Precision Standard Model Phenomenology at N3LO and Beyond

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SLAC NATIONAL ACCELERATOR LABORATORY

QCD Theory Seminar

CERN, 2 December 2022

Based on:

"Collinear expansion for color singlet cross sections"
M.Ebert, B.Mistlberger, **GV**
[2006.03055]

"TMD PDFs at N3LO"
M.Ebert, B.Mistlberger, **GV**
[2006.05329]

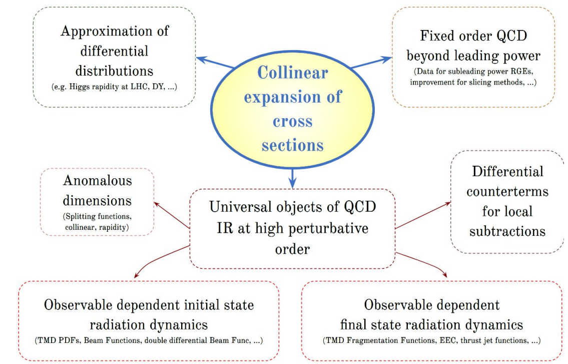
"N-jettiness beam functions at N3LO"
M.Ebert, B.Mistlberger, **GV**
[2006.03056]

"Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond"
C.Duhr, B.Mistlberger, **GV**
[2205.04493]

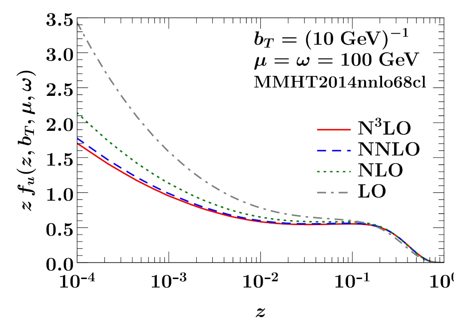
"The Four-Loop Rapidity Anomalous Dimension and Event Shapes to Fourth Logarithmic Order"
C.Duhr, B.Mistlberger, **GV**
[2205.02242]

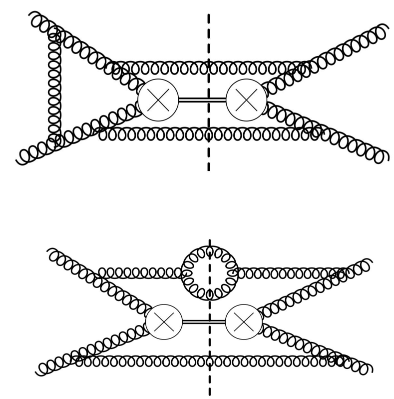
Outline

- **Introduction**
  - Why do we need higher order predictions?

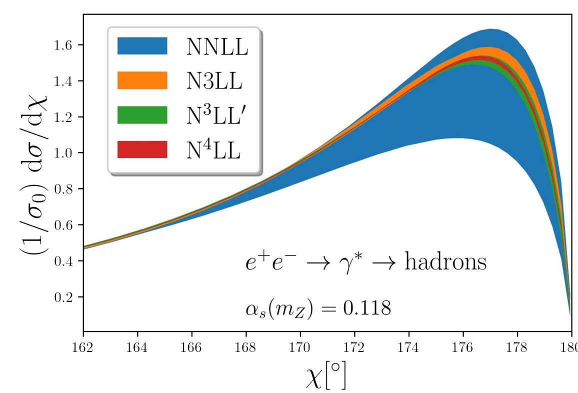


- **Beam Functions at N3LO**
  - Beam Functions from the **collinear expansion** of LHC cross sections
  - Impact of N3LO TMDPDFs



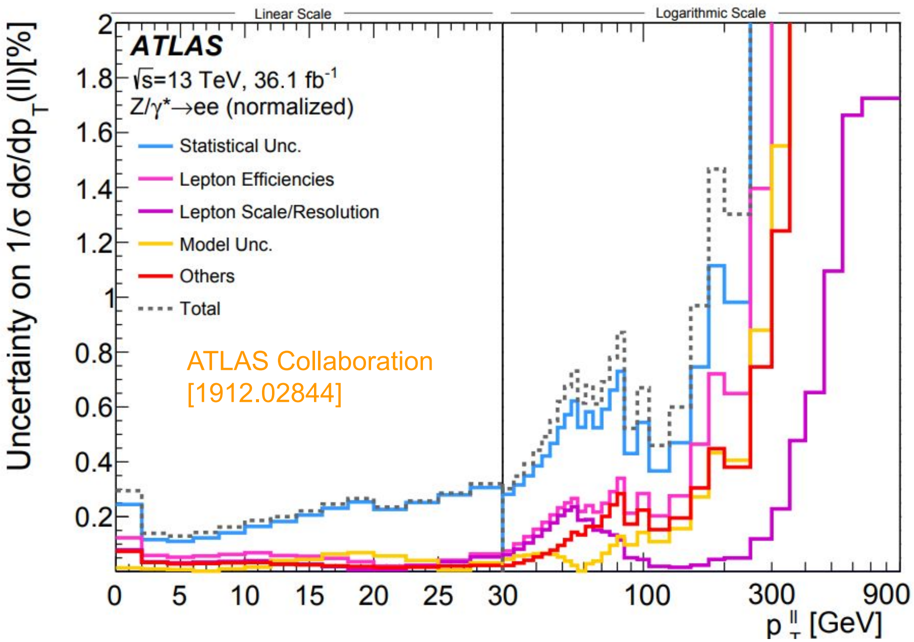


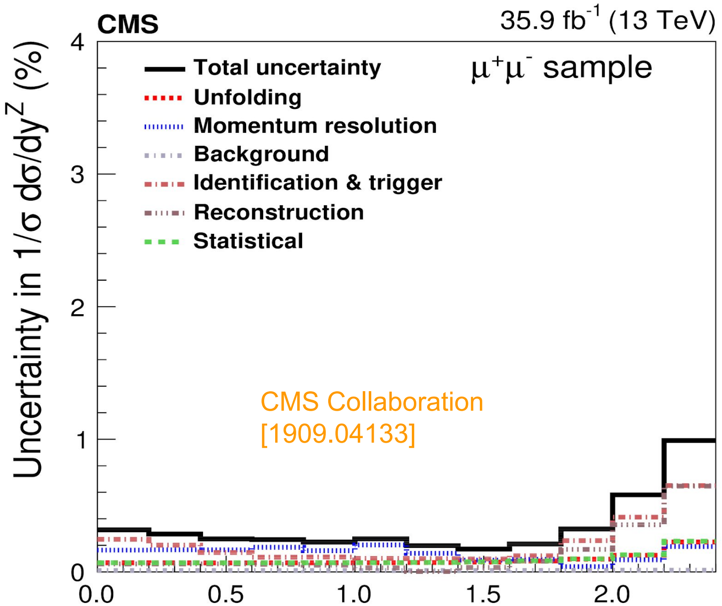
- **Going Beyond N3LO**
  - Rapidity Anomalous Dimension at four loops
  - Event Shapes at N4LL

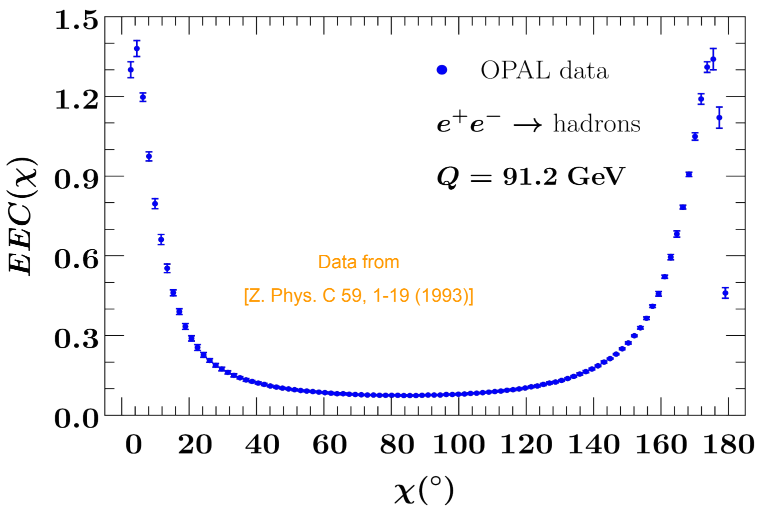


Testing the Standard Model at Colliders

- Experimental measurements of key benchmark processes have reached astonishing level of precision.



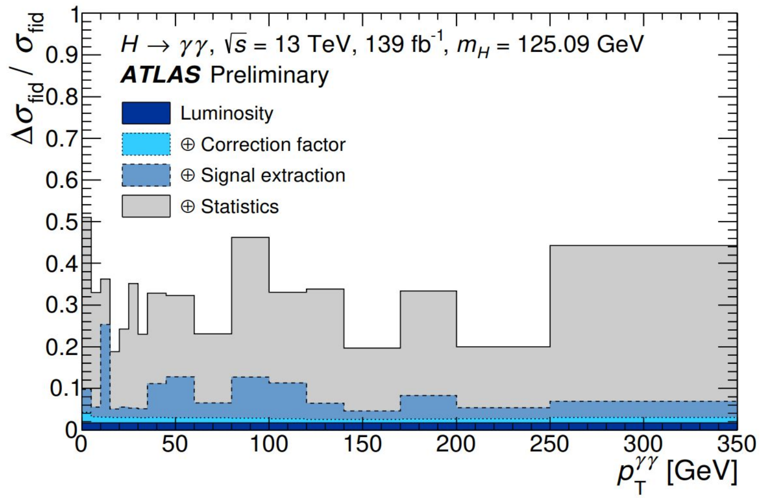




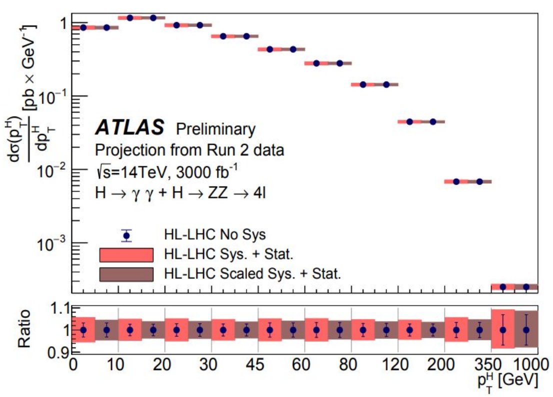
Ability to test the SM at (sub)-percent accuracy!

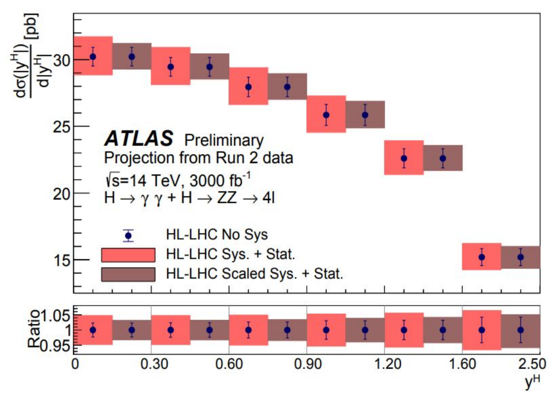
Testing the Standard Model at Colliders

Higgs measurements at the moment are limited by statistics

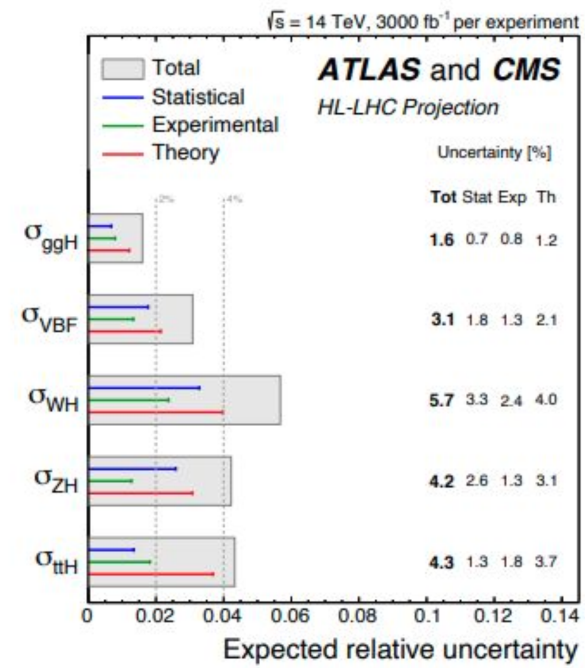


...but statistical uncertainties **will improve dramatically** with HL LHC





Theory errors are projected to be a major limiting factor for Higgs precision program



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Improving Theoretical Predictions

We should aim at comparable precision from the theory side!

