ICE-8 Santiago de Compostela / 31st May 2023

Non–unitary ground state preparation for the $\mathbb{Z}_2$ lattice gauge theory in the presence of noise

Jesús Cobos Jiménez

Universidad del País Vasco/
Euskal Herriko Unibertsitatea

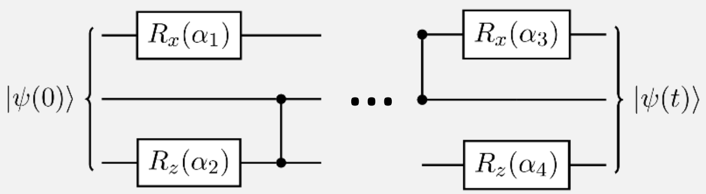
Collaborators:
David Locher (RWTH & Institute for Theoretical Electronics)
Markus Müller (RWTH & Institute for Theoretical Electronics)
Alejandro Bermúdez (IFT)
Enrique Rico (UPV/EHU & IKERBASQUE)

eman ta zabal zazu
Universidad del País Vasco
Euskal Herriko Unibertsitatea

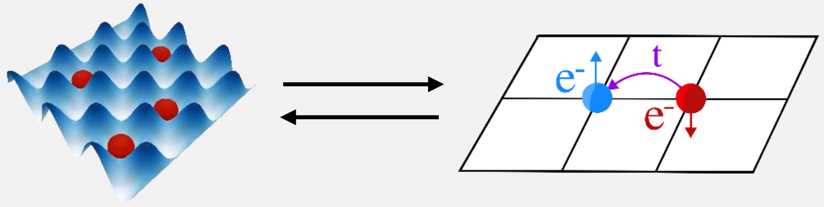
EHU QC
EHU Quantum Center

1. Introduction

- Quantum simulators are a promising tool for studying many-body quantum systems.



Digital approach (Our focus)

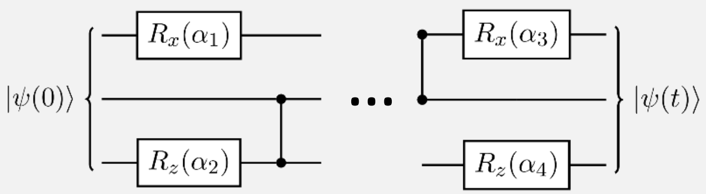


Analog approach

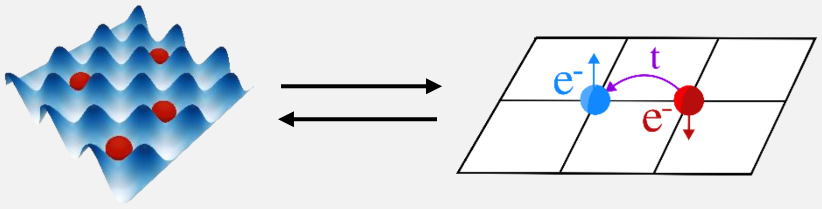
- One relevant problem is the measurement of ground state properties.

1. Introduction

- Quantum simulators are a promising tool for studying many-body quantum systems.



Digital approach (Our focus)



Analog approach

- One relevant problem is the measurement of ground state properties.
- It is widely accepted that variational quantum algorithms are the only implementable approach in the NISQ era.

$$|\psi(\boldsymbol{\alpha})\rangle = \hat{U}_k(\alpha_M)\,\hat{U}_{k-1}(\alpha_{M-1})\ldots\hat{U}_1(\alpha_1)\,|\psi(0)\rangle \qquad \boldsymbol{\alpha}^* = \underset{\boldsymbol{\alpha}}{\operatorname{argmin}}\,\langle\psi(\boldsymbol{\alpha})|\,\hat{H}\,|\psi(\boldsymbol{\alpha})\rangle$$

$|\psi(0)\rangle \longrightarrow$ Reference state (Easily prepared) $\qquad \langle\psi|\,\hat{H}\,|\psi\rangle \geq E_{\text{g.s.}}\ \forall\ |\psi\rangle$

- We propose a variational ansatz for the ground state of the pure $\mathbb{Z}_2$ lattice gauge theory.

2. The pure $\mathbb{Z}_2$ lattice gauge theory

- Its gauge degrees of freedom are two level systems (qubits). Matter not considered.
- Contains interesting non-trivial phenomena: Ising-like second order phase transition, confinement and the presence of a topological phase.
- The ground state degeneracy in the topological phase only appears in finite lattices when periodic (Toric code) or surface-code-like boundary conditions are used.

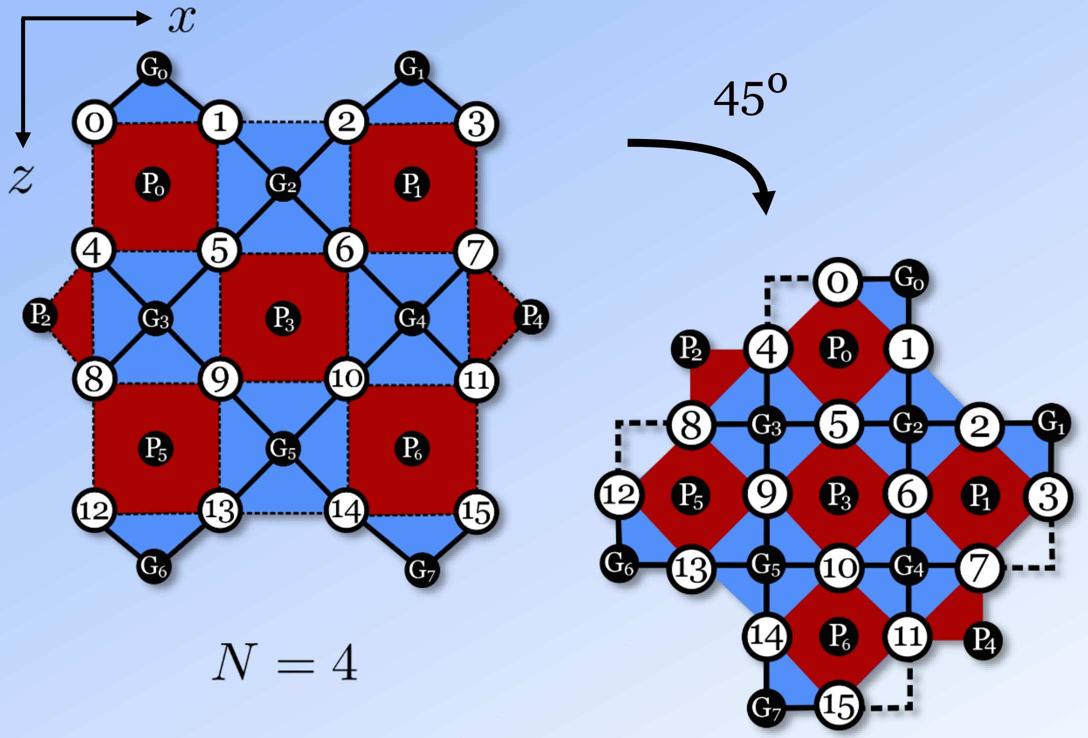
Hamiltonian

$$\hat{H} = -\underbrace{\sum_{n=0}^{N-1} \hat{\sigma}_n^x}_{\text{Electric term}} - \lambda \underbrace{\sum_{n=0}^{N_p-1} \hat{P}_n}_{\text{Magnetic term}}$$

