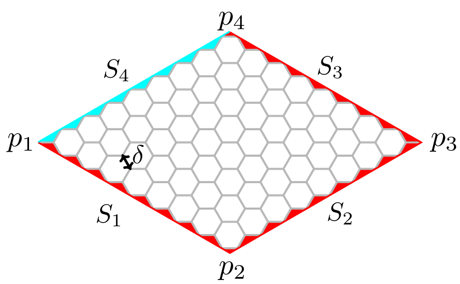
Exercises Day 1

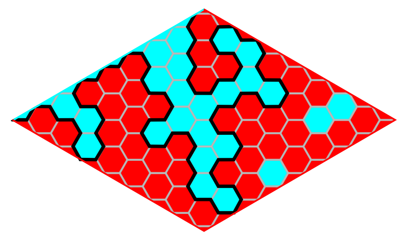
Exercise 1. *(See figures below) In this and a later exercise, we will argue why* SLE(6) *is a natural scaling limit for a percolation interface. Denote by* Λ *the hexagonal lattice, and let* Ω *be a rhombus with corners* p_1, p_2, p_3, p_4 *counterclockwise. Denote by* $\Omega^\delta := \Omega \cap \delta\Lambda$ *the hexagonal* δ*-discretization of* Ω*. Consider a site percolation on* Ω^δ *with Dobrushin boundary conditions:*

- *Color the hexagons on* S_4 *blue, and the hexagons on* S_1, S_2, S_3 *red.*
- *Color each interior hexagon independently either red or blue with equal probability.*

Denote by γ^δ *the interface between the blue and red hexagons started from the point* p_1*.*



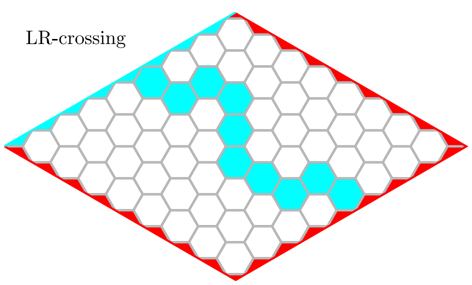
(A) Rhombus Ω and its hexagonal discretization Ω^δ with Dobrushin boundary conditions.



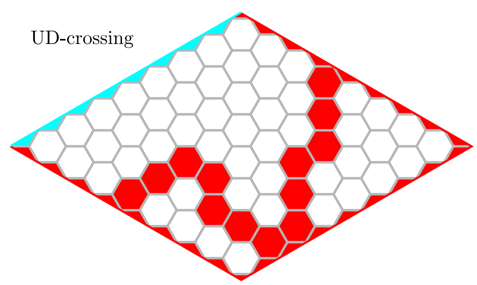
(B) A sample of percolation on Ω^δ. The highlighted interface is the curve γ^δ.

(a) An LR-crossing (UD-crossing) is a path of blue (red) hexagons from S_2 *to* S_4 *(*S_1 *to* S_3*). Show that every configuration has either an LR or UD crossing, but not both. Find a bijection between configurations with LR-crossings and UD-crossings. From this, conclude that both events have equal probability equal to* $\frac{1}{2}$*:*

$$P(\{\textit{There is an LR-crossing}\}) = P(\{\textit{There is a UD-crossing}\}) = 1/2.$$



(A) LR-crossing



(B) UD-crossing

(b) Show that* γ^δ *is a curve from* p_1^δ *to* p_4^δ*, where* p_j^δ *is a vertex in* Ω^δ *closest to* p_j*. Show that* γ^δ *hits the boundary* S_2 *if and only if there is an LR-crossing.*

(c) Assuming γ^δ *has a conformally invariant scaling limit as* $\delta \downarrow 0$*, argue why it has to be an* SLE(κ) *curve in* Ω *from* p_1 *to* p_4 *for some* $\kappa \geq 0$*. Hint: formulate and check a discrete version of the domain Markov Property for* γ^δ*.*

The Virasoro algebra $\mathfrak{Vir}$ is an infinite dimensional Lie algebra with generators $L_k, k \in \mathbb{Z}$ and C satisfying the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{12}n(n^2-1)\delta_{n,-m}C, \quad \text{for } n, m \in \mathbb{Z},$$
$$[L_n, C] = 0.$$

A $\mathfrak{Vir}$-module V is a highest-weight module if

$$V = \mathfrak{Vir}\, v_{c,h}$$

where $v_{c,h} \in V$ is a highest-weight vector of weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$ which satisfies the following relations:

$$Cv_{c,h} = cv_{c,h}, \qquad L_0 v_{c,h} = hv_{c,h}, \qquad L_n v_{c,h} = 0 \quad \forall n > 0,$$

Such a module admits a PBW-basis

$$V = \text{span}\{L_{-n_1} \dots L_{-n_k} v_{c,h} \mid n_1 \geq n_2 \geq \dots \geq n_k > 0, k \in \mathbb{Z}_{\geq 0}\}$$

Exercise 2. *Show that the L_0 eigenvalue of a given basis vector $v = L_{-n_1} \dots L_{-n_k} v_{c,h} \in V$ is given as follows*

$$L_0 v = \left(h + \sum_{i=1}^{k} n_i\right) v.$$

Exercise 2 shows that we can write $V = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} V_\ell$, where

$$V_\ell = \{v \in V : L_0 v = (h+\ell)v\}$$

is the $(h+\ell)$-eigenspace of L_0. Vectors in V_ℓ are called ℓ-level vectors. An ℓ-level vector $w \in V_\ell$ is called singular if it satisfies

$$L_k w = 0 \qquad \forall k \in \{1, \dots, \ell\}.$$

Exercise 3. *The goal of this and a later exercise is to derive the BPZ equation for the correlation functions of a null field at level 2 and relate them to SLE-theory. Fix the central charge $c \in \mathbb{C}$.*

(a) Using the commutation relations show that $w = L_{-1}v_{c,h}$ is a singular vector at level 1 iff $h = 0$.

(b) Show that the generic (up to scaling) two-level vector $w_{c,h} = \left(L_{-2} + aL_{-1}^2\right) v_{c,h}$ is singular if and only if

$$h = h_\pm := \frac{1}{16}\left(5 - c \pm \sqrt{(c-1)(c-25)}\right) \qquad \textit{and} \qquad a = a_\pm := \frac{-3}{2\,(2h_\pm + 1)}.$$

(c) Writing $a_- = \frac{\kappa}{4}$, show that c and h_- can be expressed in terms of κ as $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ and $h_- = \frac{6-\kappa}{2\kappa}$.