Non Perturbative   **Perturbative**

$$\sigma_{pp\to X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab\to X}$$

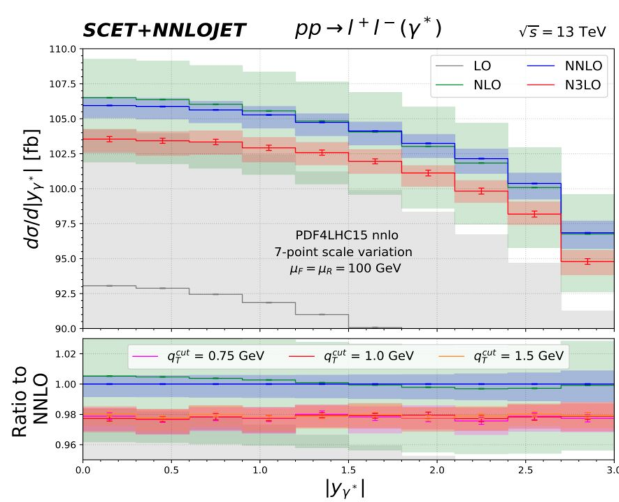
$$\hat{\sigma}_{ab\to X} = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s \sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2 \sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3 \sigma_3}_{\text{N}^3\text{LO}} + \ldots$$

"The Path Forward to N3LO"
Snowmass Whitepaper
[Caola, Chen, Duhr, Liu, Mistlberger, Petriello, GV, Weinzierl]

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	$\delta\sigma^{\text{NNLO}}$
$gg \to$ Higgs	m_H	3.5%	30%
$b\bar{b} \to$ Higgs	m_H	-2.3%	2.1%
NCDY	30	-4.8%	−0.34%
	100	-2.1%	−2.3%
CCDY(W^+)	30	-4.7%	−0.1%
	150	-2.0%	−0.1%
CCDY(W^-)	30	-5.0%	−0.1%
	150	-2.1%	−0.6%

n3loxs [Baglio, Duhr, Mistlberger, Szafron '22]

CAVEAT!
Often times convergence turns out to be slower than naive estimate
=> **N3LO gives few <u>percent</u> (not per-mille) shift**



[Chen, Gehrmann, Glover, Huss, Yang, Zhu '21]

Differential Distributions via Slicing

- Cross sections have IR divergences due to soft and collinear radiation at intermediate steps of the calculation => complicate to automatize higher order calculations
- One way to deal with this problem semi-numerically is to use **EFT-based subtractions**

q_T subtraction

[Catani, Grazzini '07]

N-Jettiness subtraction

[Boughezal, Focke, Liu, Petriello '15]
[Gaunt, Stahlhofen, Tackmann, Walsh '15]

$$\sigma(X) = \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma^{\mathrm{sing}}(X)}{\mathrm{d}q_T} + \int_{q_{T\,\mathrm{cut}}} \mathrm{d}q_T \frac{\mathrm{d}\sigma(X)}{\mathrm{d}q_T} + \Delta\sigma(X, q_{T\,\mathrm{cut}})$$

Below the cut region:

- Singular distribution
- Contains most complicated cancellation of IR divergences
- Control it analytically via factorization theorems

Above the cut region:

- Resolved extra radiation
- No events in Born configuration
- Lower number of loops
- Calculate numerically and/or with lower order subtraction schemes

Residual:

Non singular terms from below the cut.
Often neglected.

Differential Distributions via Slicing at NNLO

- Extremely successful program for many color singlet LHC processes at NNLO

$pp \to Z,\ pp \to W,\ pp \to H,\ pp \to \gamma\gamma,\ pp \to Z\gamma,\ pp \to W\gamma,\ pp \to ZZ,$
$pp \to WW,\ pp \to WZ$

[Matrix collaboration]

- With N-Jettiness ability to tackle also processes with jets in the final state

[Boughezal, Focke, Liu, Petriello + Campbell, Ellis, Giele '15, '16]

[Campbell, Ellis, Williams '16]

[Mondini, Williams '21]

[Campbell, Ellis, Seth '19]

- Error due to higher order terms in q_T expansion

$$\Delta\sigma(X, q_{T\,\mathrm{cut}}) \equiv \sum_{i>0} \int_0^{q_{T\,\mathrm{cut}}} \mathrm{d}\tau \frac{\mathrm{d}\sigma^i(X)}{\mathrm{d}q_T}$$

  - In principle: made negligible by pushing cut to small values
  - In practice: tradeoff between numerical stability and size of power corrections

- Interesting prospects of improving them by computing power corrections analytically

See for example: [Ebert, Moult, Stewart, Tackmann, **GV**, Zhu, 1807.10764, 1812.08189] [Boughezal, Isgro', Petriello '19]

Singular Region for q_T Slicing

- **Singular region** (i.e. below the cut) can be understood at all orders as

Leading power **factorization** for Transverse-Momentum Distributions in pp

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\,\mathrm{d}^2\vec{q}_T} = \sigma_0 \sum_{i,j} H_{ij}(Q^2,\mu) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, \tilde{B}_i\Big(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\Big)\tilde{B}_j\Big(x_2^B, b_T, \mu, \frac{\nu}{\omega_b}\Big)\tilde{S}(b_T,\mu,\nu)$$

q_T Beam Functions

- At each order:
  - Log-enhanced terms (predicted by RGE/anomalous dims. and lower order results)
  - Boundary values (non-log enhanced terms, need explicit calculation)

- Boundary value for Hard and Soft are **constants.**
  - Known at N3LO for Hard since 2010 and for Soft since 2016. [Li, Zhu] [Gehrmann, Glover, Huber, Ikizlerli, Studerus]

- **Beam function** boundary values are **full functions** (of the collinear splitting variable)
  - More complicated objects.
  - Different for quark vs gluons

Last missing ingredients for qT subtraction at N3LO

Beam Functions

- **Beam Functions: universal gauge invariant matrix elements** encoding collinear dynamics of **protons** and initial state radiation in the presence of multiple measurements

- They can be understood as **generalization** of **Parton Distribution Functions** (PDFs)

Longitudinal momentum fraction

PDF:

$$f_q(x) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n} \cdot \mathcal{P}) \chi_n \right] | p_n \rangle$$

Longitudinal momentum fraction

Beam Function:

$$B_q(x, q_T) = \langle p_n | \bar{\chi}_n \frac{\bar{n}\!\!\!/}{2} \left[\delta(p^- - \bar{n} \cdot \hat{\mathcal{P}}) \delta(q_T - \hat{k}_T) \chi_n \right] | p_n \rangle$$

Additional observable (q_T, beam thrust, etc...)

$$\sigma_{pp \to X} \sim \int f_a(x_1) f_b(x_2) \otimes \hat{\sigma}_{ab \to X}$$

Inclusive

$$\frac{d\sigma}{d\mathcal{T}} \sim \int H \mathcal{B}_n \otimes \mathcal{B}_{\bar{n}} \otimes S + \dots$$

More differential

Beam Functions

- Beam functions are **non-perturbative** objects!

- However, in **perturbative** regime of the observable $\mathcal{T} \gg \Lambda_{\rm QCD}$, can be matched perturbatively onto PDF, via **matching kernel** $\mathcal{I}_{ij}(x, \mathcal{T}, \mu)$

$$B_i(x, \mathcal{T}, \mu) = \sum_j \mathcal{I}_{ij}(x, \mathcal{T}, \mu) \otimes_x f_j(x, \mu) \times \left[1 + \mathcal{O}(\Lambda_{\rm QCD}/\mathcal{T})\right]$$

- Bare matching kernel can be calculated using **collinear expansion of differential partonic cross sections for LHC processes!**

$$\mathcal{I}_{ij}^{\rm bare}(z, \mathcal{T}) = \int_0^1 {\rm d}x \int_0^\infty {\rm d}w_1 {\rm d}w_2 \, \delta\left[z - (1 - w_1)\right] \times \lim_{\text{strict } n-\text{coll.}} \left\{ \delta\left[\mathcal{T} - \mathcal{T}(Q, Y, w_1, w_2, x)\right] \frac{{\rm d}\eta_{j\bar{i}}}{{\rm d}Q^2 {\rm d}w_1 {\rm d}w_2 {\rm d}x} \right\}$$

$$w_1 = -\frac{\bar{n} \cdot k}{\bar{n} \cdot p_1}, \qquad w_2 = -\frac{n \cdot k}{n \cdot p_2} \qquad x = \frac{k^2}{(\bar{n} \cdot k)(n \cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n} \cdot k)(n \cdot k)}$$

"Collinear expansion for color singlet cross sections" [2006.03055]

Ebert, Mistlberger, **GV**

Collinear expansion of cross sections: Applications

"Collinear expansion for color singlet cross sections"
Ebert, Mistlberger, **GV**

Collinear expansion of cross sections

Approximation of differential distributions
(e.g. Higgs rapidity at LHC, DY, ...)

Fixed order QCD beyond leading power
(Data for subleading power RGEs, improvement for slicing methods, ...)

Universal objects of QCD IR at high perturbative order

Anomalous dimensions
(Splitting functions, collinear, rapidity)

Differential counterterms for local subtractions

Observable dependent initial state radiation dynamics
(TMD PDFs, Beam Functions, double differential Beam Func, ...)

Observable dependent final state radiation dynamics
(TMD Fragmentation Functions, EEC, thrust jet functions, ...)

Beam Functions

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$$\mathcal{I}_{ij}^{\rm bare}(z, \mathcal{T}) = \int_0^1 {\rm d}x \int_0^\infty {\rm d}w_1 {\rm d}w_2\, \delta\left[z - (1 - \ldots)\right]$$

$$\ldots x)] \frac{{\rm d}\eta_{j\bar{i}}}{{\rm d}Q^2 {\rm d}w_1 {\rm d}w_2 {\rm d}x} \Big\}$$

$$\ldots \frac{\ldots}{n \cdot p_2} \qquad x = \frac{k^2}{(\bar{n} \cdot k)(n \cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n} \cdot k)(n \cdot k)}$$

What are these differential cross sections?
How do we calculate them?

"Collinear expansion for color singlet cross sections"
[2006.03055]
Ebert, Mistlberger, **GV**

Analytic cross sections for collider observables

- **Important!** **Analytic** (not numerical) computations of **cross sections** (not amplitudes)
- Integral over **phase space** of final state particles
- Sum over all Real and Virtual corrections
- **Analytic** control of IR divergences

Trade phase space integrals for loop integrals with **reverse unitarity**

[Anastasiou, Melnikov] [Anastasiou, Dixon, Melnikov]

$$\int \mathrm{d}^d p \, \delta_+(p^2)$$

← See phase space constraints as "cut" propagators

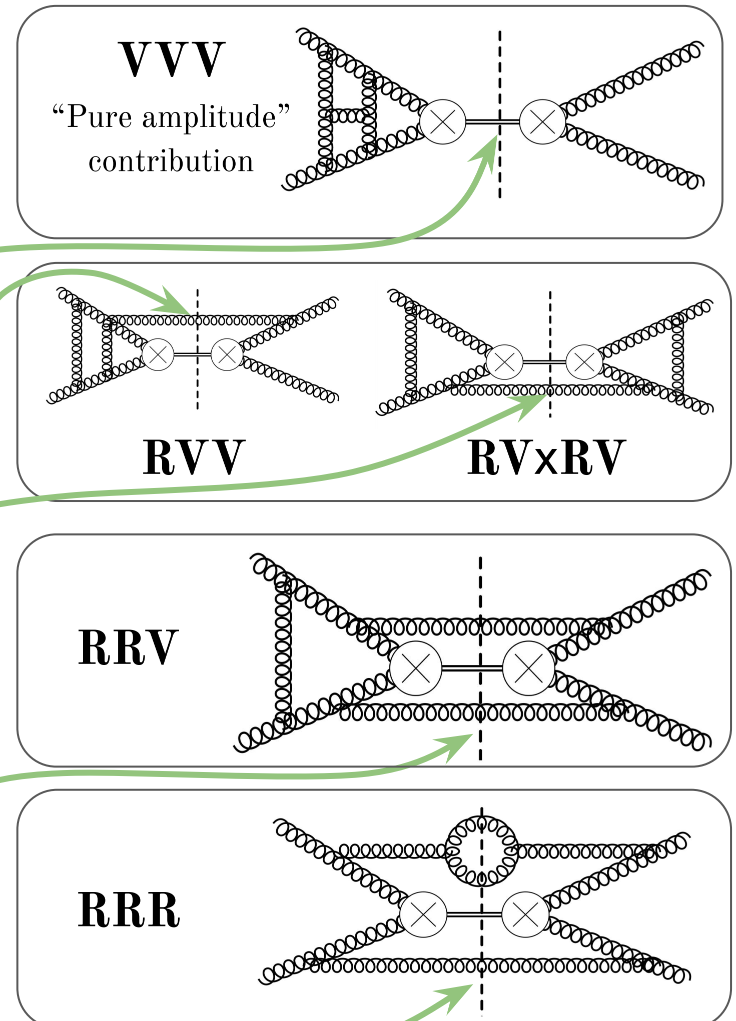
↑ See this as a loop integral

$$\delta_+(p^2) \sim \lim_{\varepsilon \to 0} \left[\frac{1}{p^2 + i\varepsilon} - \frac{1}{p^2 - i\varepsilon} \right]$$

More measurements, more cut propagators, more difficult integrals

Example:

Higgs production at N3LO in gg



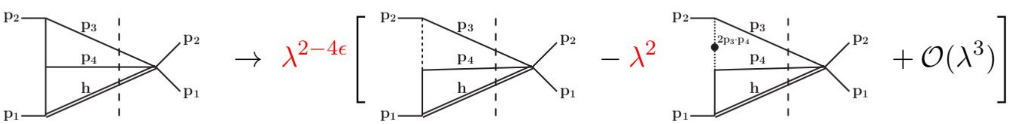
Beam Functions calculation at N3LO

(Ebert, Mistlberger, **GV**) [2006.05329], [2006.03056]

- We calculated the **collinear expansion of the partonic cross section** for DY and Higgs @N3LO **differential in (Q_T, τ, z)**

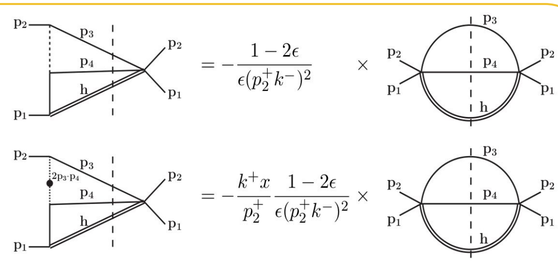
  - Collinear Expansion at the XS level

    "Collinear expansion for color singlet cross sections" [Ebert, Mistlberger, **GV**]