$$\lambda \in [0, \infty)$$

$$\hat{P}_k = \begin{cases} \hat{\sigma}^z_{\tilde{n}_p} \hat{\sigma}^z_{\tilde{n}_p+N} & \text{if } N \text{ even and } k \mod (N+1) = N/2,\ N \\ \hat{\sigma}^z_{\tilde{n}_p} \hat{\sigma}^z_{\tilde{n}_p+N} & \text{if } N \text{ odd and } k \mod (N+1) = 0,\ N \\ \hat{\sigma}^z_{\tilde{n}_p} \hat{\sigma}^z_{\tilde{n}_p+1} \hat{\sigma}^z_{\tilde{n}_p+N} \hat{\sigma}^z_{\tilde{n}_p+N+1} & \text{otherwise} \end{cases}$$

$\tilde{n}_p \longrightarrow$ Plaquette's top left qubit



$N = 4$

2. The pure $\mathbb{Z}_2$ lattice gauge theory

- Its gauge degrees of freedom are two level systems (qubits). Matter not considered.
- Contains interesting non-trivial phenomena: Ising-like second order phase transition, confinement and the presence of a topological phase.
- The ground state degeneracy in the topological phase only appears in finite lattices when periodic (Toric code) or surface-code-like boundary conditions are used.

Hamiltonian

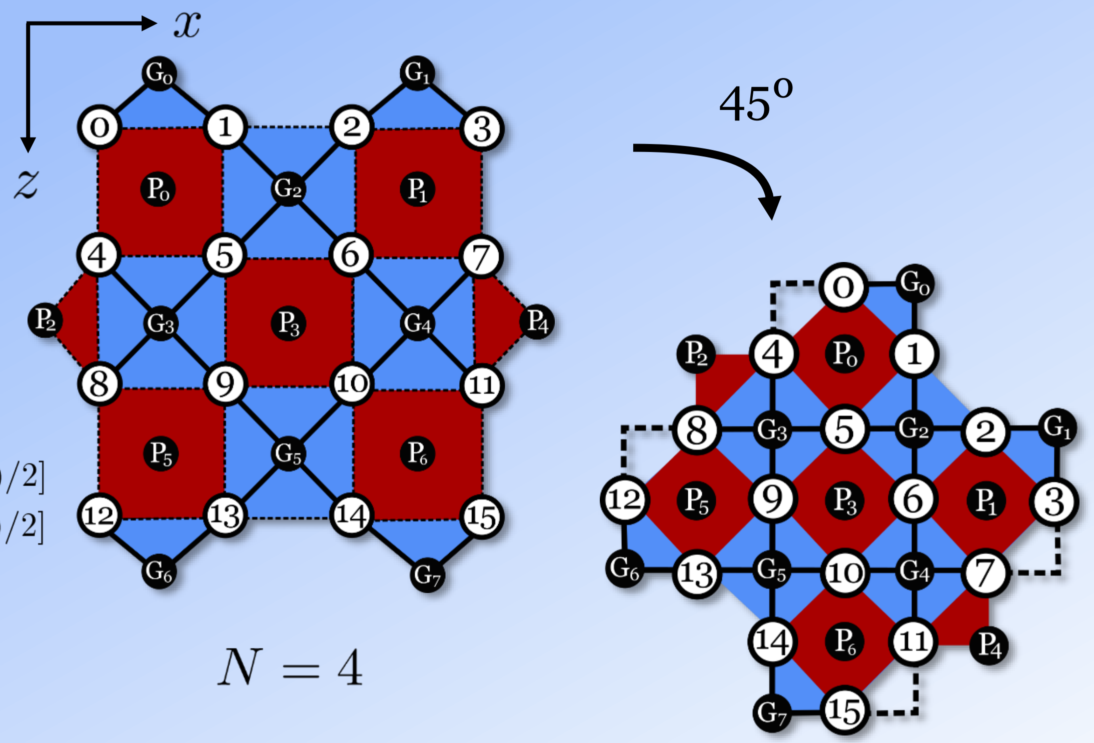
$$\hat{H} = -\underbrace{\sum_{n=0}^{N-1} \hat{\sigma}_n^x}_{\text{Electric term}} - \lambda \underbrace{\sum_{n=0}^{N_p-1} \hat{P}_n}_{\text{Magnetic term}}$$

Gauge invariance

$$\left[\hat{G}_k, \hat{H}\right] = 0 \qquad \forall\, k = 0, 1 \ldots N_g$$

$$\hat{G}_k = \begin{cases} \hat{\sigma}^x_{\tilde{n}_g} \hat{\sigma}^x_{\tilde{n}_g+1} & \text{if } N \text{ even and } [0 \leq k < N/2 \quad \text{or } k \geq N(N-1)/2] \\ \hat{\sigma}^x_{\tilde{n}_g} \hat{\sigma}^x_{\tilde{n}_g+1} & \text{if } N \text{ odd and } [0 \leq k < (N-1)/2 \text{ or } k \geq N(N-1)/2] \\ \hat{\sigma}^x_{\tilde{n}_g} \hat{\sigma}^x_{\tilde{n}_g+1} \hat{\sigma}^x_{\tilde{n}_g+N} \hat{\sigma}^x_{\tilde{n}_g+N+1} & \text{otherwise} \end{cases}$$

$\tilde{n}_g \longrightarrow$ Node's top left qubit



$N = 4$

2. The pure $\mathbb{Z}_2$ lattice gauge theory

- Its gauge degrees of freedom are two level systems (qubits). Matter not considered.
- Contains interesting non-trivial phenomena: Ising-like second order phase transition, confinement and the presence of a topological phase.
- The ground state degeneracy in the topological phase only appears in finite lattices when periodic (Toric code) or surface-code-like boundary conditions are used.

