	5	6	13	13				
2	2	1	3	3	2	2	1	1
2	3	1	3	3	2	2	1	1

#constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} -\sum_{i,j} w_i \, x_{i,j}$

8

0	1	3	1	2	3
1	1	3	1	2	3
2	1	3	1	2	3
3	1	3	1	2	3
4	2	1	13		
5	2	1	13		
6	2	1	13		
7	2	1	13		

#requests

weight

#cameras

list of cameras

7

2	1	0	3	3	2	2	1	1
2	2	0	3	3	2	2	1	1
2	3	0	3	3	2	2	1	1
2	5	4	13	13				
2	5	6	13	13				
2	2	1	3	3	2	2	1	1
2	3	1	3	3	2	2	1	1

#constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} -\sum_{i,j} w_i\, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\, ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\, ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} \; -\sum_{i,j} w_i \, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \le 1 \quad \forall i \qquad \text{(each photo once)}$$

$$x_{p,j_p} + x_{q,j_q} \le 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \le 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4[th]" camera represents stereo setup "13"
- $\min_{x_{i,j}} -\sum_{i,j} w_i \, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \leq 1 \quad \forall i \qquad \text{(each photo once)}$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\left((p,j_p),(q,j_q)\right) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\left((p,j_p),(q,j_q),(r,j_r)\right) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests
weight
#cameras
list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints
multiplicity
photos involved
constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} \; -\sum_{i,j} w_i \, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \leq 1 \quad \forall i \qquad \text{(each photo once)}$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\, ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\, ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras

```
2 51 41 3 3 2 2 1 1
3 40 58 55 13 13 2
3 58 55 41 13 2 3
2 51 48 3 3 2 2 1 1
2 48 46 3 13 1 13
2 7 4 13 13
3 44 58 55 3 13 2
```

constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} \; -\sum_{i,j} w_i \, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \leq 1 \quad \forall i \qquad \text{(each photo once)}$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M \qquad \text{(mono)}$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests

weight

#cameras

list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints

multiplicity

photos involved

constrained cameras

```
2 51 41 3 3 2 2 1 1
3 40 58 55 13 13 2
3 58 55 41 13 2 3
2 51 48 3 3 2 2 1 1
2 48 46 3 13 1 13
2 7 4 13 13
3 44 58 55 3 13 2
```

constraints

multiplicity

photos involved

constrained cameras

THE FIXED-LENGTH ENCODING

- For each photo request i, four binary variables $x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}$
- "4th" camera represents stereo setup "13"
- $\min_{x_{i,j}} \; -\sum_{i,j} w_i \, x_{i,j}$
- Under constraints

$$\sum_j x_{i,j} \leq 1 \quad \forall i \qquad \text{(each photo once)}$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\, ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\, ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M \qquad \text{(mono)}$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\} \qquad \text{(stereo)}$$

```
8
0 1 3 1 2 3
1 1 3 1 2 3
2 1 3 1 2 3
3 1 3 1 2 3
4 2 1 13
5 2 1 13
6 2 1 13
7 2 1 13
```

#requests
weight
#cameras
list of cameras

```
7
2 1 0 3 3 2 2 1 1
2 2 0 3 3 2 2 1 1
2 3 0 3 3 2 2 1 1
2 5 4 13 13
2 5 6 13 13
2 2 1 3 3 2 2 1 1
2 3 1 3 3 2 2 1 1
```

#constraints
multiplicity
photos involved
constrained cameras

```
2 51 41 3 3 2 2 1 1
3 40 58 55 13 13 2
3 58 55 41 13 2 3
2 51 48 3 3 2 2 1 1
2 48 46 3 13 1 13
2 7 4 13 13
3 44 58 55 3 13 2
```

constraints
multiplicity
photos involved
constrained cameras

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$
- $x_3 + x_4 + x_5 \leq 2$ → ?

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$

- $x_3 + x_4 + x_5 \leq 2 \rightarrow ?$
- Natively quadratic: $P(x_3 + x_4 + x_5 + s_1 + 2s_2 - 2)^2$

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$
- $x_3 + x_4 + x_5 \leq 2$ → ?
- Natively quadratic: $P(x_3 + x_4 + x_5 + s_1 + 2s_2 - 2)^2$

Extra slack variables s_1, s_2, and 16 terms

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$

- $x_3 + x_4 + x_5 \leq 2 \rightarrow ?$
- Natively quadratic: $P(x_3 + x_4 + x_5 + s_1 + 2s_2 - 2)^2$

Extra slack variables s_1, s_2, and 16 terms

- Cubic: $x_3 + x_4 + x_5 \leq 2 \rightarrow P\, x_3 x_4 x_5$

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$

- $x_3 + x_4 + x_5 \leq 2 \rightarrow ?$
- Natively quadratic: $P(x_3 + x_4 + x_5 + s_1 + 2s_2 - 2)^2$

Extra slack variables s_1, s_2, and 16 terms

- Cubic: $x_3 + x_4 + x_5 \leq 2 \rightarrow P\, x_3 x_4 x_5$

Single-term, no slack, but beyond quadratic

PENALTIES (WRITING CONSTRAINTS INDIRECTLY)

- $x_1 + x_2 \leq 1 \quad \rightarrow \quad P\, x_1 x_2$

- $x_3 + x_4 + x_5 \leq 2$ → ?
- Natively quadratic: $P(x_3 + x_4 + x_5 + s_1 + 2s_2 - 2)^2$

Extra slack variables s_1, s_2, and 16 terms

- Cubic: $x_3 + x_4 + x_5 \leq 2$ → $P\, x_3 x_4 x_5$

Single-term, no slack, but beyond quadratic

Need for reduction to quadratic!