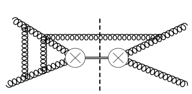


  - Reduction to basis of **Master Integrals** via Integration By Parts (IBPs) using Water

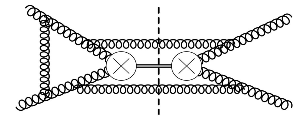
Expanded diagrams admit (simplified) **IBPs** identities



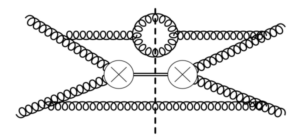
  - RVV: known in full kinematics [Duhr, Gehrmann] [Duhr, Gehrmann, Jaquier] [Dulat, Mistlberger]



  - **RRV**: 170 Collinear Master Integrals



  - **RRR**: 320 Collinear Master Integrals



  - Derived system of Differential Equations for the Master Integrals
  - System has 2 non trivial scales with algebraic dependence on the variables (not something solvable algorithmically)
  - Algebraic sectors: constructed dlog integrand basis via calculation of **leading singularities** of candidate integrals on maximal cut surface
  - Boundaries from soft integrals [Anastasiou, Duhr, Dulat, Mistlberger] and constraints on singular behavior

Beam Functions at N3LO

Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO **differential in (Q_T, τ, z)**

project to $\boldsymbol{\tau}$

project to $\mathbf{q_T}$

$$B_a(t_a, x_1^B, \mu)$$

$$\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$$

"N-Jettiness Beam Functions at N3LO"

M.Ebert, B.Mistlberger, **GV**
[2006.03056]

- Quark $\boldsymbol{\tau}$ **beam functions**
  (Quark N-Jettiness Beam Function)
- Gluon $\boldsymbol{\tau}$ **beam functions**
  (Gluon N-Jettiness Beam Function)

"Transverse Momentum Dependent PDFs at N3LO"

M.Ebert, B.Mistlberger, **GV**
[2006.05329]

- Quark **TMDPDF**
  (Quark q_T Beam Function)
- Unpolarized **Gluon TMDPDF**
  (Gluon q_T Beam Function)

Bare Beam Functions and Renormalization

N-Jettiness Beam Function

q_T Beam Function

Collinear expansion of the partonic cross section for Drell Yan and Higgs at N3LO **differential in (Q_T, τ, z)**

project to $\boldsymbol{\tau}$: $B_a(t_a, x_1^B, \mu)$

project to $\mathbf{q_T}$: $\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

Bare Results

N-Jettiness Beam Function:

- Poles in dimensional regularization (up to $1/\varepsilon^6$)
- Logs/Plus Distributions in $\boldsymbol{\tau}$
- Iterated Integrals up to weight 5, with alphabet

$$\mathcal{A} = \left\{\frac{1}{z}, \frac{1}{1-z}, \frac{1}{2-z}, \frac{1}{1+z}, \frac{1}{z}, \frac{1}{\sqrt{4-z}\sqrt{z}}\right\}$$

- Constants to weight 6

q_T Beam Function:

- Poles in dimensional regularization
- Rapidity divergences regulated by exponential regulator
- Logs/Plus Distributions in $\mathbf{b_T/q_T}$
- HPLs in z up to weight 5
- Constants to weight 6

Renormalization

N-Jettiness Beam Function:

- Coupling renormalization
- SCET_I renormalization
- IR poles subtracted via NNLO PDF counterterms

q_T Beam Function:

- Coupling renormalization
- Zero-bin subtraction via calculation of bare q_T Soft Function at N3LO
- SCET_II renormalization
- IR poles subtracted via NNLO PDF counterterms

Checks

N-Jettiness Beam Function

q_T Beam Function

$B_a(t_a, x_1^B, \mu)$ ← project to τ — **Collinear expansion of the partonic cross section** for Drell Yan and Higgs at N3LO **differential in (Q_T, τ, z)** — project to q_T → $\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

Checks

- 6 orders of poles cancel in all channels
- Terms involving $\mathcal{L}_n\left(\frac{t}{\mu^2}\right)$ $n = 0, \ldots, 5$ vs RGE prediction
- Eikonal limit vs threshold consistency
  [Billis, Ebert, Michel and Tackmann]
- Generalized leading color approx
  [Behring, Melnikov, Rietkerk, Tancredi, Wever]

- All rapidity divergences regulated
- 3 orders of ε poles cancel for all channels
- Log terms vs RGE prediction
  [Billis, Ebert, Michel and Tackmann]
- Eikonal limit vs threshold consistency
- Quark channels vs [Luo, Yang, Zhu, Zhu 1912.05778] (found small discrepancy)

Checks

N-Jettiness Beam Function

q_T Beam Function

$B_a(t_a, x_1^B, \mu)$ ← project to $\boldsymbol{\tau}$ ← **Collinear expansion of the partonic cross section** for Drell Yan and Higgs at N3LO **differential in (Q_T, τ, z)** → project to $\mathbf{q_T}$ → $\tilde{B}_i\left(x_1^B, b_T, \mu, \frac{\nu}{\omega_a}\right)$

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Confirmation of our results in later independent calculation

(Baranowski, Behring, Melnikov, Tancredi, Wever) [2211.05722]

- All rapidity divergences regulated
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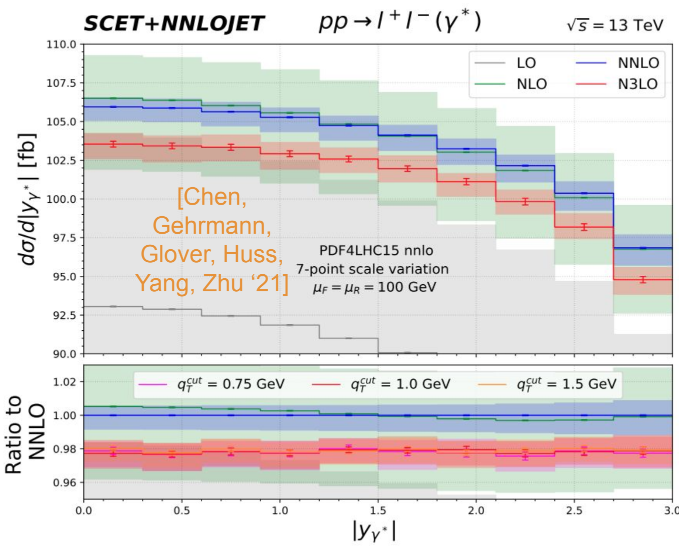
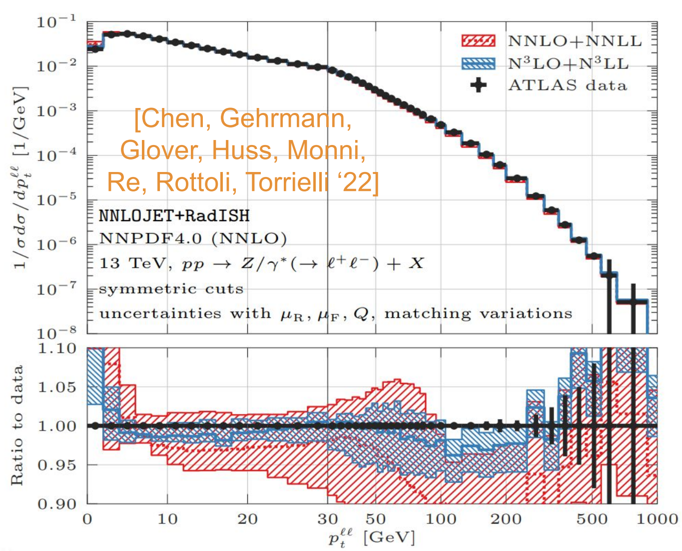
Confirmation of our results in later independent calculation

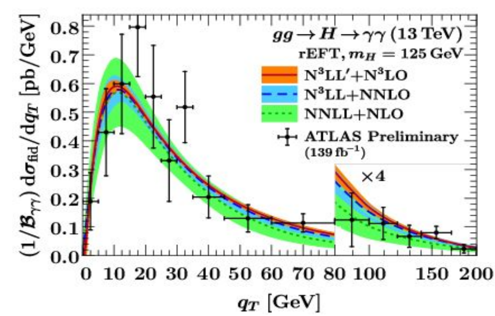
(Luo, Yang, Zhu, Zhu) [2012.03256]

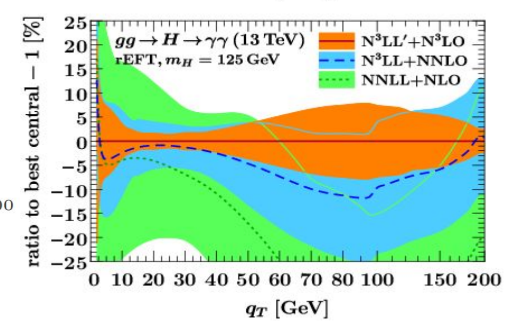
Slicing at N3LO

- **q_T beam functions** at N3LO were **last missing ingredient** for:
  - q_T subtraction for differential and fiducial Drell-Yan and Higgs production at N3LO
  - q_T resummation at N3LL`
- Many new exciting phenomenological results at N3LO employing them!

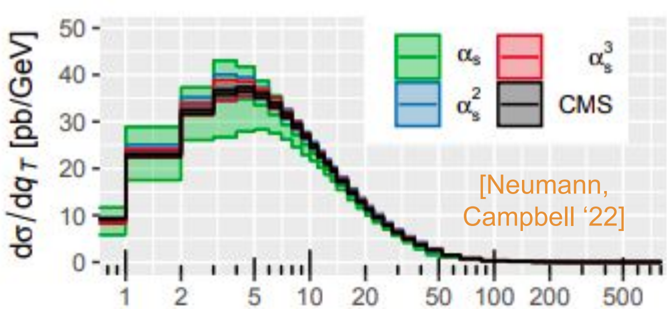








[Billis, Dehnadi, Ebert, Michel, Tackmann '21]



And many more:

[Ju, Schönherr '21]

[Camarda, Cieri, Ferrera '21]

[Re, Rottoli, Torrielli '21]

...

Going Beyond N3LO:
Rapidity Anomalous Dimension to Four Loops and Resummation at N4LL

C.Duhr, B.Mistlberger, G.Vita
[2205.02242]

$$\gamma^i_{r,4} = C_A^3 C_R \left(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6} \right) + C_R n_f^3 \left(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187} \right)$$
$$+ C_R C_A^2 n_f \left(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3} \right) + C_R C_F n_f^2 \left(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243} \right)$$
$$+ \frac{C^4_{AR}}{d_R} \left(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 - 384 + 4b_{4,d4AF} \right) + C_A C_R n_f^2 \left(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} - \frac{898033}{2916} \right)$$
$$+ \frac{C^4_{FR}}{d_R} n_f \left(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}} \right)$$
$$+ C_A C_F C_R n_f \left(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486} \right)$$
$$+ C_F^2 C_R n_f \left(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54} \right)$$

The Rapidity Anomalous dimension

- Key ingredient for resummation of large logs for transverse momentum dependent (TMD) observables is the **rapidity anomalous dimension**, AKA Collins Soper Kernel

- It appears in many contexts, eg. in RGE evolution of TMD soft function

$$\nu \frac{\mathrm{d}}{\mathrm{d}\nu} \ln S(b_T, \mu, \nu) = \gamma_r(b_T, \mu)$$

- It can be decomposed in a term directly related to the cusp anomalous dimension and a non-cusp term which contains the information intrinsic to the rapidity

$$\gamma_r^i(b_T, \mu) = -4 \int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'} \Gamma^i_{\mathrm{cusp}}[\alpha_s(\mu')] + \boxed{\gamma_r^i(\mu_0, b_T)}$$

- Non-cusp term vanishes at LO and NLO.
- NNLO: known for a long time. [Davies, Webber, Stirling '85] [de Florian, Grazzini '00]
- N3LO: determined in 2016 via bootstrap methods [Li, Zhu '16]
- N4LO: C.Duhr, B.Mistlberger, **GV** [2205.02242] This talk (see also [Moult, Zhu, Zhu '22])

Rapidity Anomalous Dimension to Four Loops

- The calculation of the Rapidity anomalous dimension to 4 loops by brute force would require calculation of some **differential object** (e.g. p_T soft function) **to 4 loops:**
  - This is beyond the current technology for fixed order calculations

- State of the art at 4 loops is:
  - **Hard/Collinear** Anomalous Dimension to 4 loops [von Manteuffel, Panzer, Schabinger - 2002.04617]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} H^B_{ij}(\mu^2) = \gamma^r_H(\alpha_S(\mu^2), \mu^2) H^B_{ij}(\mu^2),$$

$$\gamma^r_H(\alpha_S(\mu^2), \mu^2) = \Gamma^r_{\mathrm{cusp}}(\alpha_S(\mu)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2}\gamma_H(\alpha_S(\mu^2))$$

Hard anomalous dimension (2 x collinear anomalous dimension of form factors)

  - **Virtual** Anomalous Dimension to 4 loops [Das, Moch, Vogt - 1912.12920]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} f_i^{\mathrm{th}}(z, \mu^2) = \gamma^r_f(z, \alpha_S(\mu^2)) \otimes_z f_i^{\mathrm{th}}(z, \mu^2),$$

$$\gamma^r_f(z, \alpha_S(\mu^2)) = \Gamma^r_{\mathrm{cusp}}(\alpha_S(\mu^2)) \left[\frac{1}{1-z}\right]_+ + \frac{1}{2}\gamma^r_f(\alpha_S(\mu^2))\delta(1-z)$$