Hamiltonian

$$\hat{H} = -\underbrace{\sum_{n=0}^{N-1} \hat{\sigma}_n^x}_{\text{Electric term}} -\lambda \underbrace{\sum_{n=0}^{N_p-1} \hat{P}_n}_{\text{Magnetic term}}$$

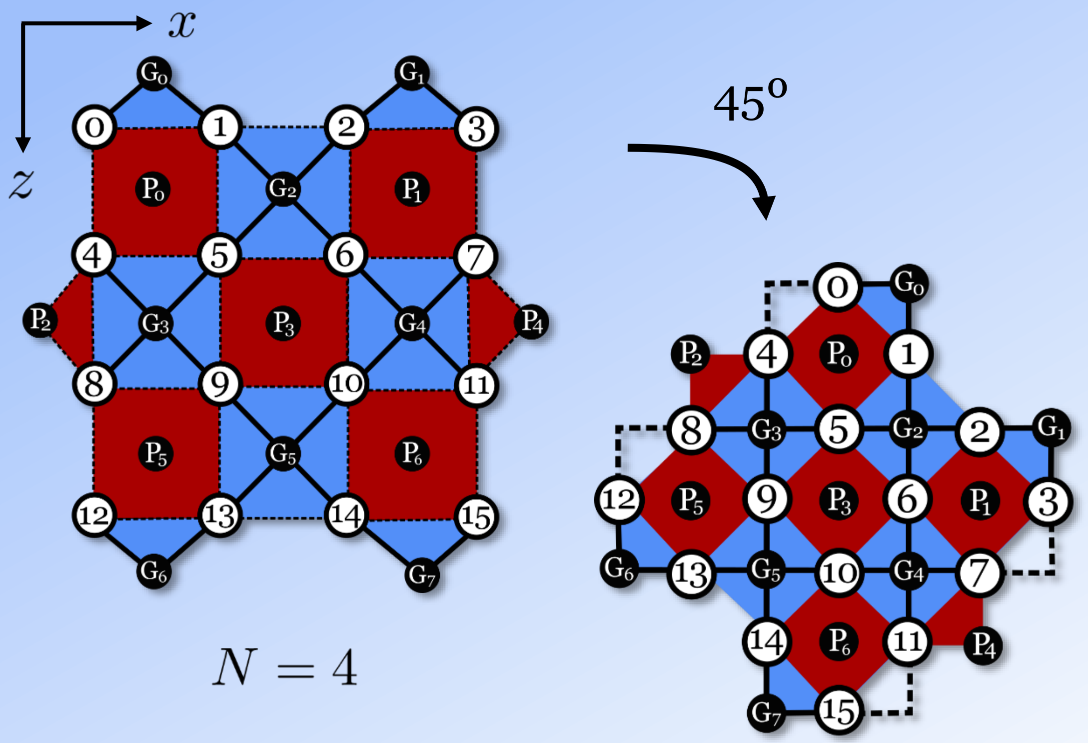
Gauge invariance

$$\left[\hat{G}_k, \hat{H}\right] = 0 \qquad \forall\, k = 0, 1 \ldots N_g$$

Gauss' Law

$$(\nabla \cdot \mathbf{E})(k) \equiv 0 \implies \hat{G}_k \left|\psi\right\rangle = \left|\psi\right\rangle$$

$\left|\psi\right\rangle \longrightarrow$ *Physical* states



3. Variational ground state preparation

- The variational ansatz that will be proposed is in part inspired in the well known QAOA.
- The QAOA, may not be suitable for the preparation of the ground state of every many-body system. Why?

3. Variational ground state preparation

- The variational ansatz that will be proposed is in part inspired in the well known QAOA.
- The QAOA, may not be suitable for the preparation of the ground state of every many-body system. Why?
- Consider the following flavour of QAOA ansatz:

$$\hat{H} = \sum_{n=1}^{N} \hat{H}_n$$

$$|\psi(\boldsymbol{\alpha})\rangle = \prod_{k=1}^{L} \prod_{n=1}^{N} e^{i\alpha_{k,n}\hat{H}_n} |\psi(0)\rangle \longrightarrow \text{(Almost) Trotterized discretized propagator}$$

$$|\psi(0)\rangle : \hat{H}_n |\psi(0)\rangle = H_n |\psi(0)\rangle \text{ For some } n$$

3. Variational ground state preparation

- The variational ansatz that will be proposed is in part inspired in the well known QAOA.
- The QAOA, may not be suitable for the preparation of the ground state of every many-body system. Why?
- Consider the following flavour of QAOA ansatz:

$$\hat{H} = \sum_{n=1}^{N} \hat{H}_n$$

$$|\psi(\boldsymbol{\alpha})\rangle = \prod_{k=1}^{L} \prod_{n=1}^{N} e^{i\alpha_{k,n}\hat{H}_n} |\psi(0)\rangle \longrightarrow \text{(Almost) Trotterized discretized propagator}$$

$$|\psi(0)\rangle : \hat{H}_n |\psi(0)\rangle = H_n |\psi(0)\rangle \ \text{For some } n$$

- The **QAOA approximates the optimal adiabatic schedule**[†] for an evolution time which is related with the number of variational parameters considered.

Guaranteed expressivity with enough parameters	Adiabaticity breaks down through phase transitions

† Phys. Rev. X 10, 021067 (2020)

4. Imaginary time evolution

- It is possible to approximate the ground state of any physical system from any state $|\psi\rangle$ fulfilling $\langle \text{g.s.}|\psi\rangle \neq 0$ using the following operator

$$\lim_{\tau\to\infty} e^{-\tau\hat{H}} |\psi\rangle \longrightarrow |\text{g.s.}\rangle$$

$$|\psi\rangle = |\text{g.s.}\rangle + |E_1\rangle + |E_2\rangle + \dots \longrightarrow e^{-\tau\hat{H}} |\psi\rangle = e^{-\tau E_0} |\text{g.s.}\rangle + \underbrace{e^{-\tau E_1} |E_1\rangle + e^{-\tau E_2} |E_2\rangle + \dots}_{\text{Exponentially supressed}}$$

- This concept is already used in classical simulation techniques. E.g. Quantum Monte Carlo.

4. Imaginary time evolution

- It is possible to approximate the ground state of any physical system from any state $|\psi\rangle$ fulfilling $\langle \text{g.s.}|\psi\rangle \neq 0$ using the following operator

$$\lim_{\tau\to\infty} e^{-\tau\hat{H}}\,|\psi\rangle \longrightarrow |\text{g.s.}\rangle$$

$$|\psi\rangle = |\text{g.s.}\rangle + |E_1\rangle + |E_2\rangle + \dots \longrightarrow e^{-\tau\hat{H}}\,|\psi\rangle = e^{-\tau E_0}\,|\text{g.s.}\rangle + \underbrace{e^{-\tau E_1}\,|E_1\rangle + e^{-\tau E_2}\,|E_2\rangle + \dots}_{\text{Exponentially supressed}}$$

- This concept is already used in classical simulation techniques. E.g. Quantum Monte Carlo.
- The non-unitarity of the operator $e^{-\tau\hat{H}}$ complicates its introduction in quantum algorithms.
- We propose combining an implementable partial imaginary time evolution followed by unitary evolution.