REDUCTION TO QUADRATIC

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin
- E.g. $-\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3 = \min_{s} -s(x_1 + x_2 + x_3 - 2)$, slack variable s

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin
- E.g. $-x_1 x_2 x_3 = \min_s -s(x_1 + x_2 + x_3 - 2)$, slack variable s
- $a\, x_1 x_2 x_3 \cdots x_d$ → ?

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin
- E.g. $-x_1 x_2 x_3 = \min_s -s(x_1 + x_2 + x_3 - 2)$, slack variable s
- $a\, x_1 x_2 x_3 \cdots x_d$ → ?

- **Boros**: based on replacing pairs for slack variables: $x_2 x_3$ for s_1;

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin
- E.g. $-x_1 x_2 x_3 \;=\; \min_s \; -s(x_1 + x_2 + x_3 - 2)$, slack variable s
- $a\, x_1 x_2 x_3 \cdots x_d$ → ?

- **Boros**: based on replacing pairs for slack variables: $x_2 x_3$ for s_1;
- **Ishikawa**: based on whole term, efficient when a<0 (1 slack), not so for a>0 (d/2 slacks)

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials, i.e. same argmin
- E.g. $-x_1x_2x_3 \ = \ \min_s \ - s(x_1 + x_2 + x_3 - 2)$, slack variable s
- $a\, x_1x_2x_3 \cdots x_d$ → ?

- **Boros**: based on replacing pairs for slack variables: x_2x_3 for s_1;
- **Ishikawa**: based on whole term, efficient when a<0 (1 slack), not so for a>0 (d/2 slacks)

- Contribution: mixed method: Ishikawa for a<0, Boros for a>0

THE STANDARD ENCODING

THE STANDARD ENCODING

- Fixed-length encoding is wasteful

THE STANDARD ENCODING

- Fixed-length encoding is wasteful
- Must reduce number of qubits!

THE STANDARD ENCODING

- Fixed-length encoding is wasteful
- Must reduce number of qubits!

- standard encoding: either $x_{i,1}, x_{i,2}, x_{i,3}$ (mono) or x_i (stereo)

THE STANDARD ENCODING

- Fixed-length encoding is wasteful
- Must reduce number of qubits!

- standard encoding: either $x_{i,1}, x_{i,2}, x_{i,3}$ (mono) or x_i (stereo)

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\,((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\,((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$~~

THE STANDARD ENCODING

- Fixed-length encoding is wasteful
- Must reduce number of qubits!

- standard encoding: either $x_{i,1}, x_{i,2}, x_{i,3}$ (mono) or x_i (stereo)

$$\sum_j x_{i,j} \le 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \le 1 \quad \forall\, ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \le 2 \quad \forall\, ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

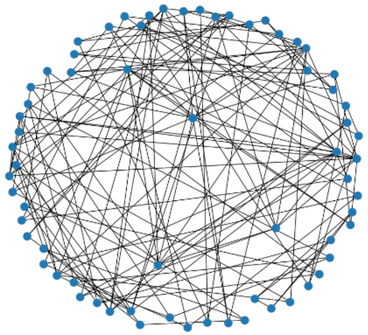
~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$~~

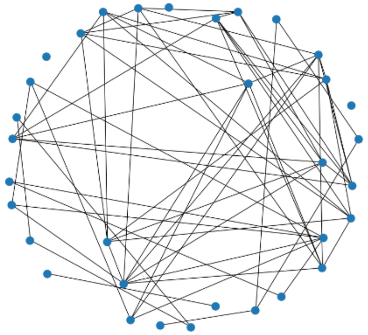
SPOT5 Instance 15

15 requests
9 mono
6 stereo

14 constraints
11 binary
3 ternary



fixed-length



standard

THE STANDARD ENCODING

- Fixed-length encoding is wasteful
- Must reduce number of qubits!