DGLAP at threshold

Rapidity Anomalous Dimension to Four Loops

- There is a **Rapidity**/**Threshold** **correspondence** for conformal theories, which holds at the critical dimension of QCD [Vladimirov - 1610.05791]

$$\gamma_r^i[\alpha_s,\epsilon^*]+\gamma_{\rm th}^i[\alpha_s,\epsilon^*]=0$$

$$\beta[\alpha_s,\epsilon]=-2\alpha_s\left[\epsilon+\frac{\alpha_s}{4\pi}\beta_0+\left(\frac{\alpha_s}{4\pi}\right)^2\beta_1+\dots\right]$$

$$\beta[\alpha_s,\epsilon^*]=0$$

$$\epsilon^*=-\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0+\left(\frac{\alpha_s}{4\pi}\right)^2\beta_1+\dots\right]$$

Critical dimension of QCD

- **Threshold anomalous dimension** is part of RGE of soft function

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln S_i(\vec{b}_T,\mu,\nu)=4\Gamma_{\rm cusp}^i[\alpha_s(\mu)]\ln\mu/\nu+\gamma_{\rm th}^i[\alpha_s]$$

$$\nu\frac{\mathrm{d}}{\mathrm{d}\nu}\ln S_i(\vec{b}_T,\mu,\nu)=-4\int_{b_0/b_T}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\Gamma_{\rm cusp}^i[\alpha_s(\mu')]+\gamma_r^i[\alpha_s]$$

- Via SCET I consistency relations, relate **Threshold** to **Virtual** and **Collinear** anomalous dimensions

$$\gamma_{\rm thr.}^r(\alpha_S(\mu^2))=-2\gamma_f^r(\alpha_S(\mu^2))-\gamma_H^r(\alpha_S(\mu^2))$$

Rapidity Anomalous Dimension to Four Loops

- Difference between **threshold** and **rapidity** anomalous dimension comes from higher orders in dimensional regularization evaluated at critical point!

$$\epsilon^* = -\left[\left(\frac{\alpha_s}{4\pi}\right)\beta_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\beta_1 + \dots\right]$$

- To obtain these terms it is necessary to calculate the **TMD Soft Function at N3LO** to **higher orders in dimensional regularization**

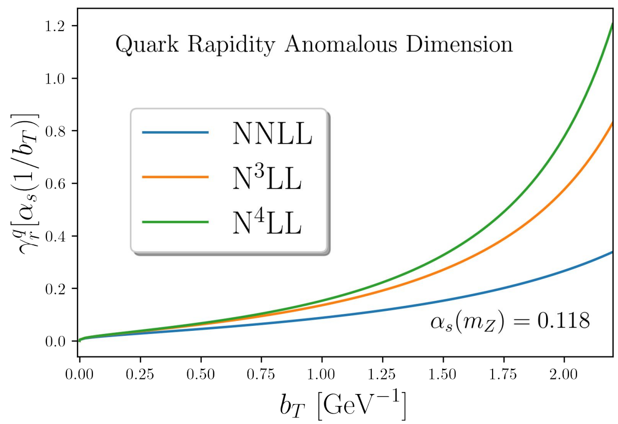
- We obtained this in

"Soft Integrals and Soft Anomalous Dimensions at N3LO and Beyond"
C.Duhr, B.Mistlberger, **GV** [2205.04493]

- Key point: Use method of differential equations and fix boundaries by relations between differential and inclusive threshold integrals

Rapidity Anomalous Dimension to Four Loops

- Obtained results at N4LO
- Quark and gluon related by **generalized casimir scaling**
- Well behaved series (stable coefficients) (see also [Moult, Zhu, Zhu])



$$\gamma_r^q(n_f\!=\!5) = 0.53929\alpha_s^2 + 0.68947\alpha_s^3 + \left(0.61753 \pm 5\cdot 10^{-5}\right)\alpha_s^4$$

$$\gamma_r^g(n_f\!=\!5) = 1.21341\alpha_s^2 + 1.55130\alpha_s^3 + \left(1.6041 \pm 5\cdot 10^{-4}\right)\alpha_s^4$$

- 4 coefficients are not known analytically but only numerically (very well)

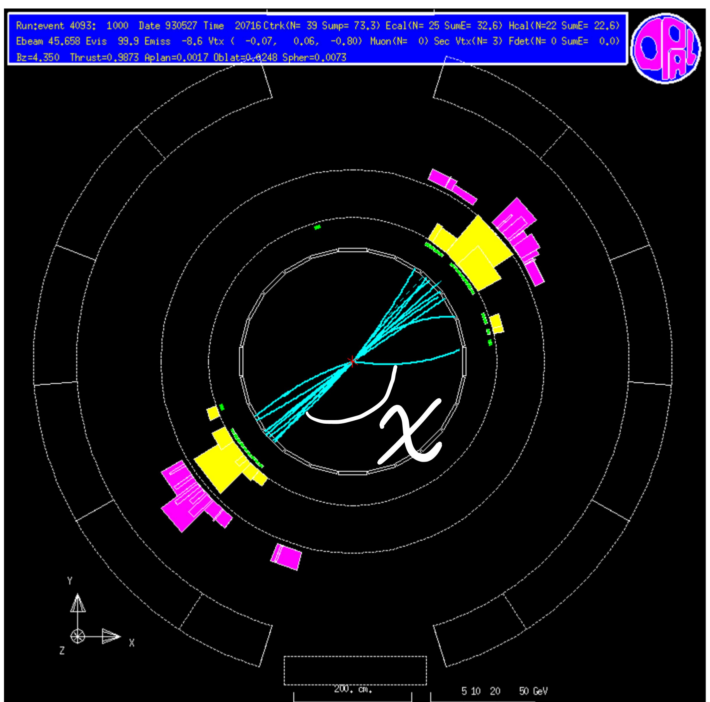
$$\begin{aligned}
\gamma_{r,4}^i = {} & C_A^3 C_R \Big(-\frac{21164}{9}\zeta_3^2 - \frac{26104}{9}\zeta_2\zeta_3 + \frac{4228}{3}\zeta_4\zeta_3 + \frac{2752}{3}\zeta_2\zeta_5 \\
& + \frac{1201744\zeta_3}{81} + \frac{778166\zeta_2}{243} + \frac{8288\zeta_4}{9} - \frac{181924\zeta_5}{27} \\
& - \frac{63580\zeta_6}{27} + \frac{11071\zeta_7}{3} - \frac{28290079}{2187} - \frac{b^4_{q,C^4_{AF}}}{6}\Big) \\
& + C_A C_R n_f^2 \Big(\frac{224}{9}\zeta_3\zeta_2 + \frac{6752\zeta_2}{243} - \frac{22256\zeta_3}{81} + \frac{160\zeta_4}{9} + \frac{1472\zeta_5}{9} \\
& - \frac{898033}{2916}\Big) + C_R n_f^3 \Big(\frac{160\zeta_3}{9} - \frac{16\zeta_4}{9} + \frac{10432}{2187}\Big) \\
& + C_R C_A^2 n_f \Big(-\frac{8584}{9}\zeta_3^2 + \frac{2080}{3}\zeta_2\zeta_3 - \frac{247652\zeta_3}{81} - \frac{182134\zeta_2}{243} \\
& + \frac{43624\zeta_4}{27} - \frac{17936\zeta_5}{27} + \frac{1582\zeta_6}{27} + \frac{10761379}{2916} \\
& - \frac{b^4_{q,C^4_{FF}}}{12} - 2b^4_{q,n_f C_F^2 C_A} - b^4_{q,n_f C_F^3}\Big) \\
& + C_R C_F n_f^2 \Big(\frac{6928\zeta_3}{27} + \frac{160\zeta_4}{3} + 32\zeta_5 - \frac{110059}{243}\Big) \\
& + \frac{C^4_{AR}}{d_R} \Big(\frac{6688\zeta_3^2}{3} + 3584\zeta_2\zeta_3 + 736\zeta_4\zeta_3 + \frac{15616\zeta_3}{9} - \frac{224\zeta_4}{3} \\
& + \frac{4352\zeta_2}{3} - 2048\zeta_2\zeta_5 + \frac{3680\zeta_5}{9} - \frac{6952\zeta_6}{9} - 6968\zeta_7 \\
& - 384 + 4b_{4,d4AF}\Big) \\
& + \frac{C^4_{FR}}{d_R} n_f \Big(-\frac{2432}{3}\zeta_3^2 - 256\zeta_2\zeta_3 + \frac{10624\zeta_3}{9} - \frac{9088\zeta_2}{3} \\
& + \frac{1600\zeta_4}{3} + \frac{43520\zeta_5}{9} - \frac{2368\zeta_6}{9} + 768 + 4b^4_{q,C^4_{FF}}\Big) \\
& + C_A C_F C_R n_f \Big(4b_{4,n_f C_F^2 C_A} + \frac{6800\zeta_3^2}{3} - \frac{8864}{9}\zeta_2\zeta_3 - \frac{1892\zeta_3}{9} \\
& + \frac{5122\zeta_2}{27} - \frac{122216\zeta_4}{27} + \frac{21904\zeta_5}{9} - 1436\zeta_6 + \frac{2149049}{486}\Big) \\
& + C_F^2 C_R n_f \Big(4b^4_{q,n_f C_F^3} - 736\zeta_3^2 + \frac{1024}{3}\zeta_2\zeta_3 + \frac{2240\zeta_3}{9} - 648\zeta_2 \\
& + 668\zeta_4 - \frac{7744\zeta_5}{3} + \frac{29336\zeta_6}{9} - \frac{27949}{54}\Big)
\end{aligned}$$

Event Shapes Resummation at N4LL

Energy-Energy Correlation

- Interesting observable involving measurements on QCD final state radiation is the Energy-Energy Correlation (EEC)

$$\mathrm{EEC}(\chi) = \frac{\mathrm{d}\sigma}{\mathrm{d}\chi} = \sum_{i,j} \int \mathrm{d}\sigma_{e^+e^- \to ij+X} \frac{E_i E_j}{Q^2} \delta(\cos\theta_{ij} - \cos\chi)$$



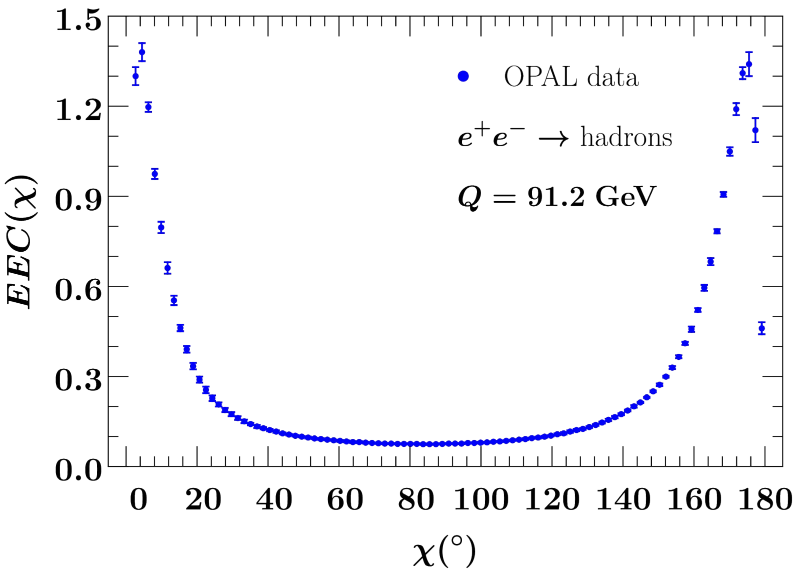
- Measures **angle** χ between pairs of color charged particles, weighted by energy

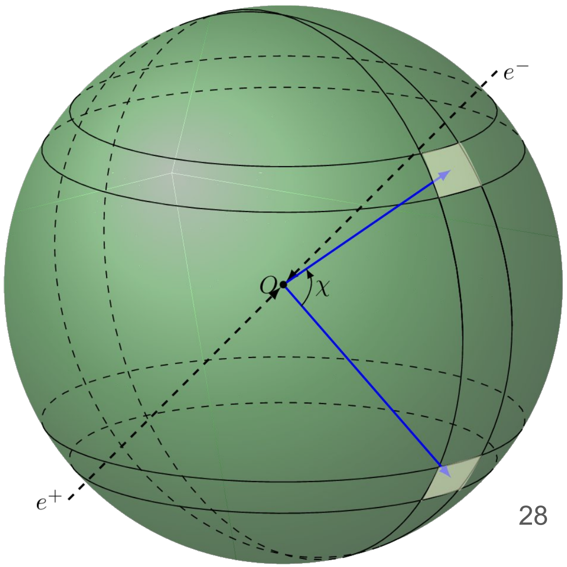
- One of the oldest IRC safe observables proposed [Basham, Brown, Ellis, Love, PRL **41**, 1585 **(1978)**]

Energy-Energy Correlation: Motivations

Interest in EEC for a variety of reasons:

- Very well measured event shape at electron-positron colliders **GOOD DATA!**
- Arguably the event shape with the simplest analytic structure **GOOD THEORY!**
- Can be expressed as a four point correlator in terms of energy flow operators [Maldacena, Hofman; Korchemsky;]
- Substantial recent progress in understanding it both in QCD and CFTs. [Moult, Dixon, Zhu; Korchemsky; Chicherin, Henn, Sokatchev, Yan; Simmons Duffin, Kologlu, Kravchuk, Zhiboedov; Moult, Vita, Yan; Luo, Shtabovenko, Yang;...]
- Natural playground for connections between $\mathcal{N}=4$ and QCD
- Allows precise extraction of α_s [OPAL Collaboration; Kardos, Kluth, Somogyi, Tulipant, Verbytskyi; ...]
- ...





