Gradient magnetometry with various types of spin ensembles

Single atomic ensembles, chain of spins & two different ensembles

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- May 30, 2023 -

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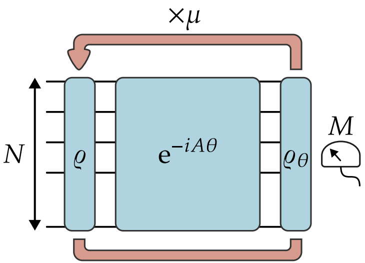
Outline

1. Multiparametric Quantum Metrology
   - Cramér-Rao precision bound and quantum Fisher information
   - Multiparametric qFI matrix and simultaneous estimation
2. System setup and precision bounds of the gradient parameter estimation for various states
   - Gradient magnetometry and basic setup of the system
   - Precision bounds for various systems and different spin states
3. Conclusions

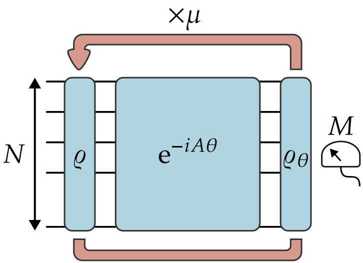


- The *quantum Cramér-Rao (qCR) bound* provides an upper bound for the precision

$$\frac{1}{(\Delta\theta)^2} \leqslant \mu \mathcal{F}_{\mathrm{Q}}[\varrho, A].$$

Quantum Metrology

Quantum Fisher Information



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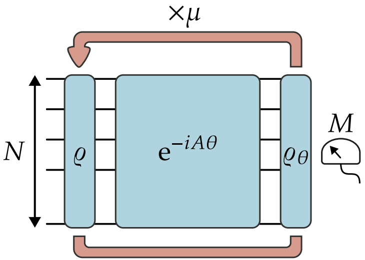

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- Quantum Fisher information

$$\mathcal{F}_{\mathrm{Q}}[\varrho, A] = 2 \sum_{\lambda \neq \mu} \frac{(p_\lambda - p_\mu)^2}{p_\lambda + p_\mu} |\langle\lambda|A|\mu\rangle|^2$$

written on the eigenbasis of the state, $\varrho = \sum p_\lambda |\lambda\rangle\langle\lambda|$.

[M.G.A. Paris (2009), IJQI **7**, 125]

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Properties of the qFI for a single parameter estimation problem

1. It is independent of the measurement. An optimal measurement exists though, which saturates the qCR bound.

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3. For pure states $\mathcal{F}_{\mathrm{Q}}[|\Psi\rangle, A] = 4(\Delta A)^2_\Psi$.

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Entanglement

1. Separable states can achieve at most the so called Shot-noise limit (SNL),

$$\mathcal{F}_{\mathrm{Q}}[\varrho_{\mathrm{sep}}, H] \sim N.$$

2. An ultimate limit is obtained maximizing the qFI over all pure states

$$\max_{|\Psi\rangle} \mathcal{F}_{\mathrm{Q}}[|\Psi\rangle, H] = N^2,$$

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Hence, entanglement is needed to overcome the SNL.

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E.g. entanglement criteria based on qFI

- Due to its tight relation with the variance, qFI has been used to improve some entanglement conditions.

[G Tóth (2022), PRR **4** 013075]

- Ion chains can be used to estimate the magnetic field as a function of position, $B(x)$.

[Matteo Fadel *et al.*, arXiv.org:2201.11081]

Multiparametric Quantum Metrology

Motivation

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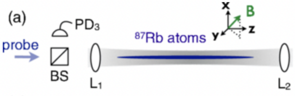

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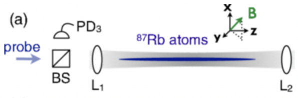

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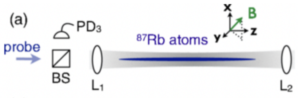

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We assume that the magnetic field is pointing in the z-direction and its Taylor expansion around the origin is

$$\boldsymbol{B} = (0, 0, B_0) + (0, 0, xB_1) + \mathcal{O}(x^2).$$

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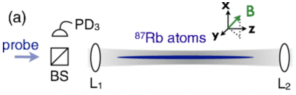

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In general, one cannot avoid a global rotation of the state.