- standard encoding: either $x_{i,1}, x_{i,2}, x_{i,3}$ (mono) or x_i (stereo)

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall\, ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall\, ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

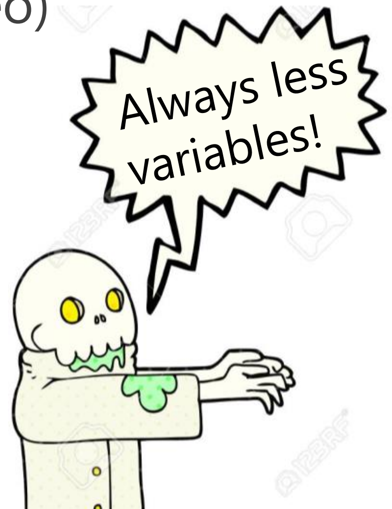
$$x_{i,4} = 0 \quad \forall i \in M$$

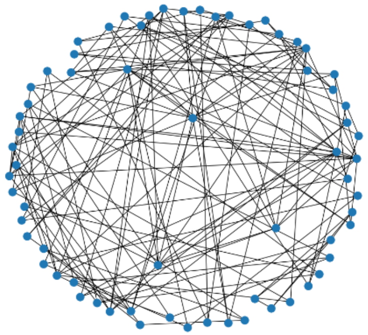
$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

SPOT5 Instance 15

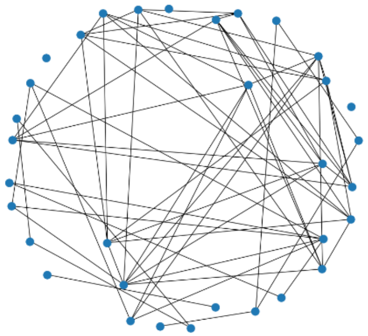
15 requests
9 mono
6 stereo

14 constraints
11 binary
3 ternary





fixed-length



standard

THE DENSE ENCODING

THE DENSE ENCODING

- Then let’s make an even denser encoding!

THE DENSE ENCODING

- Then let's make an even denser encoding!
- Stereo: x_i only

THE DENSE ENCODING

- Then let's make an even denser encoding!
- Stereo: x_i only
- Mono: only 2 bits, $x_{i,0}, x_{i,1}$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

THE DENSE ENCODING

- Then let's make an even denser encoding!
- Stereo: x_i only
- Mono: only 2 bits, $x_{i,0}, x_{i,1}$
- Removes more constraints:

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

THE DENSE ENCODING

- Then let's make an even denser encoding!
- Stereo: x_i only
- Mono: only 2 bits, $x_{i,0}, x_{i,1}$
- Removes more constraints:

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

~~$\sum_j x_{i,j} \leq 1 \quad \forall i$~~

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$~~

THE DENSE ENCODING

- Then let's make an even denser encoding!
- Stereo: x_i only
- Mono: only 2 bits, $x_{i,0}, x_{i,1}$
- Removes more constraints:

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$

THE DENSE ENCODING, COMPARED

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$ (struck through)

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$ (struck through)

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$ (struck through)

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

THE DENSE ENCODING, COMPARED

- However… how do C_2, C_3 look like now?

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

THE DENSE ENCODING, COMPARED

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

- However… how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1

THE DENSE ENCODING, COMPARED

- However… how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\ x_{4,3} x_{5,3}$

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

THE DENSE ENCODING, COMPARED

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

- However... how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\ x_{4,3}x_{5,3}$
- dense encoding:

$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

- However... how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\ x_{4,3} x_{5,3}$
- dense encoding:

  e.g. $P\ x_{4,0}\ x_{4,1}\ x_{5,0}\ x_{5,1}$

$$\sum_j x_{i,j} \leq 1 \quad \forall i$$ (struck through)

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$x_{i,4} = 0 \quad \forall i \in M$$ (struck through)

$$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$$ (struck through)

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

$x_{5,0}x_{5,1}$ \\ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

- However... how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\, x_{4,3} x_{5,3}$
- dense encoding:

  e.g. $P\, x_{4,0}\, x_{4,1}\, x_{5,0}\, x_{5,1}$

- Quartic terms for quadratic constraint!

~~$\sum_j x_{i,j} \leq 1 \quad \forall i$~~

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$~~

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

~~$\sum_j x_{i,j} \le 1 \quad \forall i$~~

$$x_{p,j_p} + x_{q,j_q} \le 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \le 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$~~

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

- However… how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\ x_{4,3} x_{5,3}$
- dense encoding:

  e.g. $P\ x_{4,0}\ x_{4,1}\ x_{5,0}\ x_{5,1}$
- Quartic terms for quadratic constraint!
- C_3 : 3 7 8 9 2 3 13

$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

- However… how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\ x_{4,3} x_{5,3}$
- dense encoding:

  e.g. $P\ x_{4,0}\ x_{4,1}\ x_{5,0}\ x_{5,1}$
- Quartic terms for quadratic constraint!
- C_3 : 3 7 8 9 2 3 13
- standard encoding: $P\ x_{7,2}\ x_{8,3}\ x_9$ (cubic)

$$~~\sum_j x_{i,j} \le 1 \quad \forall i~~$$

$$x_{p,j_p} + x_{q,j_q} \le 1 \quad \forall ((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \le 2 \quad \forall ((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

$$~~x_{i,4} = 0 \quad \forall i \in M~~$$

$$~~x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}~~$$

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

~~$\sum_j x_{i,j} \le 1 \quad \forall i$~~

$$x_{p,j_p} + x_{q,j_q} \le 1 \quad \forall((p,j_p),(q,j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \le 2 \quad \forall((p,j_p),(q,j_q),(r,j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1,2,3\}$~~

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

- However… how do C_2, C_3 look like now?
- C_2: 2 4 5 3 3 2 2 1 1
- standard encoding (also fixed-length):
  e.g. $P\, x_{4,3} x_{5,3}$
- dense encoding:

  e.g. $P\, x_{4,0}\, x_{4,1}\, x_{5,0}\, x_{5,1}$

- Quartic terms for quadratic constraint!