No longer adiabatic
Shorter circuit

Measurement &
feedforward required

4.5. Partial imaginary time evolution

- The non-unitary operation included in the ansatz provides it with the ability to reproduce the true ground state of the $\mathbb{Z}_2$ lattice gauge theory in the limits $\lambda \to 0, \infty$ ($\beta \to 0, \infty$).

$$|\psi(\beta)\rangle = \frac{e^{-\beta \hat{H}_B}}{(\cosh 2\beta_{N_p})^{-N_p/2}} |\Omega_E\rangle = \left[\prod_n \frac{1 + \hat{P}_n \tanh\beta}{\sqrt{1+\tanh^2\beta}}\right] |\Omega_E\rangle$$

$$|\Omega_E\rangle = \bigotimes_l |+\rangle_l$$

$$|\psi(0)\rangle = |\Omega_E\rangle \qquad |\psi(\infty)\rangle = \left[\prod_n \frac{1+\hat{P}_n}{\sqrt{2}}\right] |\Omega_E\rangle$$

- This variational ansatz was first proposed by Cardy and Hamber †. We provide its deterministic implementation in the circuit model.

† Nuclear Physics B 170, 1, p 79-90 (1980)

5. Variational ansatz proposal

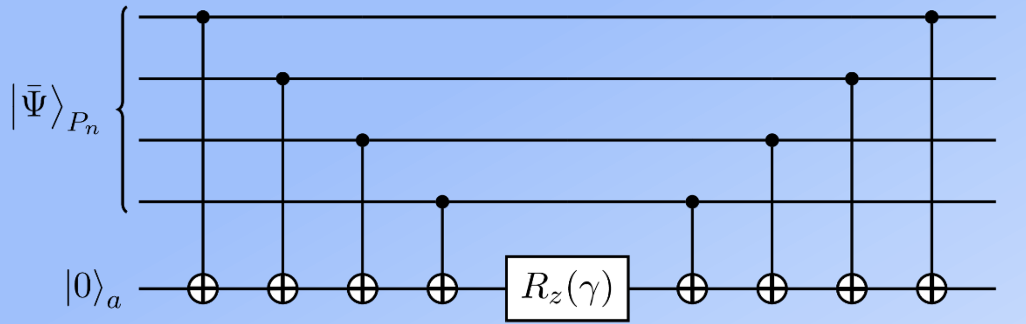
$$|\psi(\boldsymbol{\alpha},\boldsymbol{\beta})\rangle = \left[\prod_{k=2}^{L} e^{i\alpha'_k \hat{H}'_E} e^{i\alpha_k \hat{H}_E} e^{i\beta'_k \hat{H}'_B} e^{i\beta_k \hat{H}_B}\right] e^{i\alpha'_1 \hat{H}'_E} e^{i\alpha_1 \hat{H}_E} \frac{e^{\beta'_1 \hat{H}'_B}}{(\cosh 2\beta)^{N'_p/2}} \frac{e^{\beta_1 \hat{H}_B}}{(\cosh 2\beta)^{N_p/2}} |\Omega_E\rangle$$

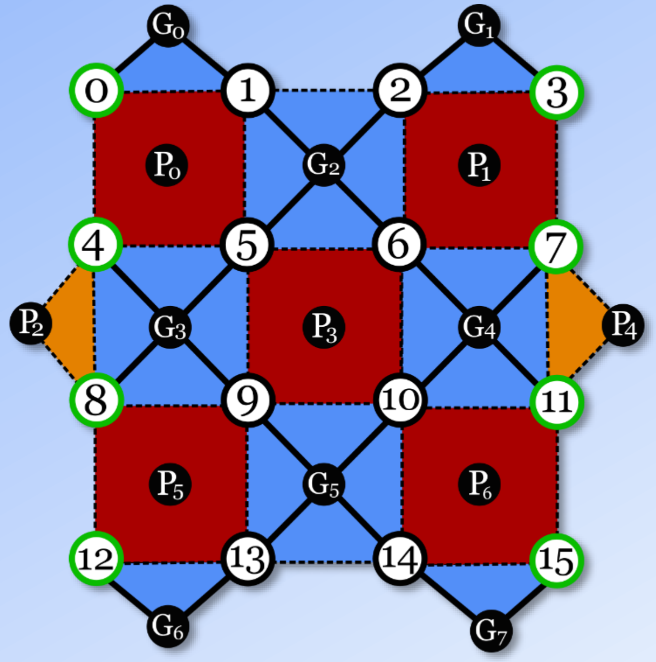
$$\hat{H}_E = \sum_{n\in \text{bulk}} \hat{\sigma}^x_n \qquad \hat{H}'_E = \sum_{n\in \text{boundary}} \hat{\sigma}^x_n \qquad \frac{\partial \langle \hat{O}\rangle(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial \alpha_k} = \frac{\langle \hat{O}\rangle(\alpha_k+s) - \langle \hat{O}\rangle(\alpha_k - s)}{2\sin(s)}$$

$$\hat{H}_B = \sum_{n\in \text{bulk}} \hat{P}_n \qquad \hat{H}'_B = \sum_{n\in \text{boundary}} \hat{P}_n \qquad \frac{\partial \langle \hat{O}\rangle(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial \beta_k} = \frac{\langle \hat{O}\rangle(\beta_k+s) - \langle \hat{O}\rangle(\beta_k - s)}{2\sinh(s)}$$