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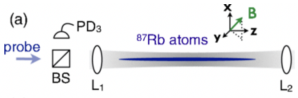

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We want to estimate B_1.

Consider the following evolution for the state

$$\varrho_{\boldsymbol{\theta}} = \mathrm{e}^{-i\sum_k A_k\theta_k}\,\varrho\,\mathrm{e}^{+i\sum_k A_k\theta_k}\,.$$

- In this case the CR bound is a matrix inequality for the covariance matrix

$$\mathrm{Cov}[\theta_i,\theta_j] \geqslant \frac{1}{\mu}(\mathcal{F}_{\mathrm{Q}}{}^{-1})_{i,j},$$

where $\mathrm{Cov}[\theta_i,\theta_j] = \langle\theta_i\theta_j\rangle - \langle\theta_i\rangle\langle\theta_j\rangle$.

Multiparametric Quantum Metrology

Cramér-Rao matrix inequality

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- When $[A_i, A_j] = 0$, the bounds can be saturated.

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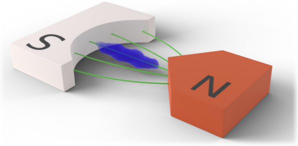
Gradient magnetometry and basic setup



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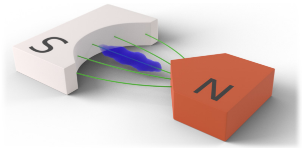

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- In this work we assume that the position state is an statistical mixture of point-like particles

$$\varrho^{(\mathrm{x})} = \int \frac{P(\boldsymbol{x})}{\langle \boldsymbol{x}|\boldsymbol{x}\rangle} |\boldsymbol{x}\rangle\langle \boldsymbol{x}|.$$

Gradient magnetometry and basic setup

- The atoms interact only with the magnetic field, $h^{(n)} = \gamma B_z{}^{(n)} \otimes j_z{}^{(n)}$, where $\gamma = g\mu_{\mathrm{B}}$. The collective Hamiltonian is

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- The two unknown parameters are B_0 and B_1 are encoded in b_0 and b_1 acting onto the state with the following unitary operator

$$U = \mathrm{e}^{-i(b_0 H_0 + b_1 H_1)},$$

where

$$H_0 := J_z = \sum_n j_z{}^{(n)} \qquad \text{and} \qquad H_1 = \sum_n x^{(n)} \otimes j_z{}^{(n)}.$$

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In the following we are interested on the precision bound for b_1, the gradient parameter.

Gradient magnetometry and basic setup

Precision bounds for states **insensitive** to the homogeneous B_0

For states that commute with the homogeneous field, $[\varrho, J_z] = 0$, the precision bound is

$$\frac{1}{(\Delta b_1)^2} \leqslant \mathcal{F}_{\mathrm{Q}}[\varrho, H_1],$$

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For states sensitive to global rotations of the spin state, the precision bound is

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Chain of qubits

$$P(\boldsymbol{x}) = \prod_n \delta(x_n - na).$$

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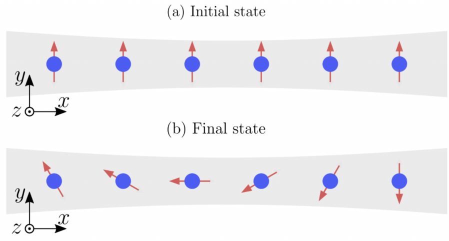
Totally polarized $|0\rangle_y^{\otimes N}$ state under a magnetic field pointing towards the z-direction



$$\frac{1}{(\Delta b_1)^2} \leqslant \sum_{n,m} nma^2 \mathcal{F}_Q[|0\rangle_y^{\otimes N}, j_z{}^{(n)}, j_z{}^{(m)}] - \frac{\left(\sum_n na \mathcal{F}_Q[|0\rangle_y^{\otimes N}, j_z{}^{(n)}, J_z]\right)^2}{\mathcal{F}_Q[|0\rangle_y^{\otimes N}, J_z]}$$