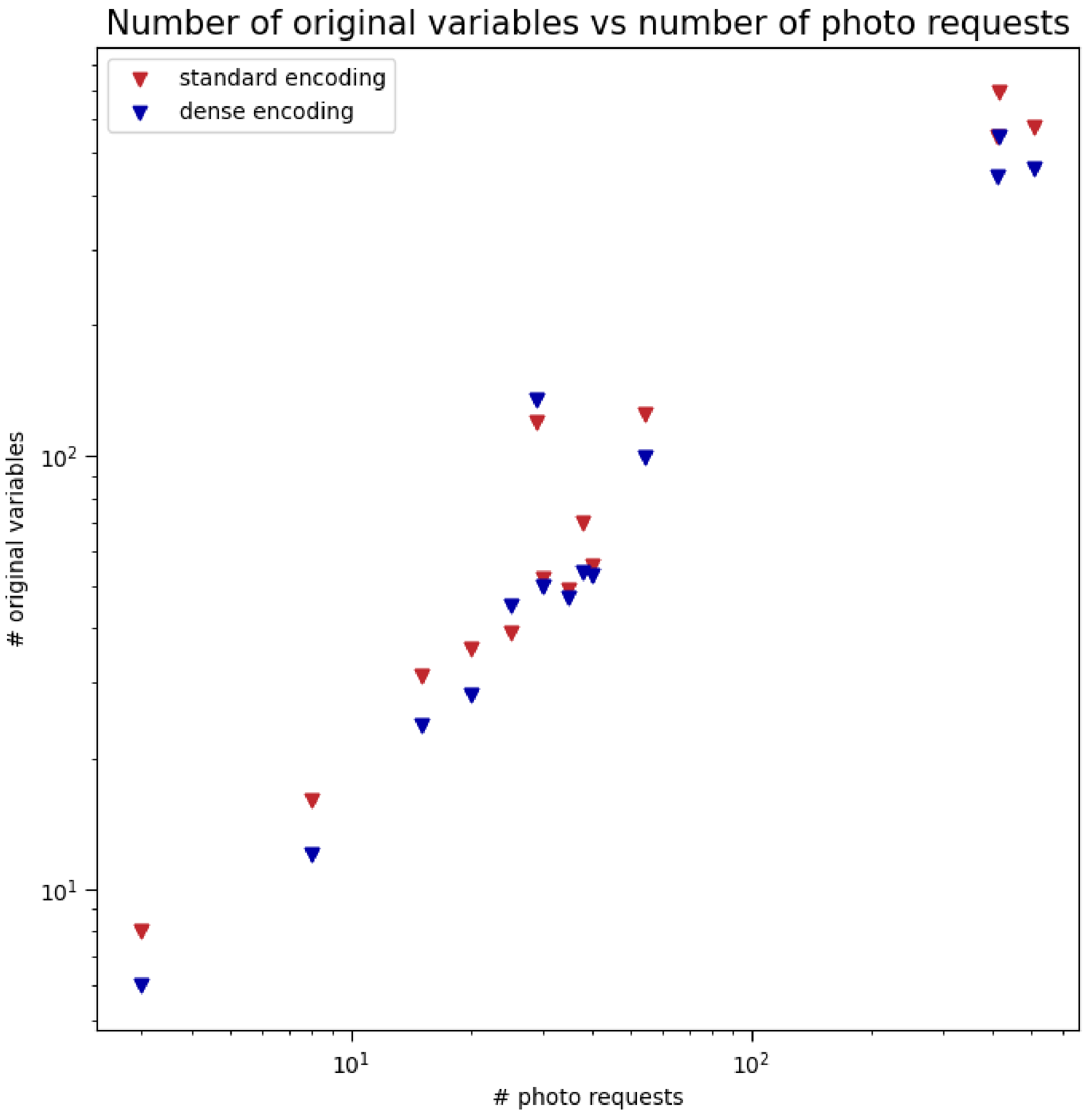
$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

- C_3 : 3 7 8 9 2 3 13
- standard encoding: $P\, x_{7,2}\, x_{8,3}\, x_9$ (cubic)
- dense encoding: $P\, x_{7,0}(1 - x_{7,1}) x_{8,0} x_{8,1} x_9$ (quintic!)