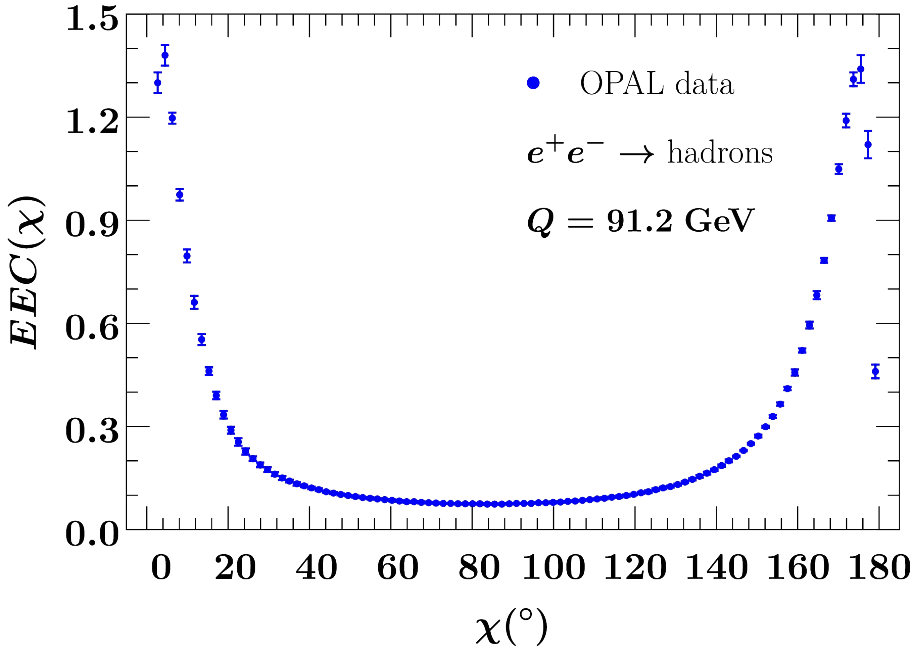
Energy-Energy Correlation: End Points

- It has singular structure and logarithmic enhancement at both end points

$\chi \to 0$
$z \to 0$
Collinear/Forward/ small angle Limit

$$z \equiv \frac{1}{2}(1-\cos\chi)$$

$\chi \to \pi$
$z \to 1$
Back-to-Back Limit



- The two limits have very different structure (no symmetry between them)
- **Single logarithmic** series in small angle limit

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 0}{\sim} \sum_{L=1}^{\infty}\sum_{m=0}^{L-1}\left(\frac{\alpha_s}{4\pi}\right)^L c_{L,m}\,\frac{\log^m z}{z}$$

- **Double logarithmic** series at $z \to 1$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} \overset{z\to 1}{\sim} \sum_{L=1}^{\infty}\sum_{m=0}^{2L-1}\left(\frac{\alpha_s}{4\pi}\right)^L d_{L,m}\,\frac{\log^m(1-z)}{(1-z)}$$

- Plus **contact terms** $\sim \delta(1-z)\,,\,\delta(z)$

- We can derive factorization theorems at both ends in SCET for resummation

[Moult, Zhu] [Moult, Dixon, Zhu] [Ebert, Mistlberger, **GV**]

EEC in the back to back limit to N4LL

Back-to-back region of EEC obeys TMD-like fact. thm and resummation ("crossed version of q_T")

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\, \mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T} \overbrace{\delta\left(1 - z - \frac{q_T^2}{Q^2}\right)}^{1-z\equiv(\cos\frac{\chi}{2})^2\approx\frac{q_T^2}{Q^2}} \underbrace{\mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu,Qb_T\upsilon\right)}_{\text{Pure Rapidity EEC Jet Functions}}$$

Standard RGE

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln H_{q\bar{q}}(Q,\mu) = \gamma_H^q(Q,\mu)\,,$$

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = \gamma_{\mathcal{J}_q}(\mu,\upsilon\mu/Q)$$

Rapidity RGE

$$\upsilon\frac{\mathrm{d}}{\mathrm{d}\upsilon}\ln \mathcal{J}_q\left(b_T,\mu,\frac{Qb_T}{\upsilon}\right) = -\frac{1}{2}\gamma_r^q(b_T,\mu)$$

Resummed cross section to all orders (at LP)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{8}\int_0^\infty \mathrm{d}(b_TQ)^2\, J_0\left(b_TQ\sqrt{1-z}\right) H_{q\bar{q}}(Q,\mu_H)$$
$$\times\, \mathcal{J}_q\left(b_T,\mu_J,\frac{Qb_T}{\upsilon_n}\right)\mathcal{J}_{\bar{q}}\left(b_T,\mu_J,Qb_T\upsilon_{\bar{n}}\right)\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}$$
$$\times\, \exp\left[4\int_{\mu_J}^{\mu_H}\frac{\mathrm{d}\mu'}{\mu'}\Gamma^q_{\mathrm{cusp}}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma_H^q[\alpha_s(\mu')]\right]$$

Logarithmic Accuracy for Resummed Predictions

- **Resummation accuracy** is determined by perturbative accuracy of ingredients entering resummed cross section

- For **N4LL resummation:**
  - 3 Loop Hard Function
    [Gehrmann, Glover, Huber, Ikizlerli, Studerus '10]
  - 3 Loop EEC Jet Function
    [Ebert, Mistlberger, **GV** 2012.07859]
  - 4 Loop Collinear Anom. Dim.
    [von Manteuffel, Panzer, Schabinger '20]
  - 4 Loop Rapidity Anomalous Dimension NEW!
  - 5 Loop Beta function
    [Baikov, Chetyrkin, Kuhn '16]
  - 5 Loop Cusp (approx)
    [Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt '18]

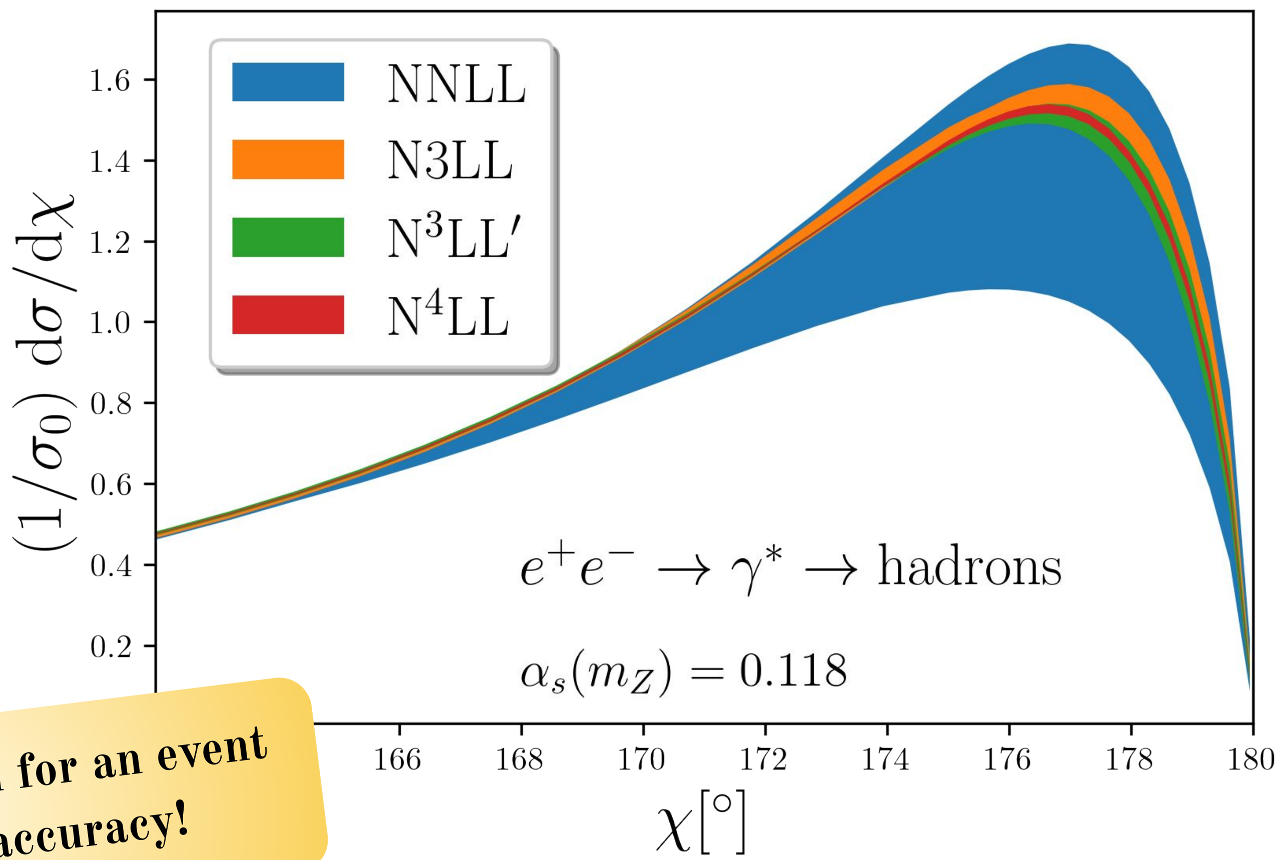
Resummed cross section to all orders (at LP)

$$\frac{d\sigma}{dz} = \frac{\hat{\sigma}_0}{8}\int_0^\infty d(b_T Q)^2\, J_0(b_T Q\sqrt{1-z})\, H_{q\bar{q}}(Q,\mu_H)$$
$$\times\, \mathcal{J}_q\Big(b_T,\mu_J,\frac{Qb_T}{v_n}\Big)\mathcal{J}_{\bar{q}}\big(b_T,\mu_J,Qb_T v_{\bar{n}}\big)\Big(\frac{v_n}{v_{\bar{n}}}\Big)^{\frac{1}{2}\gamma_r^q(b_T,\mu_J)}$$
$$\times\, \exp\left[4\int_{\mu_J}^{\mu_H}\frac{d\mu'}{\mu'}\Gamma^q_{\rm cusp}[\alpha_s(\mu')]\ln\frac{\mu'}{Q} - \gamma^q_H[\alpha_s(\mu')]\right]$$

Accuracy	H, $\mathcal{J}$	$\gamma_H^q(\alpha_s)$	$\gamma_r^q(\alpha_s)$	$\beta(\alpha_s)$	$\Gamma_{\rm cusp}(\alpha_s)$
LL	Tree level	–	–	1-loop	1-loop
NLL	Tree level	1-loop	1-loop	2-loop	2-loop
NLL′	1-loop	1-loop	1-loop	2-loop	2-loop
NNLL	1-loop	2-loop	2-loop	3-loop	3-loop
NNLL′	2-loop	2-loop	2-loop	3-loop	3-loop
N^3LL	2-loop	3-loop	3-loop	4-loop	4-loop
N^3LL′	3-loop	3-loop	3-loop	4-loop	4-loop
N^4LL	3-loop	4-loop	4-loop	5-loop	5-loop

EEC in the back to back limit to N4LL

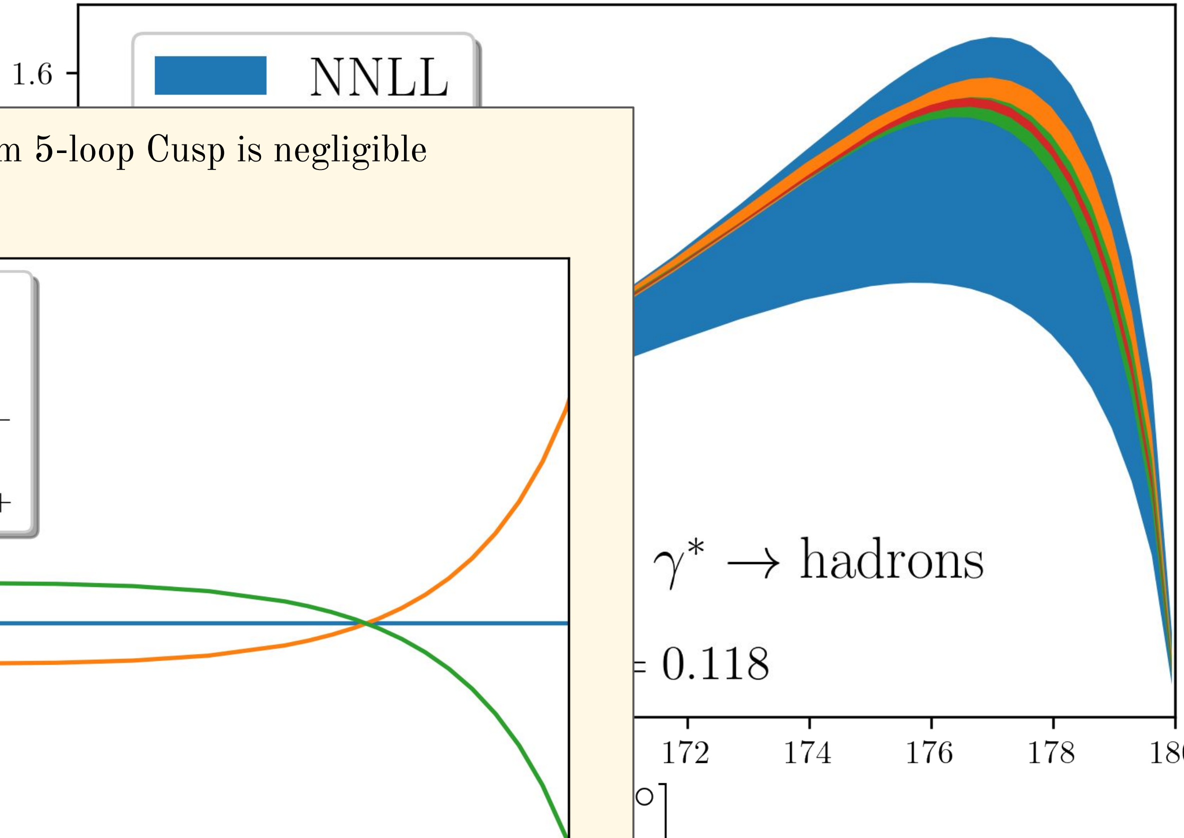
- Implemented the resummation of this event shape at **N4LL** in new numerical framework: **pySCET**
- Nice convergence of perturbative result
- Uncertainties obtained by 15 point scale variation in SCET