Chain of qubits

$$P(\boldsymbol{x}) = \prod_n \delta(x_n - na).$$

- Mean particle position:

$$\mu = a\frac{N+1}{2}$$

- Variance of the particle positions:

$$\sigma^2 = a^2\frac{N^2-1}{12}$$

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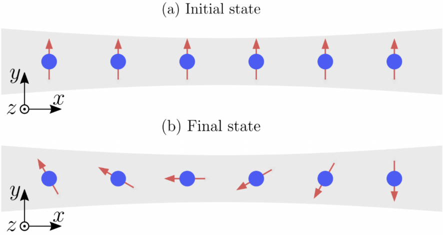

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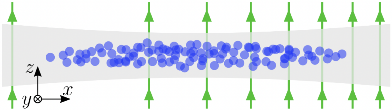
$$= a^2\frac{N^2-1}{12}N = \sigma^2 N.$$

Single ensemble of point-like spin-$\frac{1}{2}$ atoms

Permutationally invariant PDF

$$P(\boldsymbol{x}) = \frac{1}{N!} \sum_{k \in S_N} \mathcal{P}_k[P(\boldsymbol{x})]$$



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- $\sigma^2 = \int x_n^2 P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, if the origin is at 0.
- $\eta = \int x_n x_m P(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$ for $n \neq m$.
  $\eta \in [-\sigma^2/(N-1), \sigma^2]$.

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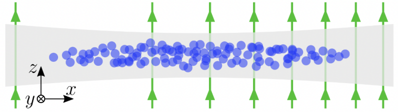

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  $\eta \in [-\sigma^2/(N-1), \sigma^2]$.

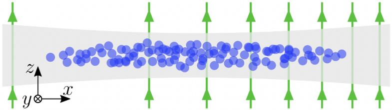
Precision CR bound

$$\frac{1}{(\Delta b_1)^2} \leqslant (\sigma^2 - \eta)\sum_n \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, j_z{}^{(n)}] + \eta\, \mathcal{F}_{\mathrm{Q}}[\varrho^{(\mathrm{s})}, J_z]$$

Single ensemble of point-like spin-$\frac{1}{2}$ atoms

Permutationally invariant PDF

$$P(\boldsymbol{x}) = \frac{1}{N!}\sum_{k\in S_N}\mathcal{P}_k[P(\boldsymbol{x})]$$



[N Behbood *et al.* (2014), PRL **113** 093601]

- $\mu = \int x_n P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$.
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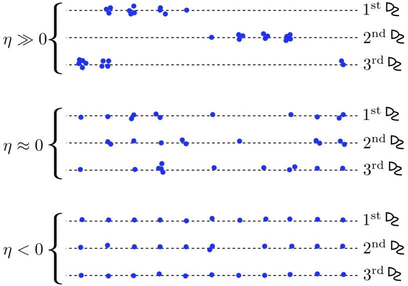

Precision CR bound

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Single ensemble of point-like spin-$\frac{1}{2}$ atoms

Precision bounds for various spin states

Singlet states

$$\varrho^{(\mathrm{s})} = \sum_{\lambda} p_{\lambda} |0,0,i\rangle\langle 0,0,i|$$

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Best separable state

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle_{\mathrm{sep}}, {j_z}^{(n)}, {j_z}^{(m)}] = \begin{cases} 4(\Delta {j_z}^{(n)})^2 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

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$|\text{GHZ}\rangle$ states

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and

$$\mathcal{F}_Q[|\text{GHZ}\rangle_x, J_z] = N^2.$$

Hence,

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*More in PRA **97**, 053603 (2018)*

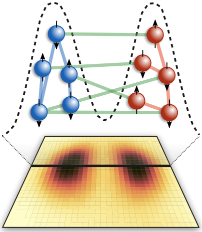
Double well of atoms

$$P(\boldsymbol{x}) = \prod_{n=1}^{N/2} \delta(x_n + a) \prod_{n=N/2+1}^{N} \delta(x_n - a)$$

The contribution of the position of the particles:

$$\int x_n P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \begin{cases} -a \\ +a \end{cases} \quad \text{and} \quad \int x_n x_m P(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \begin{cases} +a^2 \\ -a^2 \end{cases}$$

- In this case the mean position is $\mu = 0$ and the variance is $\sigma^2 = a^2$.