TRADEOFF IN ENCODINGS

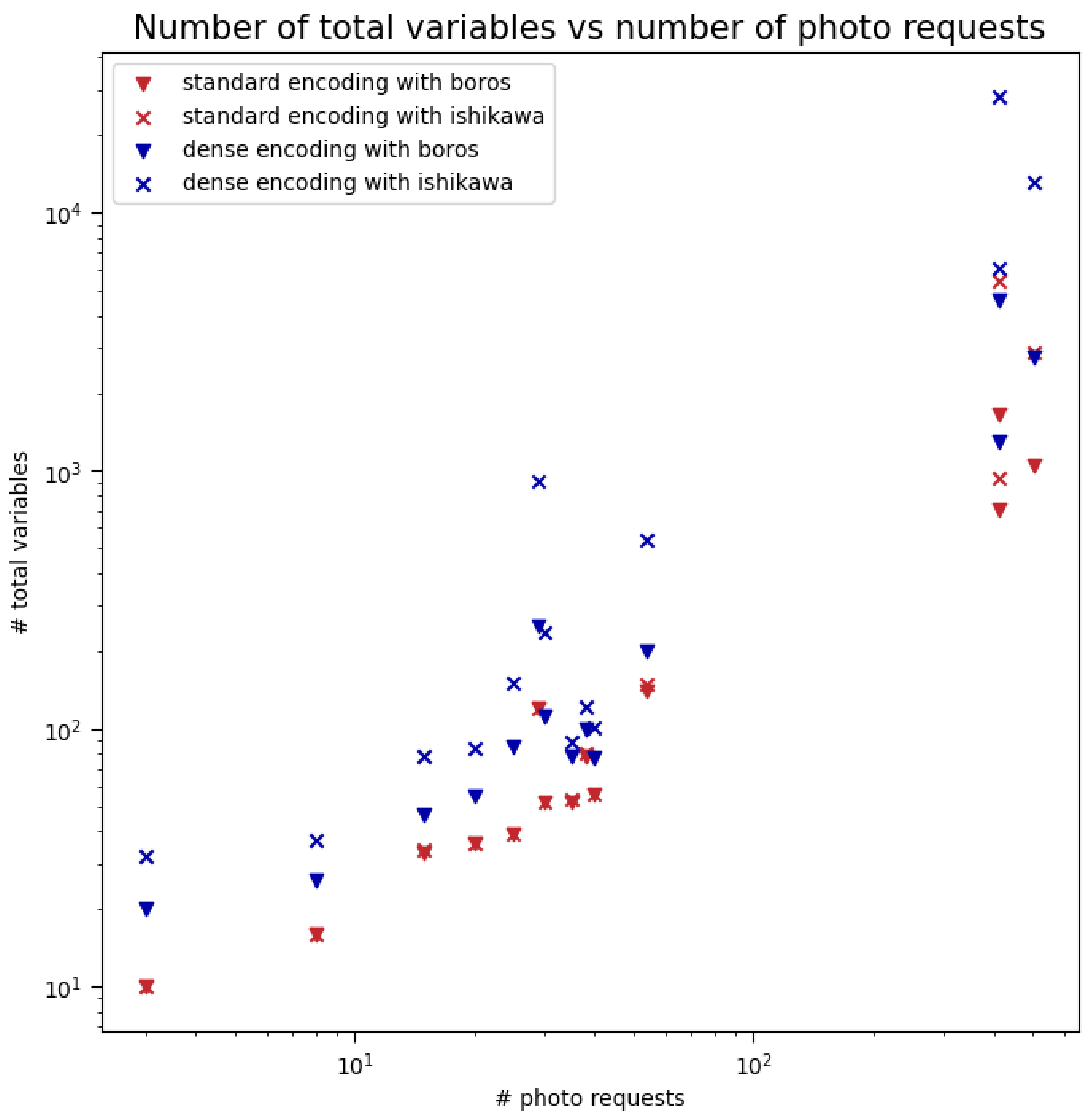
TRADEOFF IN ENCODINGS

- Dense encoding: less original variables,



TRADEOFF IN ENCODINGS

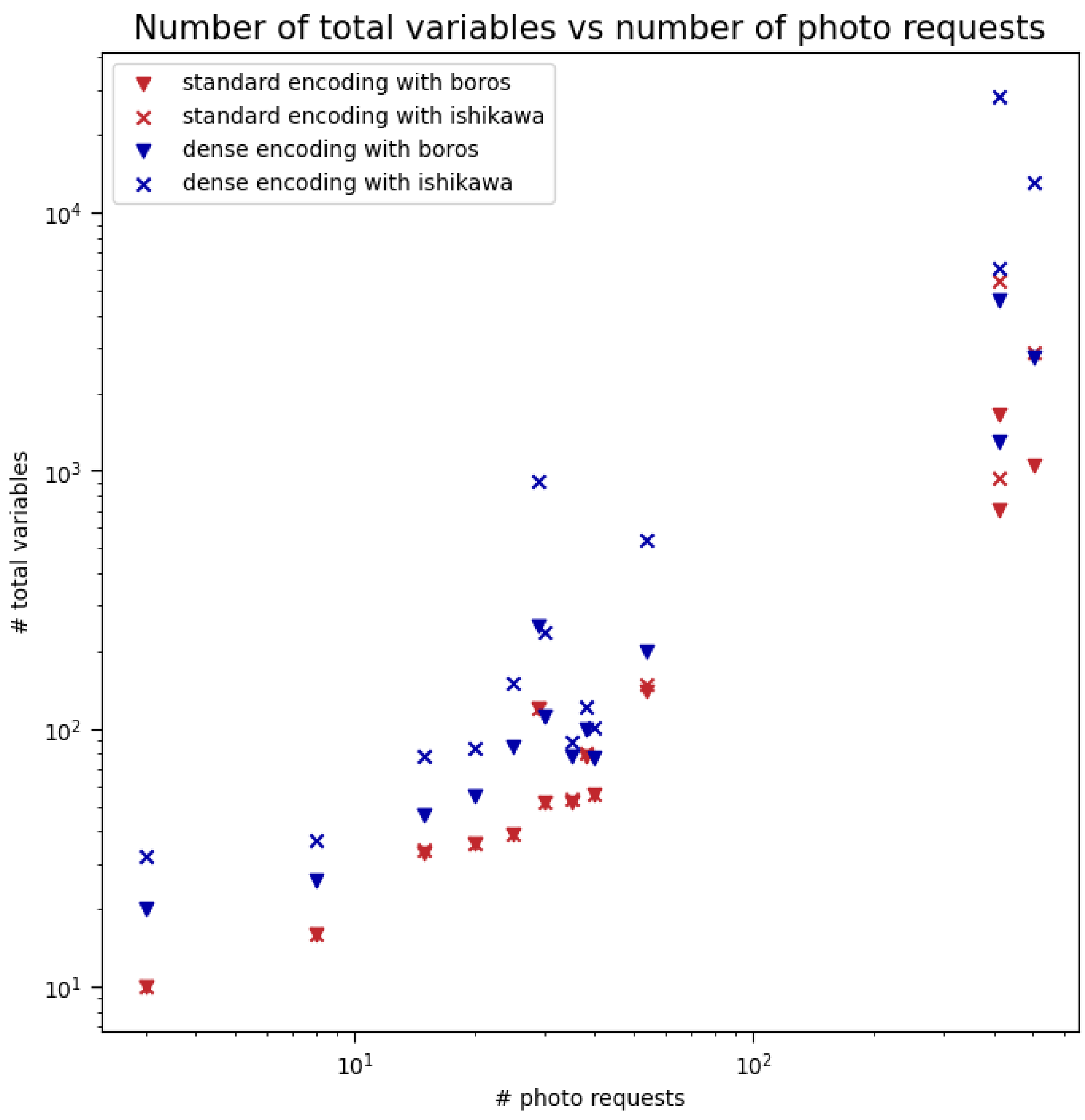
- Dense encoding: less original variables,
- but more added variables from reduction technique



TRADEOFF IN ENCODINGS

- Dense encoding: less original variables,
- but more added variables from reduction technique