$$s \neq 0$$

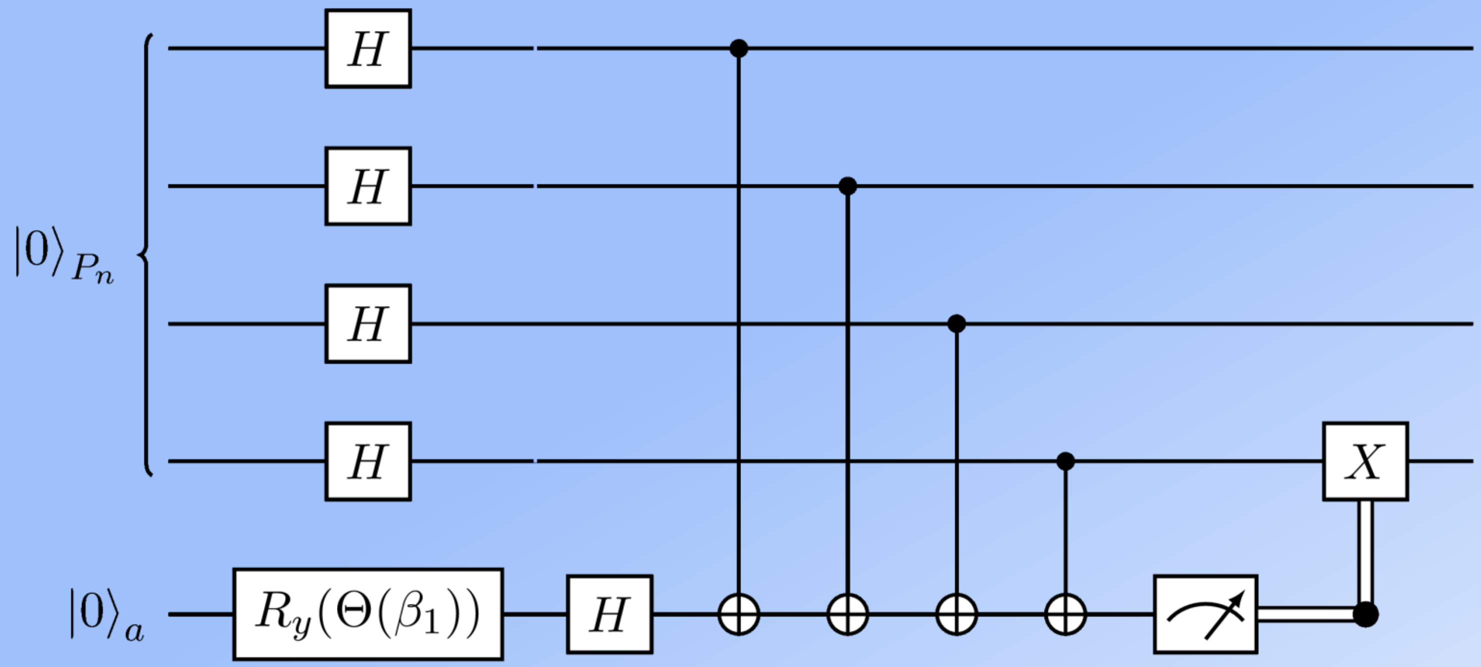
- $e^{i\alpha\hat{\sigma}^x}$ Single qubit rotations
- Circuit implementation of $e^{i\gamma_k \hat{P}_n}$





5. Variational ansatz proposal

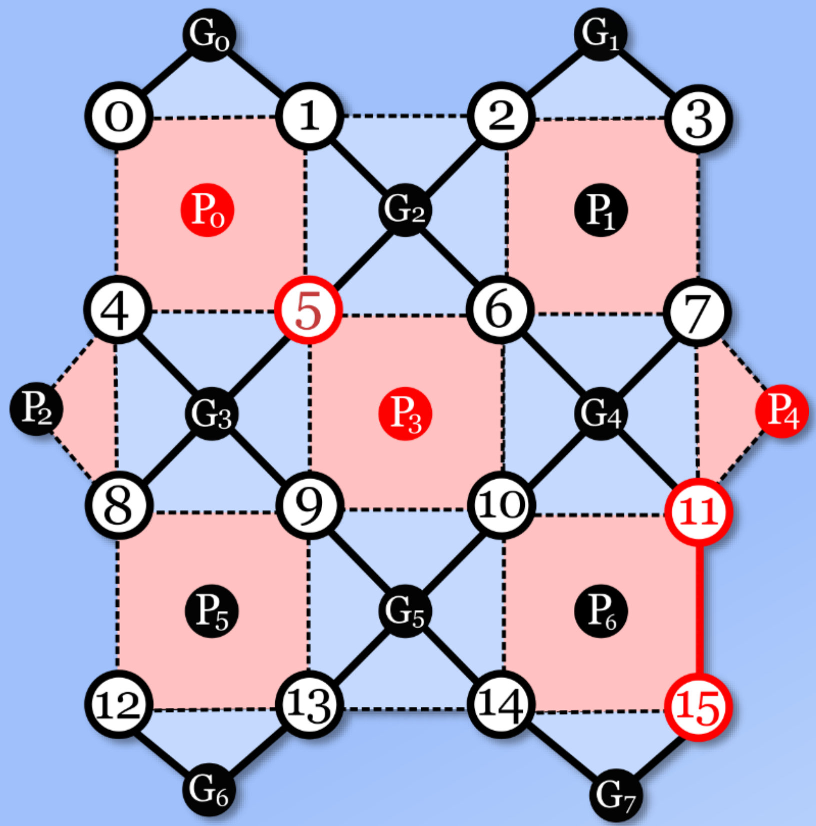
- Circuit implementation of the non-unitary exponential **acting on the reference state** $(\cosh 2\beta_1)^{-1/2} e^{\beta_1 \hat{P}_n} |\Omega_E\rangle_{P_n}$



$$\Theta(\beta_1) = \tan^{-1}(\tanh \beta_1)$$

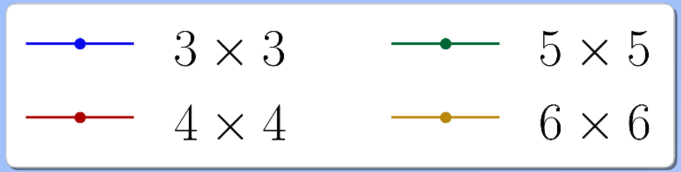
5. Variational ansatz proposal

- The correction of the sign of the exponential acting on one plaquette affects the neighbouring one.



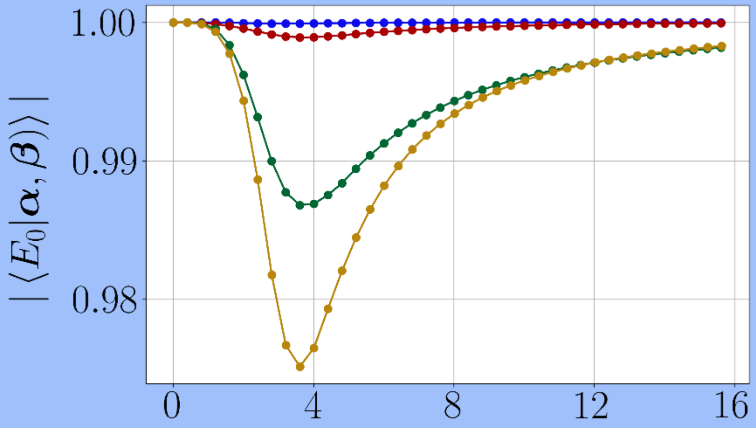
- Parallelization strategy:
  1. Execute the circuit (Without the measurement) in two rounds affecting plaquettes not sharing qubits.
  2. Measure and remember which plaquettes' ancillas were in state $|1\rangle$ (red dots).
  3. Apply $\hat{\sigma}^x$ on qubits connecting those plaquettes.