[K Langle *et al.* (2018), Sci. **360** 6387]

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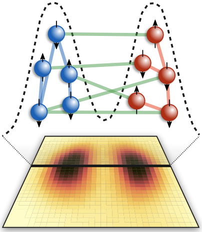

[K Langle *et al.* (2018), Sci. **360** 6387]

For spin-$\frac{1}{2}$ system, the state that maximizes the bound is

$$|\psi\rangle = \frac{|\overbrace{0,\ldots,0}^{N/2},\overbrace{1,\ldots,1}^{N/2}\rangle + |1,\ldots,1,0,\ldots,0\rangle}{\sqrt{2}}, \qquad \text{and} \qquad \frac{1}{(\Delta b_1)^2} \leqslant \sigma^2 N^2.$$

Double well of atoms

Product of two equal spin states

For states of the type $|\psi\rangle^{(\mathrm{L})}\otimes|\psi\rangle^{(\mathrm{R})}$, we have that

$$\mathcal{F}_{\mathrm{Q}}[|\psi\rangle^{(\mathrm{L})}\otimes|\psi\rangle^{(\mathrm{R})}, {j_z}^{(n)}, {j_z}^{(m)}] = \begin{cases} \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, {j_z}^{(n)}, {j_z}^{(m)}] & \text{if } n \text{ and } m \text{ same well} \\ 0 & \text{otherwise} \end{cases}$$

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Hence, the precision bounds can be simply computed for $N/2$ particles at one of the wells,

$$\frac{1}{(\Delta b_1)^2} \leqslant 2\sigma^2 \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, J_z{}^{(N/2)}] \leqslant \sigma^2 N^2/2.$$

Double well of atoms

What if we want to estimate both parameters?

If we assume $a = 1$, we have that $H_0 = J_z{}^{(\mathrm{L})} + J_z{}^{(\mathrm{R})}$ and $H_1 = J_z{}^{(\mathrm{L})} - J_z{}^{(\mathrm{R})}$.

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2\mathcal{F}_{\mathrm{Q}}[\rho, J_z{}^{(\mathrm{L})}] + 2\mathcal{F}_{\mathrm{Q}}[\rho, J_z{}^{(\mathrm{R})}]$$

Separable states

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2N_{\mathrm{L}} + 2N_{\mathrm{R}} = 2N.$$

Heisenberg limit for evenly split systems

$$\mathcal{F}_{\mathrm{Q}}[\rho, H_0] + \mathcal{F}_{\mathrm{Q}}[\rho, H_1] = 2N_{\mathrm{L}}^2 + 2N_{\mathrm{R}}^2 = N^2.$$

Examples

$$|\mathrm{GHZ}\rangle \quad \rightarrow \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_0] = N^2 \quad \text{and} \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_1] = 0.$$

$$|\psi\rangle = \frac{|\overbrace{0,\ldots,}^{N/2}\overbrace{1,\ldots}^{N/2}\rangle + |1,\ldots,0,\ldots\rangle}{\sqrt{2}} \quad \rightarrow \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_1] = N^2 \quad \text{and} \quad \mathcal{F}_{\mathrm{Q}}[|\psi\rangle, H_0] = 0.$$

Conclusions

- In principle, the **effect of an unknown global rotation** has to be considered.
- For a **single ensemble** with localized particles, a method with a huge practical advantage, the shot-noise limit can be surpassed if and only if there is a **strong statistical correlation between the particle positions**.
- There is a **trade-off** between homogeneous and gradient magnetometry if one wants to estimate both parameters at the same time.

Summary

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Thank you for your attention!