- In total, dense encoding requires more variables

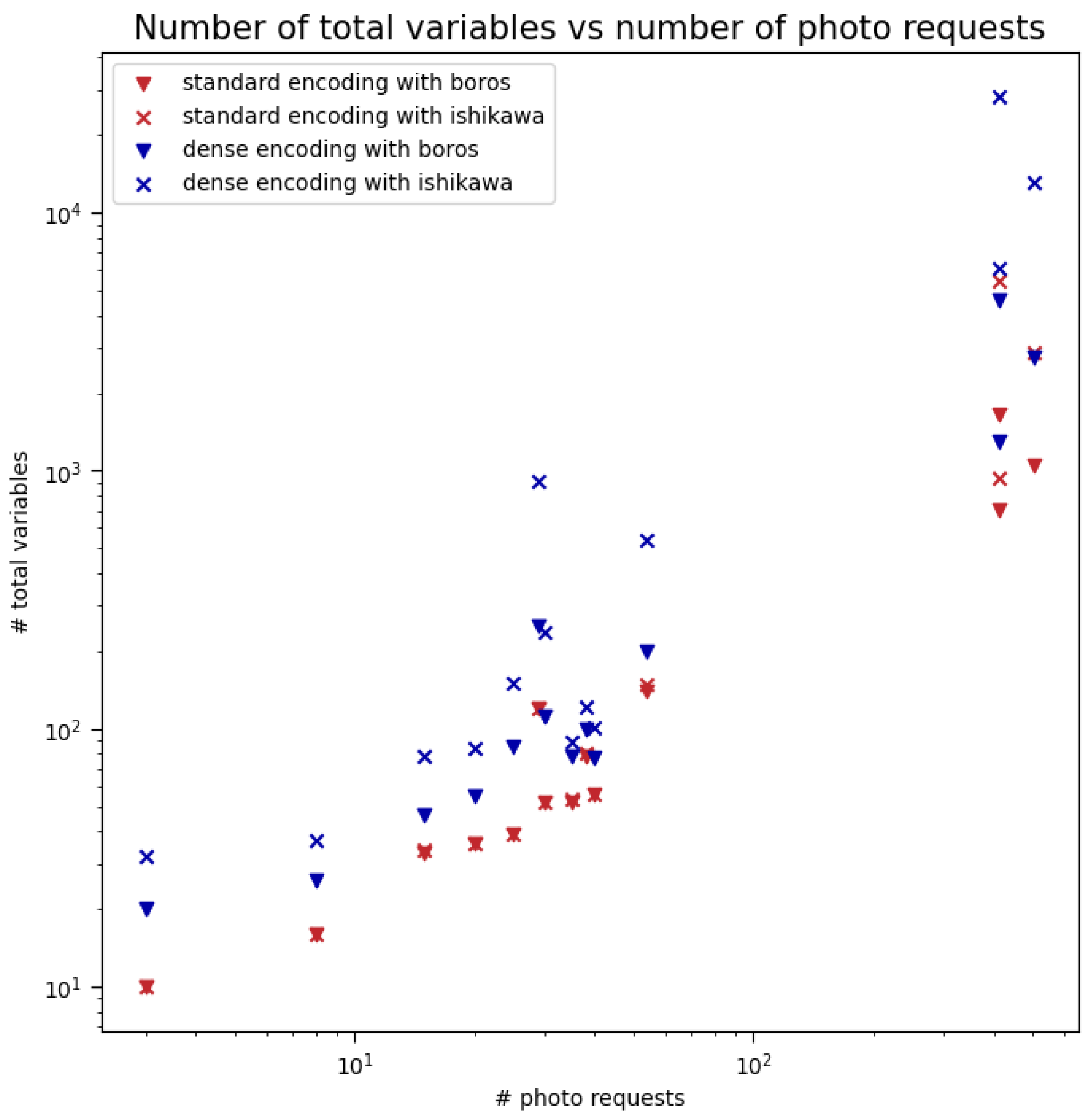


TRADEOFF IN ENCODINGS

- Dense encoding: less original variables,
- but more added variables from reduction technique

- In total, dense encoding requires more variables

- More logical qubits needed

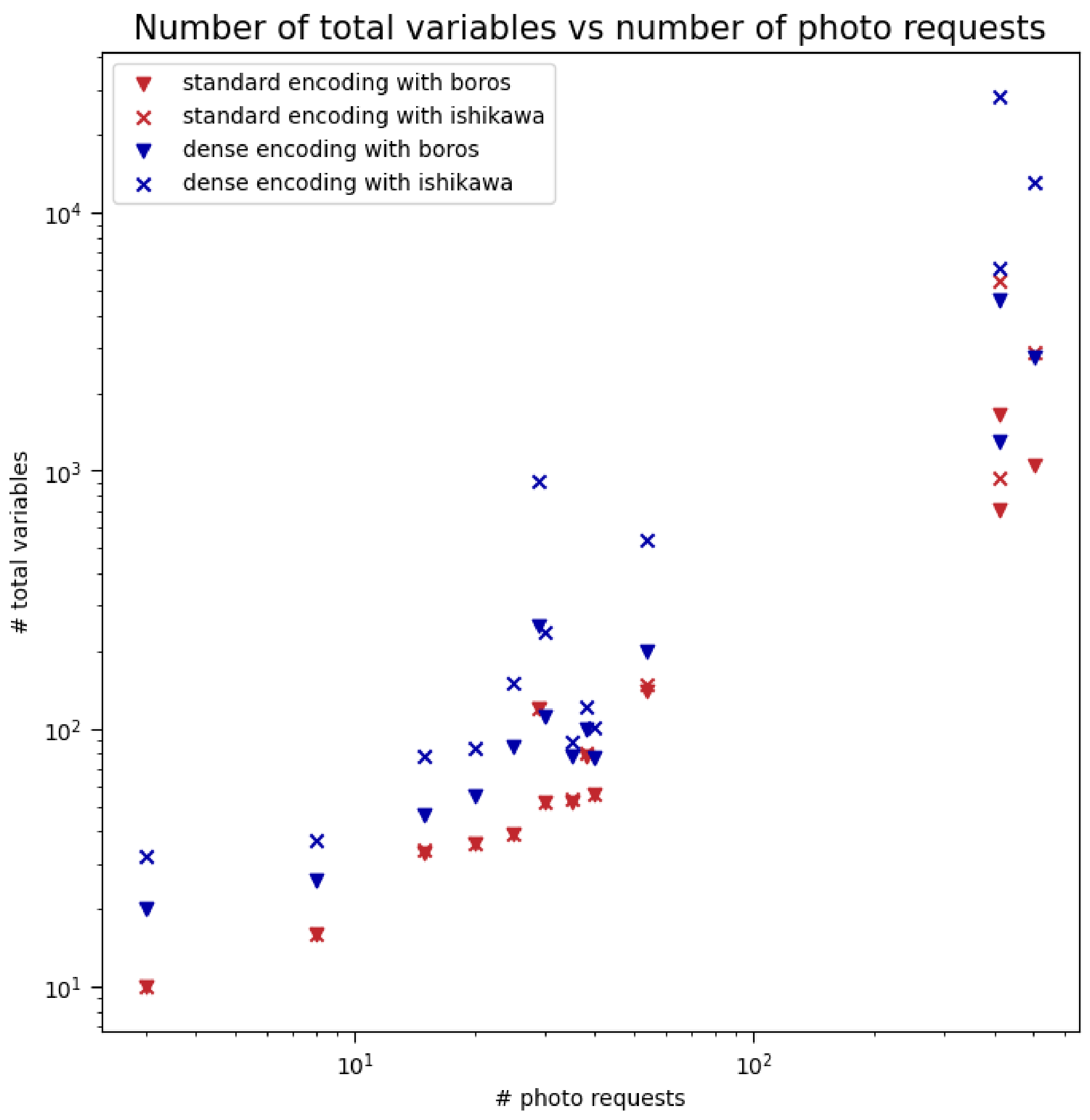


TRADEOFF IN ENCODINGS

- Dense encoding: less original variables,
- but more added variables from reduction technique