6. Variational ansatz performance



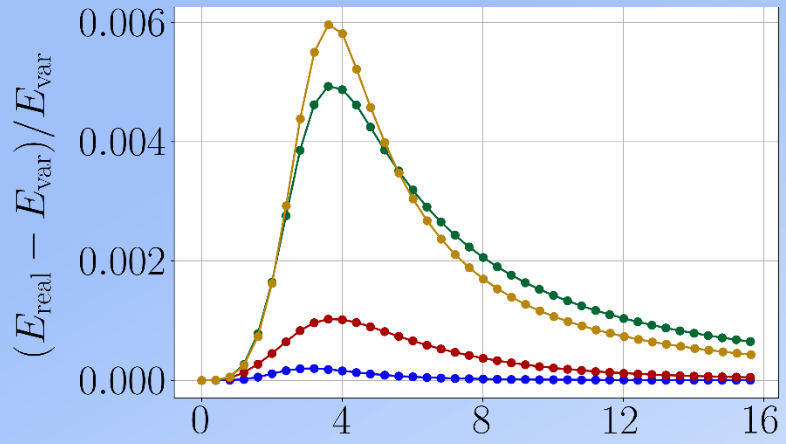
L = 2, Depth = 37,
Absence of noise

Fidelity

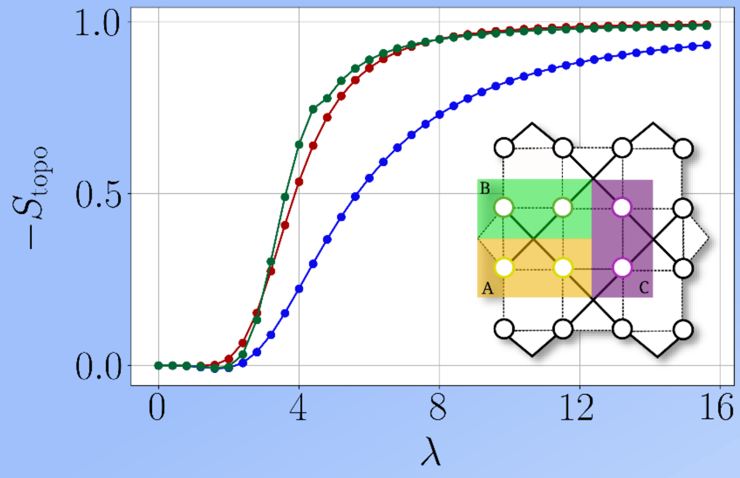


Min. value:
- 3×3: 0.99991
- 4×4: 0.9989
- 5×5: 0.987
- 6×6: 0.975

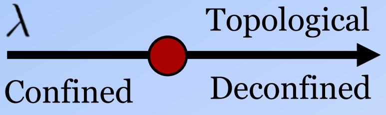
ED-Variational energy deviation



Topological entanglement entropy

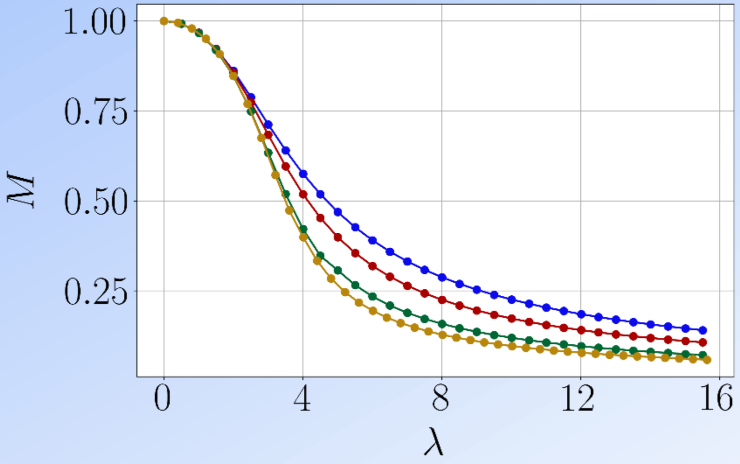


Phase diagram

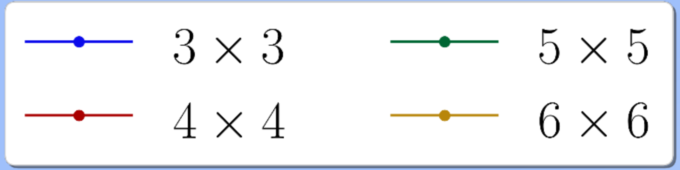


$\lambda_c = 3.04438$

Dual magnetization

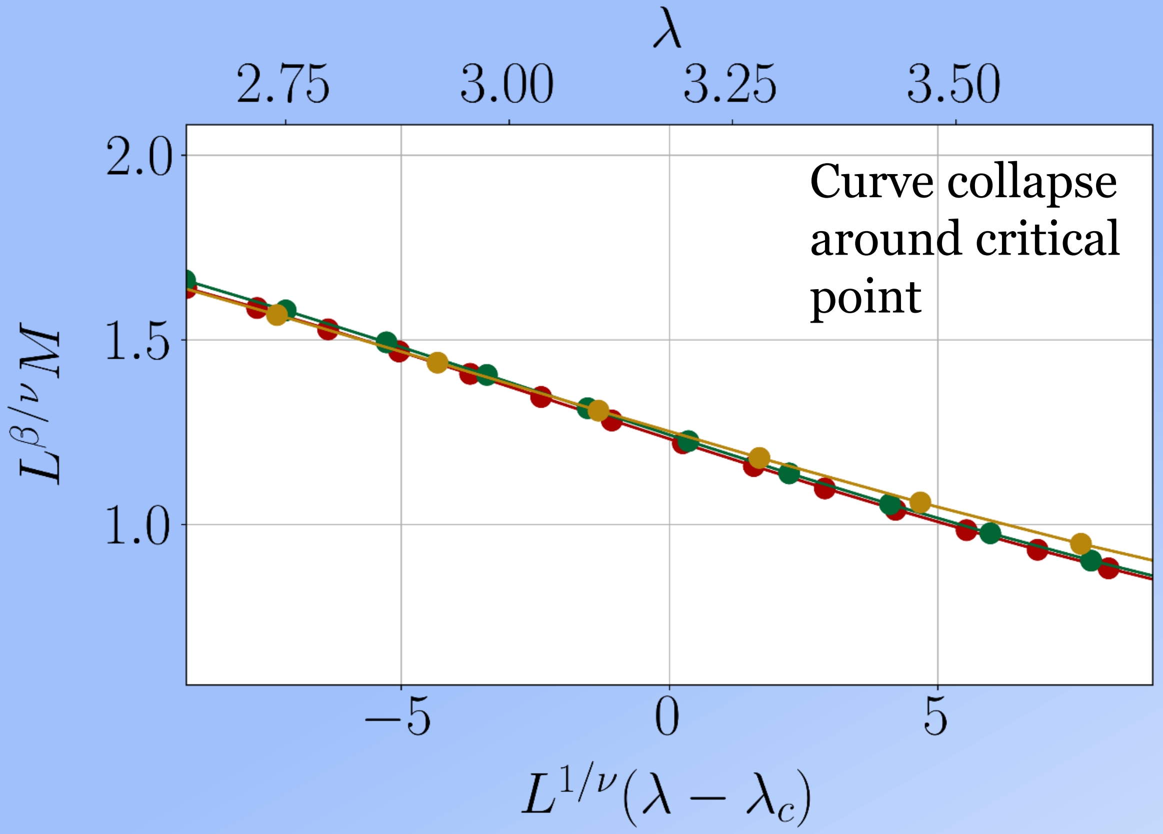


6. Variational ansatz performance

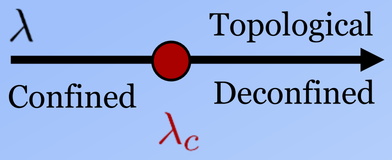


L = 2, Depth = 37,
Absence of noise