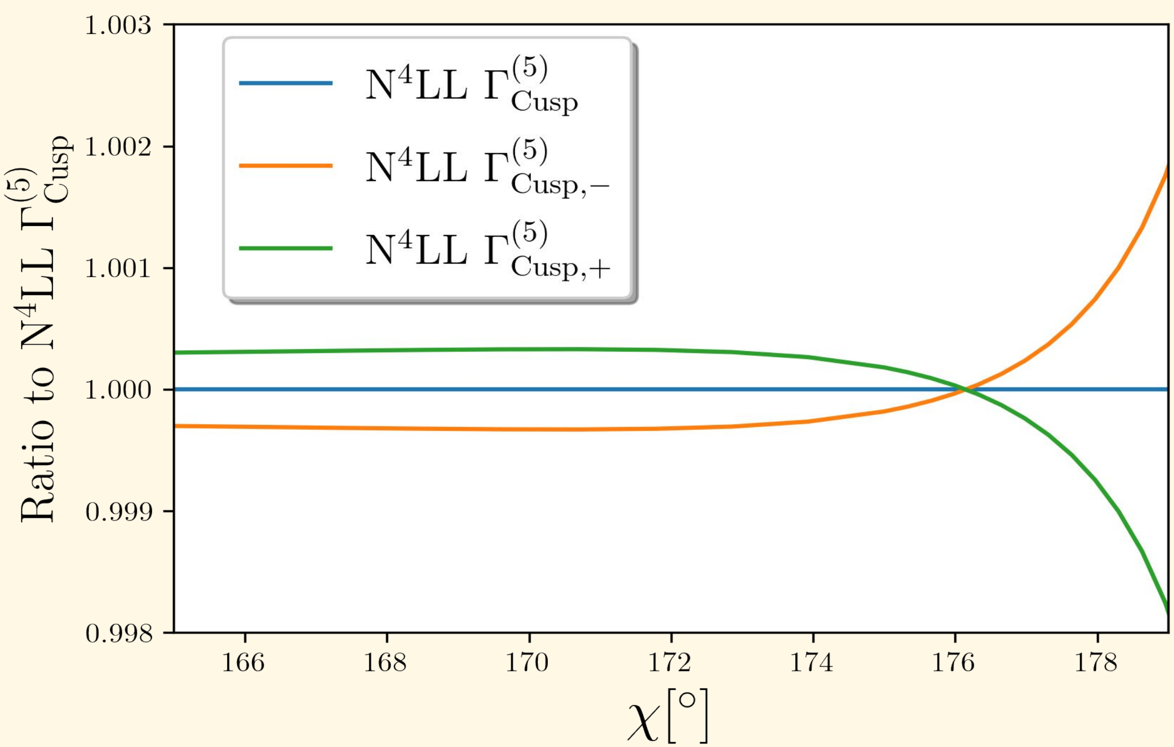
First resummation for an event shape at this accuracy!

EEC in the back to back limit to N4LL

- Implemented the resummation of this event shape at **N4LL** in
-
-

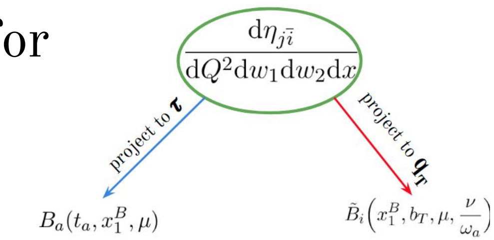


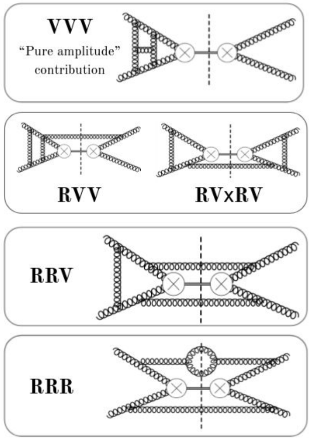
Impact of uncertainty from 5-loop Cusp is negligible



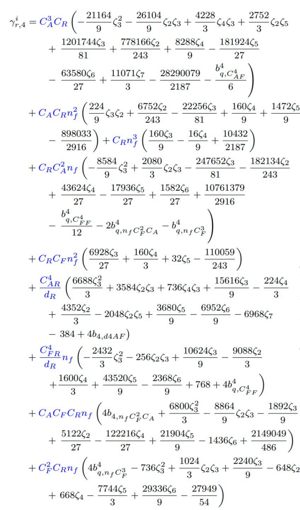
Conclusion

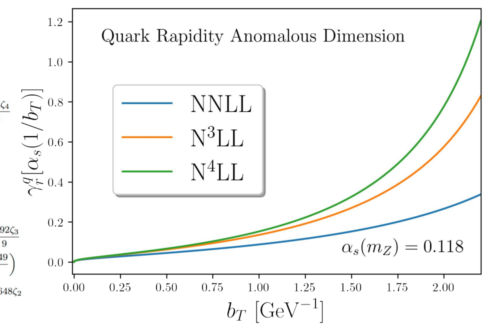
➢ Introduced motivations and techniques for theoretical predictions at N3LO



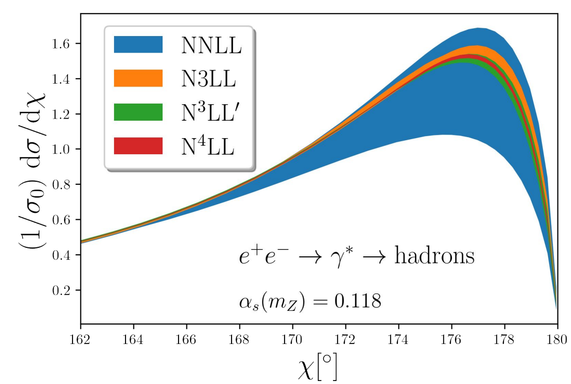


➢ Discussed the calculation of TMDPDF and N-Jettiness Beam Functions at N3LO via collinear expansion of cross sections





➢ Presented the computation of the quark and gluon Rapidity Anomalous Dimension



➢ Illustrated first results for TMD Resummation at N4LL on event shapes

Backup

More things towards percent level predictions…

$$\sigma = f_1 \circ f_2 \circ \int d\Phi |M|^2 + \mathcal{O}(\Lambda^2/Q^2)$$

1. **Accessibility and User Friendliness:** Creating frameworks that make N^3LO (and NNLO) predictions easily accessible for comparison to experimental data.

2. **Corrections beyond QCD:** EWK and masses.

3. **Factorisation Violation at N^3LO:** tops, PDFs.

4. **Parton Showers:** Consistent combination of parton showers with fixed order perturbative computations at N^3LO.

5. **Resummation:** Complementing N^3LO computations and resummation techniques for infrared sensitive observables.

6. **Uncertainties:** Deriving / defining reliable uncertainty estimates for theoretical computations at the percent level.

7. **Beyond Leading Power Factorisation:** Exploring the limitations of leading power perturbative descriptions of hadron collision cross sections.

Pure Rapidity Renormalization

[Ebert, Moult, Tackmann, Stewart, **GV**, Zhu '18]

- q_T distributions in SCET need rapidity regularization
- Use **pure rapidity regulator:** $\int d^d k \to \int d^d k \, \upsilon^\eta \left| \frac{\bar{n} \cdot k}{n \cdot k} \right|^{-\frac{\eta}{2}} = \int d^d k \, \upsilon^\eta e^{-y_k \eta}$
- Motivated by necessity of homogeneous rapidity regularization beyond leading power
- Soft function corrections (for q_T) are scaleless, hence $S = 1$ to all orders
- Rapidity divergences cancel between collinear functions only
- Shares nice features of both k^+ analytic regulator and k^z-regulator
  [Becher, Bell] [Chiu, Jain, Neill, Rothstein]
- Allows for an $\overline{\text{MS}}$-like Renormalization Scheme to derive Rapidity RGE

q_T Distributions in Pure Rapidity Renormalization

[GV, to appear]

- We apply it to the leading power q_T distribution for Higgs and Drell Yan

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2,\mu) \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \tilde{B}_a\left(x_1^B, b_T, \mu, \frac{b_T\omega_a}{\upsilon}\right) \tilde{B}_b\left(x_2^B, b_T, \mu, b_T\omega_b\upsilon\right)$$

Color singlet Hard Function

q_T-Beam Functions / TMDPDFs in Pure Rapidity Renormalization

- Soft Function corrections are scaleless => S=1 to all orders
- Rapidity divergences cancel between Beam Functions only, but **finite terms are identical**
- Similar to Collins-Soper TMDPDFs (no soft function, symmetry between collinear directions), but retain full control on rapidity scale at the matching kernel level (better handle for resummation uncertainties on RRGE)

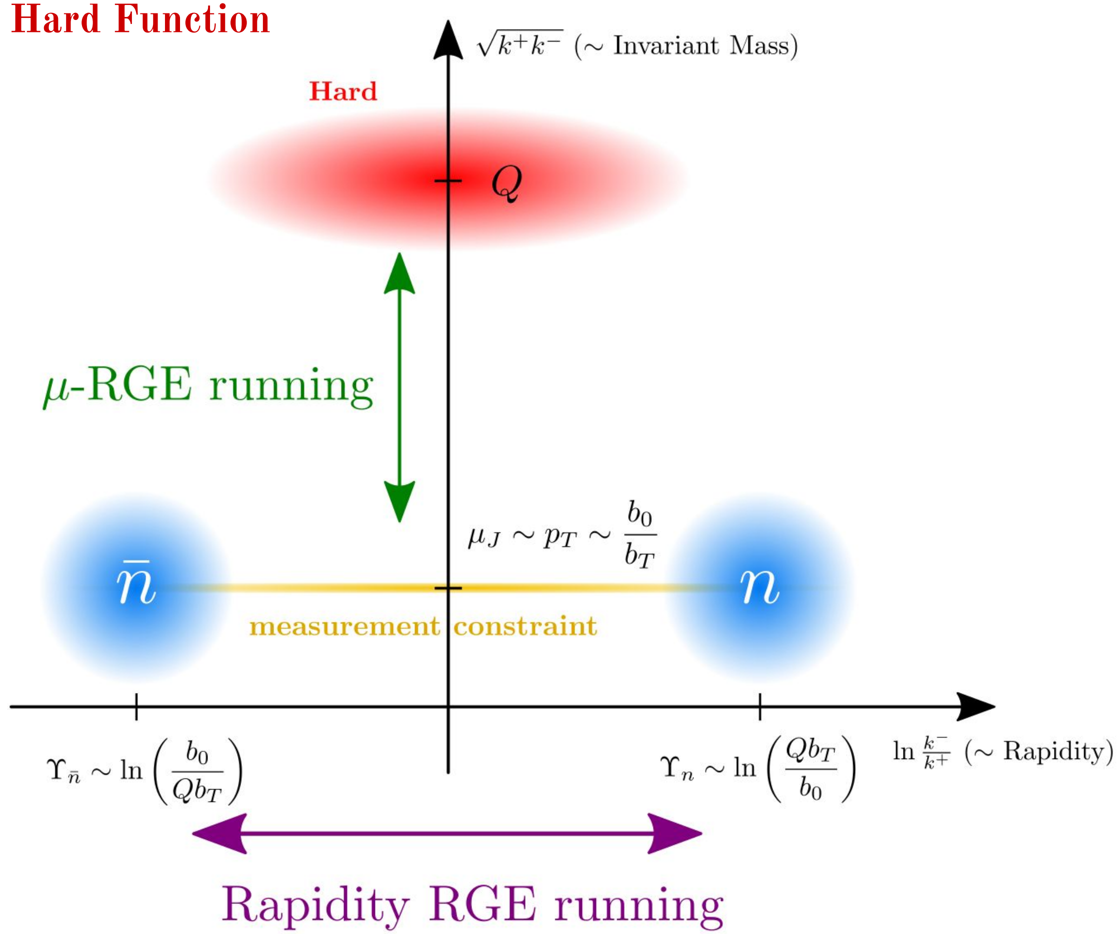
q_T Distributions in Pure Rapidity Renormalization

[GV, to appear]

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2,\mu) \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\,\vec{q}_T\cdot\vec{b}_T} \tilde{B}_a\left(x_1^B, b_T, \mu, \frac{b_T\omega_a}{\upsilon}\right) \tilde{B}_b\left(x_2^B, b_T, \mu, b_T\omega_b\upsilon\right)$$

Color singlet Hard Function

Beam Functions in Pure Rapidity Renormalization



Rapidity RGE in Pure Rapidity Renormalization

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2,\mu) \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \tilde{B}_a\Big(x_1^B, b_T, \mu, \frac{b_T\omega_a}{\upsilon}\Big) \tilde{B}_b\Big(x_2^B, b_T, \mu, b_T\omega_b\upsilon\Big)$$