- In total, dense encoding requires more variables

- More logical qubits needed



CONCLUSIONS

CONCLUSIONS

- Encoding drastically changes what can be performed in quantum computer

CONCLUSIONS

- Encoding drastically changes what can be performed in quantum computer
- Although reduction of qubits of utmost necessity, not simply “the denser the better”

THANK YOU FOR YOUR ATTENTION

FROM SIASP TO QUBO

- SIASP is not QUBO!

i) Constraints (not U)

- Penalty terms
- E.g.: constraint $x_1 + x_2 \leq 1$,   extra term $P\, x_1 x_2$

ii) Higher-order polynomial (not Q)

- Reduction to quadratic form with extra variables
- Equivalent polynomial, i.e. one that has the same argmin

REDUCTION TO QUADRATIC

- Finding **equivalent** quadratic polynomials
- E.g.: $-x_1x_2x_3$ → $-s(x_1 + x_2 + x_3 - 2)$
- Slack variable s
- In general: $a\, x_1x_2x_3 \cdots x_d$ → ?
- Boros (E. Boros et al, Discrete Applied Mathematics **123** 155 (2002))
- i) replace pair, e.g. x_2x_3, for slack s_1;
  ii) add term $M(x_2x_3 - 2x_2s_1 - 2x_3s_1 + 3s_1)$ to enforce $s_1 = x_2x_3$;
  iii) repeat until all quadratic
- 1 added variable per replaced pair

- Ishikawa (H. Ishikawa, IEEE Trans. on Patt. Analysis and Mach. Intellig., **33**, 1234 (2011))
- i) if a<0, → $a\, s_1(\sum_i x_i - d + 1)$ [above]
  ii) if a>0, → $\sim a \sum_i s_i\, [2(2i - \sum_i x_i) - 1] \;+\; a \sum_{i \neq j} x_i x_j$
- Variables added: i) 1 per term, ii) ~d/2 per term

THE DENSE ENCODING, COMPARED

~~$\sum_j x_{i,j} \leq 1 \quad \forall i$~~

$$x_{p,j_p} + x_{q,j_q} \leq 1 \quad \forall ((p, j_p), (q, j_q)) \in C_2$$

$$x_{p,j_p} + x_{q,j_q} + x_{r,j_r} \leq 2 \quad \forall ((p, j_p), (q, j_q), (r, j_r)) \in C_3$$

~~$x_{i,4} = 0 \quad \forall i \in M$~~

~~$x_{i,j} = 0 \quad \forall i \in S, \forall j \in \{1, 2, 3\}$~~

	$x_{i,0}$	$x_{i,1}$
no photo	0	0
cam 1	0	1
cam 2	1	0
cam 3	1	1

- However... how do C_2, C_3 look like now?
- C_2: 
- standard encoding (also fixed-length):

  $x_{4,3} + x_{5,3} \leq 1,$ $P\, x_{4,3}x_{5,3}$ $\quad$ $x_{4,2} + x_{5,2} \leq 1,$ $P\, x_{4,2}x_{5,2}$ $\quad$ $x_{4,1} + x_{5,1} \leq 1$ $P\, x_{4,1}x_{5,1}$

- dense encoding:

  $P\, x_{4,0}\, x_{4,1}\, x_{5,0}\, x_{5,1} \leq 1$

  $P\left(1 - x_{4,0}\right) x_{4,1} \left(1 - x_{5,0}\right) x_{5,1} \leq 1$

  $P\, x_{4,0}(1 - x_{4,1})x_{5,0}(1 - x_{5,1}) \leq 1$

- Quartic terms for quadratic constraint!
- C_3 : 3 7 8 9 2 3 13
- standard encoding: $P\, x_{7,2}\, x_{8,3}\, x_9$ (cubic) $\quad$ dense encoding: $P\, x_{7,0}\left(1 - x_{7,1}\right)x_{8,0}x_{8,1}x_9$ (quintic!)

$x_{5,0}x_{5,1}$ \ $x_{4,0}x_{4,1}$	00	01	11	10
00	0	0	0	0
01	0	P	0	0
11	0	0	P	0
10	0	0	0	P

THE DENSE ENCODING, COMPARED

- quadratic constraint
  standard: $P\ x_{4,3}x_{5,3}$
  dense encoding: $P\ x_{4,0}\ x_{4,1}\ x_{5,0}\ x_{5,1}$
- ternary constraint
  standard: $P\ x_{7,2}\ x_{8,3}\ x_9$
  dense encoding: $P\ x_{7,0}(1 - x_{7,1})x_{8,0}x_{8,1}x_9$