- Dual magnetization finite size scaling



Phase diagram

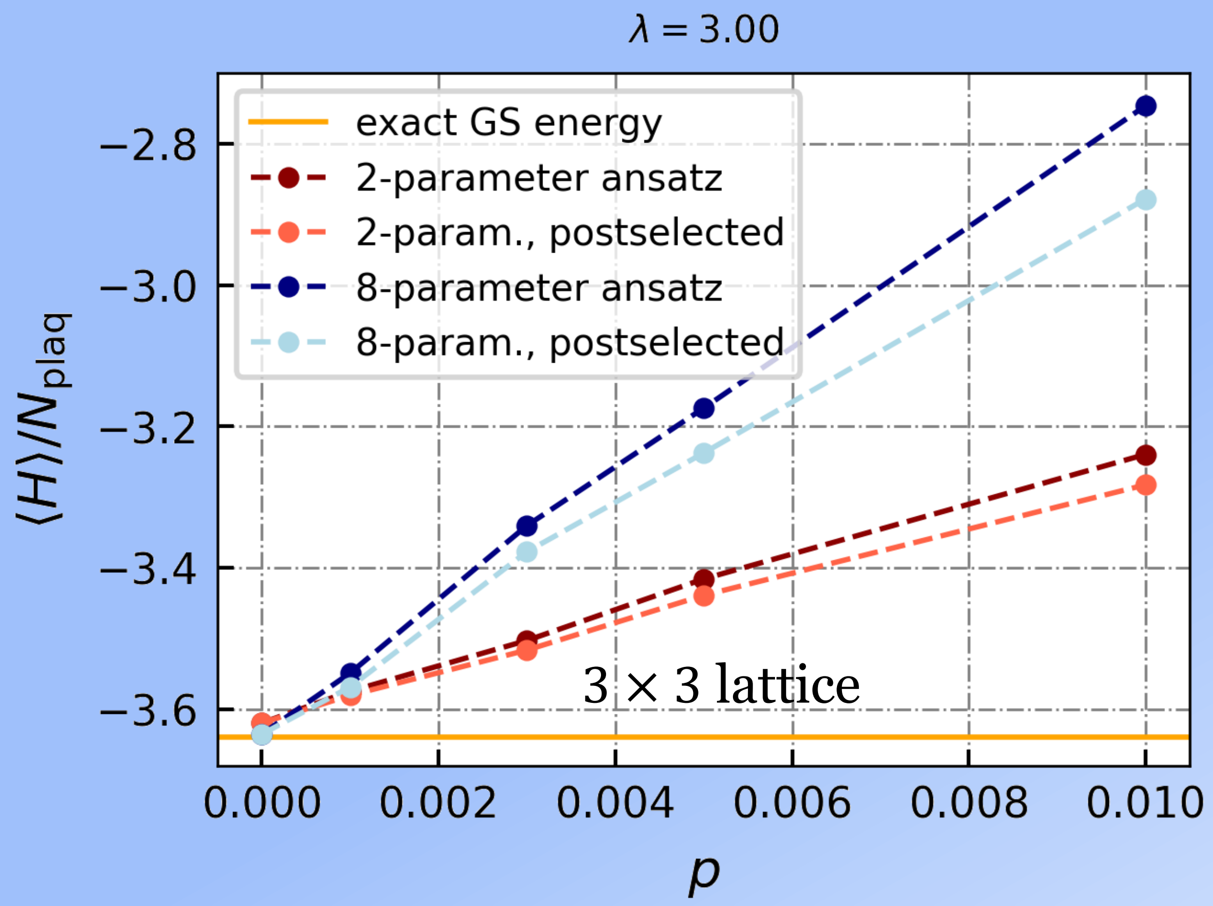


Critical exponents

Ansatz prediction	Quantum Monte Carlo
$\lambda_c = 3.18(2)$	$\lambda_c = 3.04438(2)$
$\beta = 0.317(8)$	$\beta = 0.3265(3)$
$\nu = 0.636(7)$	$\nu = 0.6301(4)$

6. Variational ansatz performance

- Ground state preparation in the presence of noise



One needs $p < 10^{-3}$ for the introduction of a second layer not to be counterproductive.

Same can be expected for other variational ansatze, be aware!

Noise model: Random Pauli before and after every gate and measurement with probability p.

Thank you!