Color singlet Hard Function

q_T-Beam Functions (TMDPDFs) in Pure Rapidity Renormalization

Functions obey RGEs in terms of anomalous dimensions

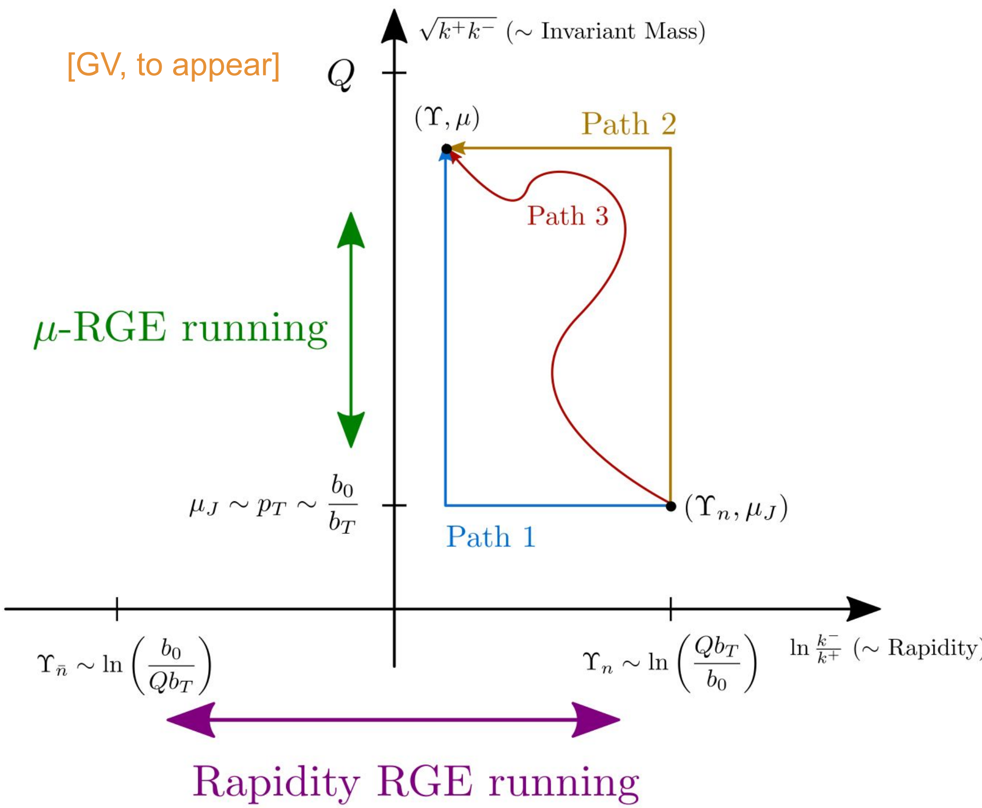
$$\frac{d}{d\ln\mu} \ln H_i(Q,\mu) = \gamma_H^i(Q,\mu),$$

quark/gluon (hard) anomalous dimension

$$\upsilon\frac{d}{d\upsilon} \ln \tilde{B}_a\Big(x, b_T, \mu, \frac{b_T\omega}{\upsilon}\Big) = -\frac{1}{2}\gamma_r^q(b_T,\mu)$$

Rapidity anomalous dimension

[GV, to appear]



RRG Evolution is path independent

[Chiu et al. '12]

$$\mu\frac{d}{d\mu}\gamma_r(b_T,\mu) = -4\Gamma_{\rm cusp}[\alpha_s(\mu)]$$

$$\left[\upsilon\frac{d}{d\upsilon}, \mu\frac{d}{d\mu}\right] J\Big(b_T,\mu,\upsilon\Big) = 0$$

Rapidity RGE in Pure Rapidity Renormalization

$$\frac{d\sigma}{dQ^2 dY d^2\vec{q}_T} = \sigma_0 \sum_{a,b} H_{ab}(Q^2,\mu) \int \frac{d^2\vec{b}_T}{(2\pi)^2} e^{i\vec{q}_T\cdot\vec{b}_T} \tilde{B}_a\left(x_1^B, b_T, \mu, \frac{b_T\omega_a}{\upsilon}\right) \tilde{B}_b\left(x_2^B, b_T, \mu, b_T\omega_b\upsilon\right)$$

Color singlet Hard Function

q_T-Beam Functions (TMDPDFs) in Pure Rapidity Renormalization

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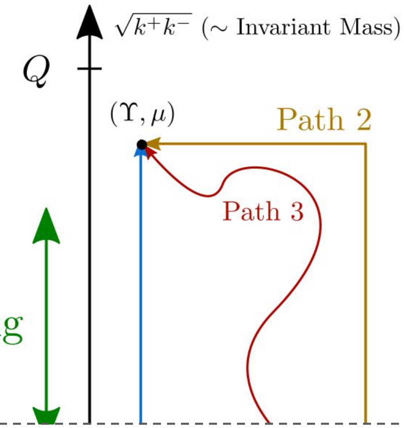
Rapidity anomalous dimension

[GV, to appear]

μ-RGE running



$$\frac{d\sigma^{res}}{dQ\,dY\,dq_T} \sim \hat{\sigma}_0 \overbrace{\int_0^\infty d(b_TQ)^2 J_0(b_Tq_T)}^{\text{Fourier Transform}} \overbrace{H_{q\bar{q}}(Q,\mu_H) B_q\left(x_1, b_T, \mu_B, \frac{Qe^{-Y}b_T}{\upsilon_n}\right) B_{\bar{q}}\left(b_T, \mu_B, Qe^{Y}b_T\upsilon_{\bar{n}}\right)}^{\text{Boundaries}}$$

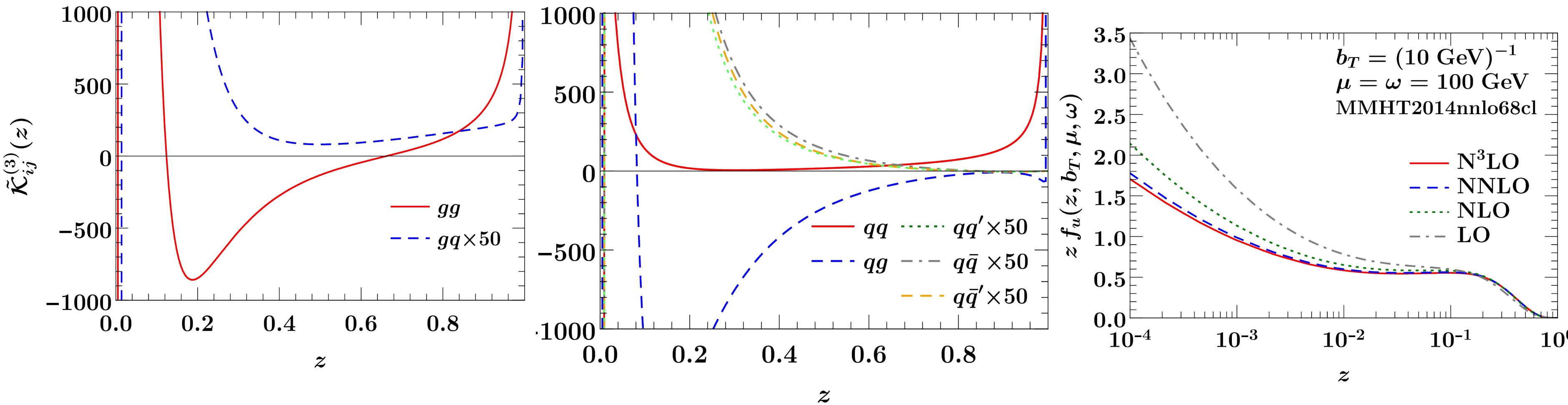
$$\times \exp\left[\underbrace{\int_{\mu_H}^{\mu} \frac{d\mu'}{\mu'}\gamma_H^q(Q,\mu')}_{\text{running of Hard function}} + \underbrace{2\int_{\mu_J}^{\mu} \frac{d\mu'}{\mu'}\tilde{\gamma}_B^q(\mu',Q/\mu)}_{\mu\text{-running of Beam functions}}\right] \underbrace{\left(\frac{\upsilon_n}{\upsilon_{\bar{n}}}\right)^{\frac{1}{2}\gamma_r^q(b_T,\mu_B)}}_{\text{running in rapidity}}$$

N3LO TMD Beam Function Results

Quark q_T beam function (necessary for Drell-Yan) and
Gluon q_T beam function (necessary for Higgs in gluon fusion) at N3LO

$$\tilde{\mathcal{I}}_{ij}^{\mathrm{TMD}}(z, b_T, \mu, \mu^2)\Big|_{\mathrm{N3LO}} = \tilde{\mathcal{I}}_{ij}^{\mathrm{TMD}\ (3,0)}(z) + \sum_{n=1}^{6} \tilde{\mathcal{I}}_{ij}^{\mathrm{TMD}\ (3,n)}(z) \left[\ln\left(\frac{b_T \mu}{b_0}\right)\right]^n$$

- Matching kernels have only HPLs
  - We provide **expansions** for kernels up to 50 orders for fast numerical evaluation
- N3LO corrections have non trivial z dependence, ~ 0.5-2% deviation from NNLO



- First calculation of gluon beam functions at N3LO
- Found small error in one color structure for all quark-quark channels in quark TMD in the literature

[Luo, Yang, Zhu, Zhu 1912.05778]

N3LO Cross sections

	Q [GeV]	$\delta\sigma^{\mathrm{N^3LO}}$	$\delta\sigma^{\mathrm{NNLO}}$	δ(scale)	$\delta(\mathrm{PDF}+\alpha_S)$	δ(PDF-TH)
$gg \to$ Higgs	m_H	3.5%	30%	$^{+0.21\%}_{-2.37\%}$	$\pm 3.2\%$	$\pm 1.2\%$
$b\bar{b} \to$ Higgs	m_H	-2.3%	2.1%	$^{+3.0\%}_{-4.8\%}$	$\pm 8.4\%$	$\pm 2.5\%$
NCDY	30	-4.8%	-0.34%	$^{+1.53\%}_{-2.54\%}$	$^{+3.7\%}_{-3.8\%}$	$\pm 2.8\%$
	100	-2.1%	-2.3%	$^{+0.66\%}_{-0.79\%}$	$^{+1.8\%}_{-1.9\%}$	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	-0.1%	$^{+2.5\%}_{-1.7\%}$	$\pm 3.95\%$	$\pm 3.2\%$
	150	-2.0%	-0.1%	$^{+0.5\%}_{-0.5\%}$	$\pm 1.9\%$	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	-0.1%	$^{+2.6\%}_{-1.6\%}$	$\pm 3.7\%$	$\pm 3.2\%$
	150	-2.1%	-0.6%	$^{+0.6\%}_{-0.5\%}$	$\pm 2\%$	$\pm 2.13\%$

n3loxs
[Baglio, Duhr, Mistlberger, Szafron ‘22]

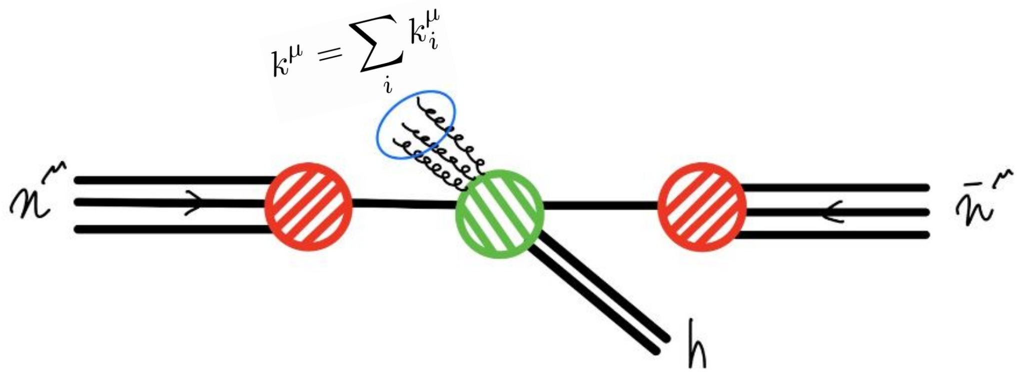
Expansion for Color Singlet Cross Sections

- Consider production of a <u>color singlet</u> state **h** in proton-proton collision
- **Measurements**: total momentum of radiation, color singlet Q and Y

Reverse Unitarity: think of **measurements** as cut propagators!

$$n^\mu = (1,0,0,1)$$

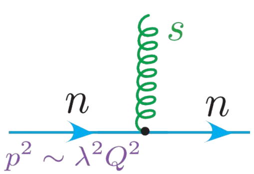
$$\bar{n}^\mu = (1,0,0,-1)$$



$$Y = \frac{1}{2}\log\left(\frac{\bar{n}\cdot p_h}{n\cdot p_h}\right)$$

$$Q^2 = p_h^2$$

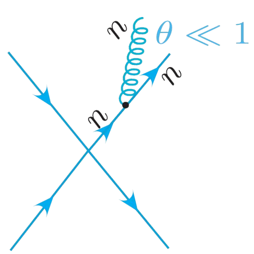
- **Limit** where total momentum of radiation is **soft** compared to Q



$$k^\mu \sim \lambda k^- \frac{n^\mu}{2} + \lambda k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Threshold expansion
(very well known in literature)

- **Limit** where total momentum of radiation is **collinear** to proton axis



$$k^\mu \sim k^- \frac{n^\mu}{2} + \lambda^2 k^+ \frac{\bar{n}^\mu}{2} + \lambda k_\perp^\mu\,, \quad \lambda \ll 1$$

Collinear Expansion
Our work!