Non–unitary ground state preparation for the $\mathbb{Z}_2$ lattice gauge theory in the presence of noise

Jesús Cobos Jiménez

Collaborators:
David Locher (RWTH & Institute for Theoretical Electronics)
Markus Müller (RWTH & Institute for Theoretical Electronics)
Alejandro Bermúdez (IFT)
Enrique Rico (UPV/EHU & IKERBASQUE)

eman ta zabal zazu
Universidad del País Vasco
Euskal Herriko Unibertsitatea
EHU QC
EHU Quantum Center

Appendix: Gauss' Law

- For Lie gauge groups:

$$W_{(n,i)} = e^{iag\mathbf{A}_i(n,i)} \qquad G_n = e^{ig^{-1}(\nabla\cdot\mathbf{E})(n)}$$

- In the temporal gauge (Implicitly fixed in the Hamiltonian formalism):

$$\mathbf{E}_i(n,i) = -\partial_t \mathbf{A}_i(n,i) \qquad [\mathbf{A}_i(n,i), \mathbf{E}_{i'}(n',i')] = \delta_{n,n'}\,\delta_{i,i'}$$

- The same interpretation is used in the $\mathbb{Z}_2$, but the discrete nature of this gauge group causes the commutation relation to not be fulfilled in this particular $\mathbb{Z}_2$ case.

$$(\nabla\cdot\mathbf{E})(n) \equiv 0 \qquad \Longrightarrow \qquad \hat{G}_n\,|\psi\rangle = |\psi\rangle$$

Appendix: Finite size scaling method

The critical exponents are obtained by minimizing the following cost function

$$S = \frac{1}{k} \sum_i \frac{1}{\mathcal{N}_i} \sum_j \frac{(y_{ij} - Y_{ij})^2}{dy_{ij}^2 + dY_{ij}^2}$$

with

$$x_{ij} = L_i^{1/\nu} (\lambda_j - \lambda_c) / \lambda_c$$

$$y_{ij} = L_i^{\beta/\nu} M_j$$

$$dy_{ij} = y_{ij} - \underbrace{\tilde{y}_{ij}}_{\text{E.D.}}$$

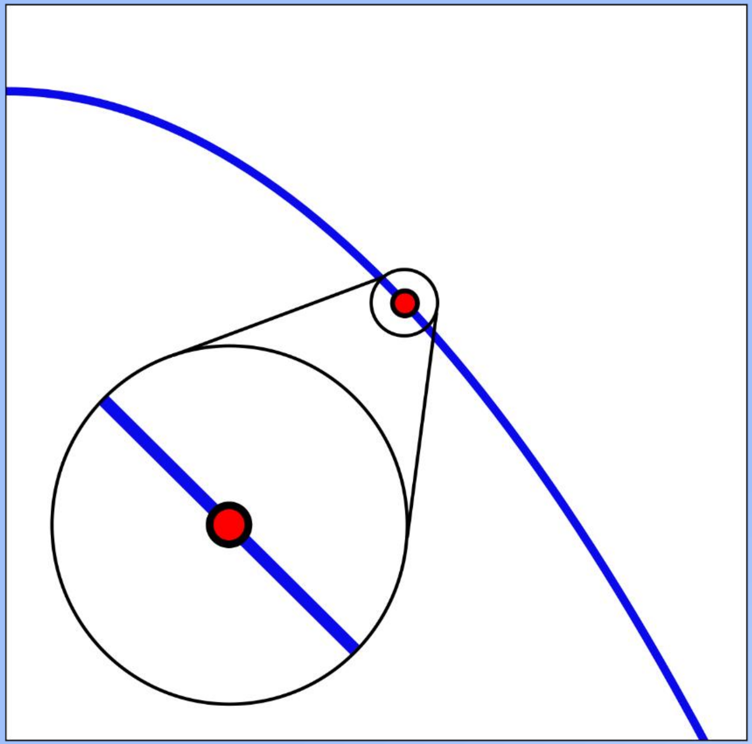
Y_{ij}, dY_{ij} → Master curve values and error

k → Nº of lattice sizes considered

$\mathcal{N}_i$ → Number of λ_j

Appendix: Finite size scaling method

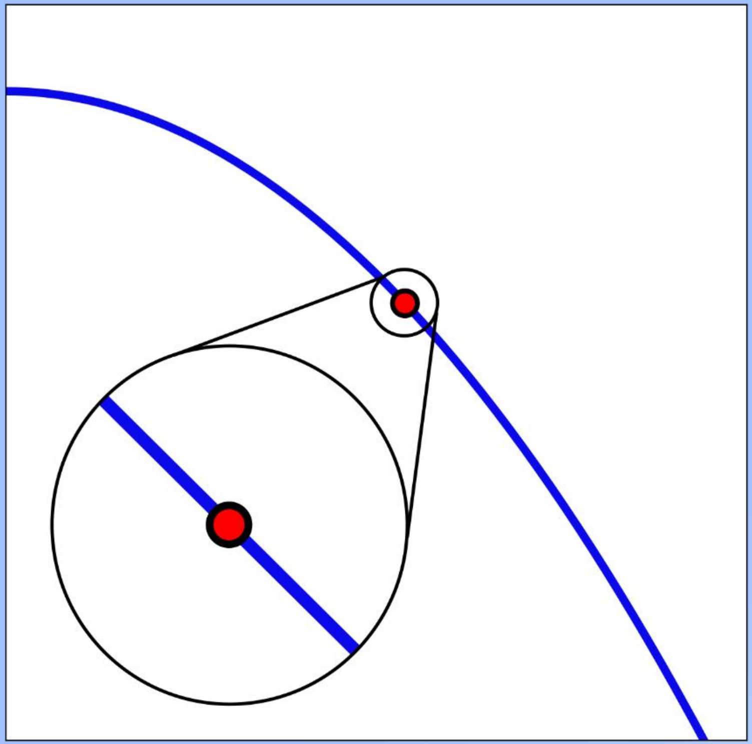
The values and error of the master curve are obtained invoking the principle of local linearity of continuous differentiable functions, assuming that $L^{\beta/\nu}M$ is a function of this kind.



- Naive approach [N. Kawashima and N. Ito, J. Phys. Soc. Jpn. **62**, 435 (1993)]
  1. Sort all the x_{ij} such that $x_k = x_{ij} \leq x_{i'j'} = x_{k+1}$.
  2. For each x_k take $x_{k\pm1}$ and perform a linear regression $\bar{y}_k(x)$ between these three points.
  3. Take $Y_{ij} = Y_k = \bar{y}_k(x_k)$ and dY_{ij} the associated statistical error.

Appendix: Finite size scaling method

The values and error of the master curve are obtained invoking the principle of local linearity of continuous differentiable functions, assuming that $L^{\beta/\nu}M$ is a function of this kind.



- Refined approach [J. Houdayer and A.K. Hartman, PRB **70**, 014418 (2004)]
  1. For each x_{ij} find for each $i' \neq i$ two points such that $x_{i'j'} \leq x_{ij} \leq x_{i'j'+1}$. If no such points can be found for certain i', ignore that lattice size.
  2. Perform a weighted linear regression $\bar{y}_{ij}(x)$ between the set of points found in the previous step. The weight associated to each point is $w_{ij} = 1/(dy_{ij})^2$.
  3. Take $Y_{ij} = \bar{y}_{ij}(x_{ij})$ and dY_{ij} the associated statistical error.