Collinear Expansion for Matrix Elements

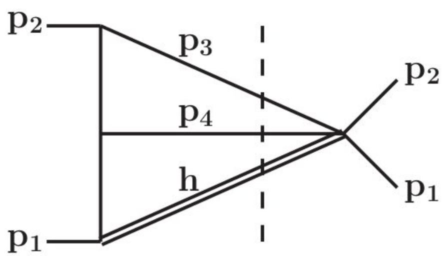
- Kinematic limit $\longrightarrow$ expansion of Feynman integrands appearing in the calculation of **partonic cross sections**

General idea has long history, see e.g.
Expansion by region [Beneke, Smirnov '97]

- Take for example double real emission (RR) scalar integral

$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2},$$

$$x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

$I_{\mathrm{RR}} =$ 

$$= \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \frac{1}{(p_2+p_3)^2(p_2+p_3+p_4)^2}$$

- Differential double real particle phase space scales homogeneously

In the **collinear** limit:

$$\int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx} \rightarrow \lambda^{2-4\epsilon} \int \frac{d\Phi_{h+2}}{dw_1 dw_2 dx}$$

- Propagators can be **expanded** easily

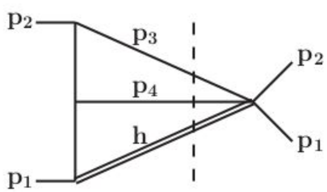
$$\frac{1}{(p_2+p_3+p_4)^2} \xrightarrow{\text{coll}} \frac{1}{2p_2\cdot(p_3+p_4) + \lambda^2\, 2p_3\cdot p_4} = \sum_{n=0}^{\infty} (\lambda^2)^n \frac{(-2p_3\cdot p_4)^n}{\left[p_2^+(p_3^- + p_4^-)\right]^{n+1}}$$

Collinear Expansion for double real graphs

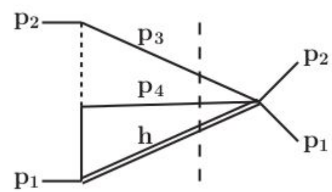
- We can perform a **collinear expansion** of the **integrand**

$$I_{\mathrm{RR}} \xrightarrow{\mathrm{coll}} \lambda^{2-4\epsilon} \int \frac{\mathrm{d}\Phi_{h+2}}{\mathrm{d}w_1 \mathrm{d}w_2 \mathrm{d}x} \left[\frac{1}{(p_2+p_3)^2 \left[p_2^+ (p_3^- + p_4^-)\right]} + \lambda^2 \frac{(-2p_3 \cdot p_4)}{(p_2+p_3)^2 \left[p_2^+ (p_3^- + p_4^-)\right]^2} + \mathcal{O}(\lambda^3) \right]$$

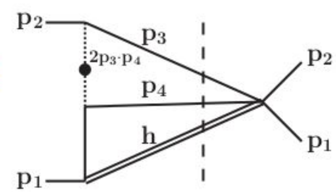
- Collinear expansion admits **diagrammatic** representation!



$\rightarrow \lambda^{2-4\epsilon} \Big[$



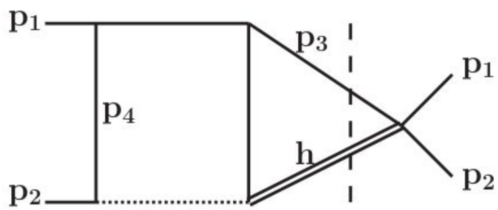
$- \lambda^2$



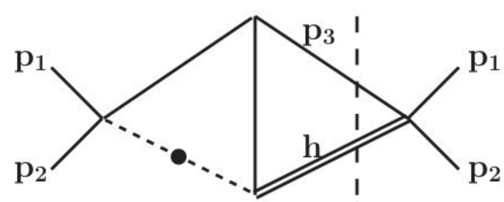
$+ \mathcal{O}(\lambda^3) \Big]$

- Same procedure can be applied for mixed loop/radiation integrals (like RV integrals at NNLO)

$I_{\mathrm{RV}} \xrightarrow{\mathrm{coll}} \lambda^{-2-4\epsilon} \Big[$



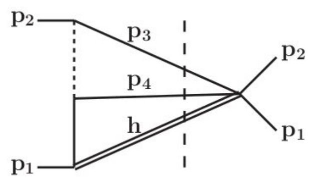
$- \lambda^2$



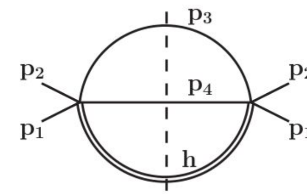
$+ \mathcal{O}(\lambda^3) \Big]$

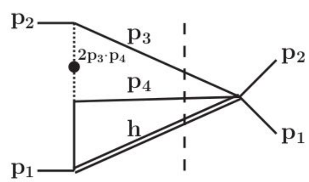
Collinear Expansion and IBPs

Key Point!
Expanded diagrams admit (simplified) integration by parts (**IBPs**) identities

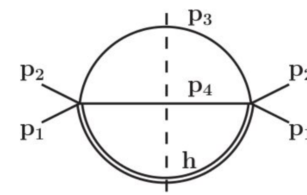


$$= -\frac{1-2\epsilon}{\epsilon(p_2^+k^-)^2} \times$$





$$= -\frac{k^+x}{p_2^+}\,\frac{1-2\epsilon}{\epsilon(p_2^+k^-)^2} \times$$



- We can make use of modern technology for multiloop calculations with simplified kinematic dependence!

IBPs

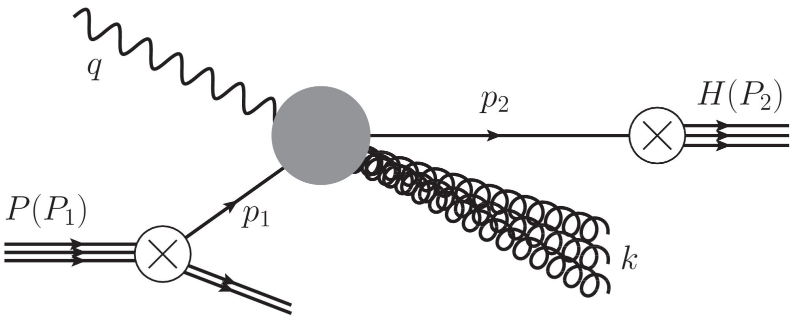
Canonical Differential Equations

Reverse Unitarity

- **Simplifications w.r.t. full kinematics are huge** and enter at each step:
  - IBPs (smaller set of MI, smaller coefficients)
  - System of DE (e.g. ~ 10 MB for differential N3LO in collinear limit vs ~ 10 GB in full kinematics)
  - Space of functions (e.g. @N3LO: Elliptic functions for inclusive color singlet production in full kinematics vs only HPL for q_T distributions in collinear limit)

SIDIS at small $\mathbf{q_T}$

- Factorization for **SIDIS at small $\mathbf{q_T}$** contains **TMD Fragmentation Functions (TMDFFs)**



Factorization for SIDIS at small $\mathbf{q_T}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x_F\mathrm{d}^2\vec{q}_T} = (2\pi)^2\alpha_{\mathrm{em}}\frac{x_B x_F^2}{Q^2}\sum_f H_{f\bar{f}}(q^2,\mu^2) \times \int \frac{\mathrm{d}^2\vec{b}_T}{(2\pi)^2}e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\tilde{B}_f(x_B,\vec{b}_T,\mu,\nu/\omega_a)\tilde{D}_{H/\bar{f}}(x_F,\vec{b}_T,\nu/\omega_b)\tilde{S}_q(b_T,\mu,\nu)$$

TMDPDF ($\mathbf{q_T}$ Beam Functions) — TMD Fragmentation Function

- **TMDFFs** are final state (time-like) analog of TMDPDFs
- **TMDFFs** can be OPEd onto longitudinal Fragmentation Functions (FF) for $q_T \gg \Lambda_{\mathrm{QCD}}$

$$\underbrace{D_{H/j}(x_F,q_T)}_{\text{TMDFF}} \sim \sum_{j'} \underbrace{\mathcal{K}_{jj'}(x_F,q_T)}_{\text{Perturbative Kernel}} \otimes_{x_F} \underbrace{d_{H/j'}(x_F)}_{\text{FF}} + \mathcal{O}(\Lambda_{\mathrm{QCD}}/q_T)$$

LP Collinear expansion of SIDIS

$$\tilde{\mathcal{K}}_{jj'}\left(\zeta,q_T\right) \sim \int_0^1 \mathrm{d}x\,\mathrm{d}w_2\,\delta\left[q_T^2 - Q^2(1-\zeta)w_2(1-x)\right] \lim_{\text{strict coll.}} \frac{\mathrm{d}\hat{\eta}_{\bar{j}+h\to j'+X}}{\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}\bigg|_{w_1=-\frac{1-\zeta}{\zeta}}$$

Expansion in kinematic limits

Kinematic limits can be exploited in different ways

Factorization theorems

- Understand **all order** structure of QCD in that limit
- Identify **universal objects**
- Improves predictions via **resummation** of infinite towers of terms

[See all SCET literature and beyond]

- Expansion of resummed results ⟶ data /cross checks for higher order calculations

Fixed order calculations

- Ingredients entering the calculation **dramatically simplify** (see for example first N3LO results for inclusive XS from Threshold expansion [Anastasiou, Duhr, Dulat, Herzog, Mistlberger 1503.06056])
- Inclusion of subleading terms is not particularly difficult ([Anastasiou, Duhr, Dulat, Herzog, Mistlberger] had >30 terms in threshold expansion, all order factorization at NLP in threshold still in progress)
- Obtain predictions at a perturbative order not reachable in full kinematics

- Kinematic expansions of XS ⟶ data for **subleading power** factorization theorems

They feed into each other: use fixed order in the kinematic limit to obtain universal objects identified by EFT. Examples: Beam Functions, TMDFFs, EEC Jet Functions at N3LO [Ebert, Mistlberger, **GV**]

Collinear Expansion of cross sections

1. Repeat procedure for all diagrams to obtain expansion of **partonic** cross section
2. With the expanded partonic cross section, construct an **expansion** of the **hadronic** cross section around collinear limit of radiation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} = \lambda^{-2}\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} + \lambda^{-1}\frac{\mathrm{d}\sigma^{(1)}}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} + \dots .$$

- **Note:** Translation to expansion of **hadronic** cross section is straightforward knowing relations to hadronic variables

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x} = \frac{\sigma_0}{\tau}\sum_{i,j} x_1 f_i(x_1)\, x_2 f_j(x_2)\, \frac{\mathrm{d}\eta_{ij}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}$$

$$w_1 = -\frac{\bar{n}\cdot k}{\bar{n}\cdot p_1}, \quad w_2 = -\frac{n\cdot k}{n\cdot p_2}$$

$$x = \frac{k^2}{(\bar{n}\cdot k)(n\cdot k)} = 1 - \frac{\vec{k}_\perp^2}{(\bar{n}\cdot k)(n\cdot k)}$$

$$x_1 = \frac{x_1^B}{z_1} = x_1^B\left[\sqrt{1+(k_T/Q)^2} - \frac{\bar{n}\cdot k}{Q}e^{-Y}\right]$$

$$x_2 = \frac{x_2^B}{z_2} = x_2^B\left[\sqrt{1+(k_T/Q)^2} - \frac{n\cdot k}{Q}e^{+Y}\right]$$

$$\frac{\mathrm{d}\eta_{ij}(y_1,y_2)}{\mathrm{d}Q^2\mathrm{d}Y\mathrm{d}\mathcal{T}} = \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}w_1\mathrm{d}w_2\, \delta(y_1 - z_1)\, \delta(y_2 - z_2) \times \delta\left[\mathcal{T} - \mathcal{T}(Q,Y,w_1,w_2,x)\right] \frac{\mathrm{d}\eta_{ij}}{\mathrm{d}Q^2\mathrm{d}w_1\mathrm{d}w_2\mathrm{d}x}$$

Resummation of the EEC in the back-to-back limit

- Extending method of collinear expansion of cross sections to processes with final state color charged particles we were able to calculate EEC Jet Function at **N3LO**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = \frac{\hat{\sigma}_0}{2} \overbrace{H_{q\bar{q}}(Q,\mu)}^{\text{Hard Function}} \int \frac{\mathrm{d}^2\vec{b}_T\,\mathrm{d}^2\vec{q}_T}{(2\pi)^2} e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\,\delta\left(1 - z - \frac{q_T^2}{Q^2}\right) \underbrace{J_q\left(b_T,\mu,\frac{\nu}{Q}\right) J_{\bar{q}}\left(b_T,\mu,\frac{\nu}{Q}\right)}_{\text{EEC Jet Functions}} \overbrace{\tilde{S}_q(b_T,\mu,\nu)}^{\text{TMD Soft Function}}$$

- SCET allows to resum large logs appearing in this limit.
- Each function obeys renormalization group equations (RGEs)

Anomalous dimensions obtained by poles of calculation in the EFT (known in the literature, rechecked in our calculation)

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln H_i(Q,\mu) = \gamma_H^i(Q,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln J_i(b_T,\mu,\nu/Q) = \tilde{\gamma}_J^i(\mu,\nu/Q)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_S^i(\mu,\nu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln J_i(b_T,\mu,\nu/Q) = -\frac{1}{2}\tilde{\gamma}_\nu^i(b_T,\mu)\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\nu}\ln \tilde{S}_i(b_T,\mu,\nu) = \tilde{\gamma}_\nu^i(b_T,\mu)\,.$$

Rapidity Renormalization Group Equations

- Running of operators resum logs as for running coupling in standard QFT